AD-784 209

MECHANICS OF A DEFORMED SOLID

L. I. Sedova, et al

Army Foreign Science and Technology Center
Charlottesville, Virginia

17 June 1974

AD784209

DEPARTMENT OF THE ARMY

U.S. ARMY FOREIGN SCIENCE AND TECHNOLOGY CENTER
220 SEVENTH STREET NE.
CHARLOTTESVILLE, VIRGINIA 22901

TRANSLATION

In Reply Refer to:
FSTC-HT-23-1825-73
DIA Task No. T74180174

Date: 17 June 1974

ENGLISH TITLE: Mechanics of a Deformed Solid

SOURCE: Vol. 3, Mekhanika v SSSR za 50 let (Fifty Years of Mechanics in the USSR), pp. 5-467)

AUTHOR: Ed. by L.I.Sedova (Chief Ed.), M.A.Lavrent'eva, G.K. Mikhaylova, N.I.Muskhelishvili, and G.G.Cherog

LANGUAGE: Russian

COUNTRY: USSR

REQUESTOR: S. D/Johnson

TRANSLATOR: LEO KANNER ASSOCIATES

The book gives a cursory survey of the major trends in the mechanics of solids in the USSR over the past 50 years. The major topics discussed are: Linear elasticity theory, nonlinear elasticity theory, plasticity theory, creep theory, the creep of aging materials, the mechanics of soils, the theory of elastic shells and plates, the dynamics of deformable solids, the stability of elastic and non-elastic systems and the mechanics of fracture.

NOTICE

Table of Contents

LINEAR THEORY OF ELASTICITY[1]

by

A. I. Kalandiya, A. I. Lur'ye, G. F. Mandzhavidze,
V. K. Prokopov, Ya. S. Uflyand

(General editor A. I. Lur'ye)

(continued)

1. This article gives a survey of studies in classical (linear) theory of elasticity in our country for the last 50 years. Only static problems in their "rigorous" formulation are considered.

Lack of space made it necessary to exclude almost completely from the survey approximate methods for the solution of problems in the theory of elasticity based on variational principles. Engineering theories (rods, plates, shells) whose construction presupposes the use of additional hypotheses of a kinematic or static content are also not discussed.

§1 and §6.5 of the survey were written by A. I. Lur'e , § 2 and 3 by V. K. Prokopov, §4 and §5.3.9 by Ya. S. Uflyand, § 5 and 6 by A. I. Kalandiya and G. F. Mandzhavidze

§1. General Solutions and Existence Theorems

1.1. General Solutions

In the equilibrium problem of an elastic body in the absence of body forces, general expressions are sought (displacements or stresses) which satisfy simple differential equations constructed in such a way that the equations of the theory of elasticity are satisfied by virtue of these simple equations. The role of the "simple" equation is played by the Laplace equation and the biharmonic equation, and it is desirable to have the smallest number of functions. A knowledge of the general solutions makes it possible when particular solutions of the equations of the theory of elasticity are set up, to use well-known "catalogs" of the solutions of "simple" equations in a particular coordinate system. However, the boundary value problems in the theory of elasticity are, of course, irreducible with the exception of the simplest problems (halfspace, torsion of a body of rotation, etc.) to problems of the Dirichlet or Neuman type for the Laplace equation. Restriction to the case when body forces are absent is not essential since a particular solution corresponding to these forces can also be constructed in the general case, and is easily satisfied when they are partially given (weight, centrifugal forces, etc.).
A survey of the early studies on general solutions was given by B. F. Papkovich (1937) and a unique method for their construction based on the use of the stress function tensor was proposed by Yu. A. Krutkov (1949).

The article of B. G. Galerkin, published in 1930, drew attention to the problem of the construction of general solutions. It was shown that the equations of elasticity theory in the stresses ($\hat{T}$ is the stress tensor, σ is its first invariant)

$$\nabla \cdot \hat{T} = 0, \quad (1 + \nu) \nabla^2 \hat{T} + \nabla\nabla\sigma = 0 \tag{1.1}$$

can be satisfied by expressing the displacement vector u in terms of the biharmonic vector G using the relation

$$2\mu u = \nabla\nabla \cdot G - 2(1 - \nu) \nabla^2 G. \tag{1.2}$$

This solution, with the remark that the earlier known solutions can be obtained from it, but without mentioning its "generality," was proposed by Zh. Bussineskiy already in 1889, and P. F. Papkovich (1937, 1939) has shown that (1.2) is a general solution of the equation of the theory of elasticity in the displacements

$$(1 - 2\nu) \nabla^2 u + \nabla\nabla \cdot u = 0. \tag{1.3}$$

It follows that the structure of equation (1.3), in the presence of a body force in the right member, repeats the structure of the solution (1.2). Therefore, taking in (1.2) for the vector u the solution of the boundary value problem in the theory of elasticity (satisfying three conditions on the surface of the body O), we are justified in expecting that also the vector G can be subjected to three additional conditions on O, which is clearly redundant.

Seeking the displacement vector in the form of the sum of the harmonic vector and the gradient of the scalar χ

$$u = 4(1 - \nu) B + \nabla\chi, \quad \nabla^2 B = 0,$$

we are led after substitution in (1.3) to the equation $\nabla^2\chi = -2\nabla \cdot B$; whose solution is represented as the sum of the particular solution χ = - R·B and the solution χ = - B_0 of the Laplace equation (R denotes the vector radius). Thus,

$$\chi = -(R\cdot B + B_0), \quad u = 4(1-\nu)B - \nabla(R\cdot B + B_0). \qquad (1.4)$$

This representation of the solution of the equation of the theory of elasticity was given by P. F. Papkovich (1932) and somewhat later by G. Neuber. According to a report from P. F. Papkovich it was known earlier to G. D. Grodskiy[1]. The displacement vector (1.4) is expressed as the sum of the harmonic vector B and the gradient of the harmonic scalar B_0, or in terms of the force harmonic functions B_0, B_s (s = 1,2,3) where B_s are the projections of B on the axes of a Cartesian Coordinate system.

Another form of writing solution (1.4) is due to I. S. Arzhanykh (1952) and M. G. Slobodyanskiy (1954), which is

$$u = 4(1-\nu)B + R\cdot\nabla B - R\nabla\cdot B. \qquad (1.5)$$

It differs from (1.4) (when $B_0 = 0$) by the harmonic vector with divergence equal to zero (i.e., the rotor of the harmonic vector). Forms of solutions expressed in terms of harmonic functions with the aid of volume integrals of Newtonian potentials have also been proposed. Such is the representation of Ter-Mkrtich'yan (1947):

$$u = 4(1-\nu)B + \frac{1}{2\pi}\nabla\int_D \frac{\nabla\cdot B}{R}\,d\tau. \qquad (1.6)$$

An integral representation of the displacement vector u in terms of its divergence and rotor and also in terms of the values of u given on the surface of the body and its normal derivative ∂u/∂n was given by I. S. Arzhany (1954).

---

1. The author of the survey is familiar with the article of G. D. Grodskiy published in 1935. The outline for the derivation of the solution (1.4) in the text was given to the author by G. Yu. Dzhanelidze.

Since only three boundary conditions are given on the surface O of the body, it is admissible to retain in the expression for the general solution (1.4) only three harmonic functions, and ignore in it, for example, B_0 or one of the functions B_s (G. Neyber). This problem was studied by B. F. Papkovich (1939) and in greater detail by M. G. Slobodyanskiy (1954).

Inclusion of B_0 in (1.4) is redundant when the vector ∇B_0 can be represented in terms of a harmonic vector B^* with the aid of the relation

$$\nabla B_0 = 4(1-\nu)\, B^* - \nabla R \cdot B^*. \tag{1.7}$$

But then $\nabla \cdot B^* = 0,\ \nabla \times B^* = 0,$ so that $B^* = \nabla\Theta,\ \nabla^2\Theta = 0$ and relation (1.7) is written in the form

$$B_0 = 4(1-\nu)\,\Theta - R\frac{\partial\Theta}{\partial R} = \sum_{n=0}^{\infty} R^n Y_n(\theta, \lambda),$$

Here the harmonic function B_0 is represented in the interior of the simply connected region by a series in the harmonic polynomials $R^n Y_n$. Then

$$\Theta = \sum_{n=0}^{\infty} \frac{R^n Y_n}{4(1-\nu)-n} \tag{1.8}$$

and for $n = 3$, $\nu = 0.25$, $B_0 = R^3 Y_3$ equation (1.7) cannot be satisifed (it is assumed that $0 < \nu < 1/2$). From the above, keeping in mind the Keldysh-Lavrent'ev theorem on the representation of a harmonic function by a uniformly converging series of harmonic polynomials in a finite simply-connected region, we must conclude that the representation (1.7) is improbable for $\nu = 0.25$. For an infinite region with a cavity, in the expression for B_0, n must be replaced by $-(n + 1)$ and the denominator $4(1-\nu) + (n+1)$ in series (1.8) does not vanish for any integer n and $0 < \nu < 1/2$. In this case, it is valid to ignore B_0. It is proved analogously that the solution (1.5) is a general solution for a finite simply connected region, including $\nu = 0.25$, and for an infinite region with a cavity for $\nu \neq 0.25$. More general results can be found in M. G. Slobodyanskiy (1954).

In our opinion, solution (1.4), as other forms of general solutions, should be seen as a useful auxiliary means for the solution of boundary value problems in the theory of elasticity, which makes it possible to use directly the classical special solutions of the Laplace equation. When the solution for a concrete problem is constructed, the retention of the fourth harmonic function makes it easier to select these solutions. Therefore, there is no need to renounce it (A. I. Lur'e, 1955).

V. I. Bloch (1958) starts out by representing the displacement vector u in the form of a sum of the harmonic vector and the gradient of the scalar χ. The solution (1.4) is included in the Bloch representation for $\chi = -(R\cdot B + B_0)$. Letting $\chi = -R^2\nabla\cdot C$, where C is the harmonic vector, Bloch obtained the expression

$$u = 2(1-2\nu)R\nabla\cdot C - 4(1-\nu)R\times(\nabla\times C) - R^2\nabla\nabla\cdot C, \tag{1.9}$$

which can be complemented with the gradient of the harmonic scalar and the rotor of the harmonic vector. The Bloch representation includes also scalar terms which are expressed in terms of three plane harmonic functions.

The forms of the solutions, given by I. S. Arzhanykh (1954) and F. S. Churikov (1953)

$$2\mu u = (1-2\nu)B - \frac{1}{2}\nabla\times(R\times B) \tag{1.10}$$

do not differ from (1.5).

A more general form is given by V. M. Deyev (1959)

$$\begin{aligned}2\mu u = [(4\nu-3)\beta + 4(1-\nu)(\varepsilon-\delta)]B + \beta(\nabla B)\cdot R +\\ + [2(4\nu-3)\varepsilon - \delta]R\cdot\nabla B + \varepsilon R^2\nabla\nabla\cdot B,\end{aligned} \tag{1.11}$$

where β, ϵ, δ are constants, which can be used arbitrarily. When these constants are selected appropriately, we return to the solutions (1.4), (1.5), (1.9). (1.11) also includes the solution

$$2\mu u = 4(1-\nu) B + \frac{2(4\nu-3)}{7-8\nu} R\nabla \cdot B + \frac{1}{7-8\nu} R\nabla\nabla^2 \cdot B, \qquad (1.12)$$

which is expressed in terms of the harmonic vector B and its divergence $\nabla \cdot B$.

The known solution of O. Loew for the axially symmetric case (about the z-axis) follows from (1.2) if we take $G_z = \chi(r, z)$, $G_x = 0$, $G_y = 0$. A more general representation of the solution in cylindrical coordinates (in terms of the harmonic and biharmonic function) is given by S. G. Gutman (1948).

For a multiply connected domain D_0, bounded outside by the surface O_0 and inside by the surface O_i $(i = 1, \ldots, k)$, which lies entirely in D and which has no common points with O_0, the displacement vector for $\nu \neq 0.25$ will be represented in the form (M. G. Slobodyanskiy, 1959)

$$u = u_{(0)} + \sum_{i=1}^{k} u_{(i)}, \quad B = B_0 + \sum_{i=1}^{k} B_{(i)},$$
$$u_0 = 4(1-\nu) B_{(0)} - \nabla R \cdot B_{(0)}, \quad u_{(i)} = 4(1-\nu) B_{(i)} - \nabla R_{(i)} \cdot B_{(i)},$$

where $B_{(i)}$ is a harmonic vector in the region external to O_i and $B_{(0)}$ is a harmonic vector in D_0 where the origin Ω_i of the vector R_i lies inside the cavity bounded by the surface O_i. The form of this solution is "complete" if the ray from Ω_i intersects O_i at a single point, and it will be "general" when O_i is a closed Lyapunov surface[1]).

1. For the distinction between a "complete" and "general" form of the solution, see M. G. Slobodyanskiy (1959).

It should also be noted that the problem of constructing "general solutions" of a system of linear differential equations of the form

$$\sum_{j=1}^{n} L_{ij} u_j = 0 \quad (i = 1, 2, \ldots, n)$$

(in it the L_{ij} are linear differential operators with constant coefficients in the variables $x_1, x_2, \ldots, x_m$) reduces (A. I. Lur'e, 1937) and later (1953) the Rumanian scientist G. Moizil) to finding the "potentials" φ_s (s = 1, . . ., n) in terms of which the solutions u_j are expressed with the aid of a relation of the form

$$u_j = \sum_{s=1}^{n} M_{sj} \varphi_s \quad (j = 1, 2, \ldots, n).$$

Here M_{sj} are the unknown linear differential operators and each potential φ_s satisfies the same differential equation

$$K \varphi_s = 0 \quad (s = 1, 2, \ldots, n).$$

It is easily seen that the operator $K = |L_{ij}|$ is the determinant of the square operator matrix L_{ij}, and that M_{sj} is the cofactor of the j-th column of this determinant. When it is applied to the equations of elasticity theory in the displacements for the isotropic body, the computation that was described leads to the Galerkin-Bussinsk solution (1.2). Clearly, the method can be applied to an anisotropic medium, to dynamic equations in the theory of elasticity, etc.

1.2. Tensor of Stress Functions

We recall that the rotor of the transposed rotor of the tensor $\hat{\Phi}$ is called the incompatibility tensor (Ink) on $\hat{\Phi}$:

$$\mathrm{Ink}\hat{\Phi} = \nabla \times (\nabla \times \hat{\Phi})^T. \tag{1.13}$$

This tensor is symmetric if the tensor $\hat{\Phi}$ is symmetric. In another representation Ink $\hat{\Phi}$ has the form

$$\text{Ink}\,\hat{\Phi} = -\nabla^2\hat{\Phi} + 2\,\text{def}\,\nabla\cdot\hat{\Phi} - \hat{E}\nabla\cdot\nabla\cdot\hat{\Phi} - (\hat{E}\nabla^2 - \nabla\nabla)\,\Phi. \quad (1.14)$$

Here $\hat{E}$ is the unit tensor and $\Phi = I_1(\hat{\Phi})$ is the first invariant of $\hat{\Phi}$, def a = 1/2 ($\nabla a + \nabla a^T$) is an operation on the vector a called the "deformation" of this vector. An example is the linear deformation tensor $\hat{\varepsilon} = \text{def}\,u$. The compactness conditions (the St.-Venant conditions) are expressed by the vanishing of this tensor:

$$\text{Ink}\,\hat{\varepsilon} = \text{Ink def}\,u = 0. \quad (1.15)$$

In general, for any tensor Ink def a = 0. Conversely, if Ink $\hat{\Phi} = 0$, $\hat{\Phi}$ = def a, a vector exists whose deformation is the tensor $\hat{\Phi}$.

In particular for the tensor $\hat{E}\Phi = \hat{E}\,I_1(\hat{\Phi})$

$$\text{Ink}\,\hat{E}\Phi = (\hat{E}\nabla^2 - \nabla\nabla)\,\Phi, \quad (1.16)$$

and for a tensor whose divergence is ($\nabla\cdot\hat{\Phi} = 0$)

$$\text{Ink}\,\hat{\Phi} = -\nabla^2\hat{\Phi} + \text{Ink}\,\hat{E}\Phi. \quad (1.17)$$

In the absence of volumetric forces, such a tensor is the stress tensor $\hat{T}$. Introducing the notation $\sigma = I_1(\hat{T})$ we have in accordance with (1.16) and (1.17)

$$\text{Ink}\,\hat{T} = -\nabla^2\hat{T} + (\hat{E}\nabla^2 - \nabla\nabla)\,\sigma \qquad (\nabla\cdot\hat{T} = 0). \quad (1.18)$$

We will now use Hooke's law for the isotropic body

$$2\mu\hat{\varepsilon} = \hat{T} - \frac{\nu}{1+\nu}\sigma\hat{E}, \quad (1.19)$$

According to (1.15) and (1.17), we have

$$-\nabla^2\hat{T}+\frac{1}{1-\nu}(\hat{E}\nabla^2-\nabla\nabla)\sigma=0, \quad \nabla^2\sigma=0, \tag{1.20}$$

and the second expression was obtained by forming the first invariant of the tensor in the left member of (1.20). Thus, we obtained the Beltrami-Mitchell relations

$$\nabla^2\hat{T}+\frac{\nabla\nabla\sigma}{1+\nu}=0. \tag{1.21}$$

It is known that a tensor with divergence equal to zero can be represented by the rotor of another tensor: if $\nabla\cdot\hat{T} = 0$, $\hat{T} = \nabla \times \hat{C}$, if, in addition, the tensor $\hat{T}$ is symmetric ($\hat{T} = \hat{T}^T$) then introducing into the discussion the symmetric tensor $\hat{\Phi}$, we must take $\hat{C} = (\nabla \times \hat{\Phi})^T$. Then $\hat{T} = \nabla \times (\nabla \times \hat{\Phi})^T$ $= \mathrm{Ink}\,\hat{\Phi}$, $\hat{T}^T = (\mathrm{Ink}\,\hat{\Phi})^T = \mathrm{Ink}\,\hat{\Phi} = \hat{T}$, since $\hat{\Phi}^T = \hat{\Phi}$. It follows that the equations for the statics of a continuous medium are satisfied in the absence of body forces ($\nabla\cdot\hat{T} = 0$, $T = \hat{T}^T$, if we take

$$\hat{T} = \mathrm{Ink}\,\hat{\Phi} \quad (\hat{\Phi} = \hat{\Phi}^T). \tag{1.22}$$

The tensor $\hat{\Phi}$ introduced by Yu. A. Krutkov (1949), V. I. Bloch (1964) and B. Fintzi is called the tensor of the stress functions. The stress tensor $\hat{T}$ remains unchanged if in the expression for $\hat{\Phi}$ a term of the form def a is introduced, where a is an arbitrary vector. This makes it possible to simplify the form in which the tensor $\hat{\Phi}$ is given and retain in its expression only three components. The Maxwell representation has this form (the tensor $\hat{\Phi}$ is diagonal) and also the Morer representation ($\hat{\Phi}$ retains only the components off the diagonal). In the book by B. I. Bloch (1964) $\hat{\Phi}$ is given in three components in Cartesian coordinates, and the book gives nine variants for the three-component representation of $\hat{\Phi}$ in cylindrical coordinates for the symmetric rotation case (see also Yu. A. Krutkov, 1949, p. 108).

The Transformation of Yu. A. Krutkov. Returning to an elastic isotropic medium and taking into consideration that according to (1.20) and (1.16) $\nabla\nabla\sigma = -\,\mathrm{Ink}\,\hat{E}\sigma$, we can write the Beltrami-Mitchell relations in the form

$$\operatorname{Ink}\left(\hat{\Phi}-\frac{\sigma}{1+\nu}\hat{E}\right)=0, \tag{1.23}$$

so that the tensor in the brackets is a deformation of some vector c:

$$\nabla^2\hat{\Phi}-\frac{\sigma}{1+\nu}\hat{E}=\operatorname{def} c. \tag{1.24}$$

At the same time, according to (1.22)

$$\sigma = I_1(\operatorname{Ink}\hat{\Phi}) = \nabla^2\Phi - \nabla\cdot\nabla\cdot\hat{\Phi} = \nabla^2\Phi - \nabla\cdot b, \quad b = \nabla\cdot\hat{\Phi}, \tag{1.25}$$

which allows us to write (1.24) in the form

$$\nabla^2\hat{\Phi}-\frac{\hat{E}}{1+\nu}(\nabla^2\Phi-\nabla\cdot b)=\operatorname{def} c. \tag{1.26}$$

The vector c is eliminated from this expression by forming the first invariants of the tensors in (1.26), and we obtain the equality

$$\nabla\cdot\left(\frac{3}{1+\nu}b-\frac{2-\nu}{1+\nu}\nabla\Phi-c\right)=0,$$

which expresses the vanishing of the divergence of the vector in the brackets. Therefore, this vector is the rotor of the second vector, but the latter can be included as part of the vector b. This determines the vector c and then def c. Substitution in (1.25) leads to the equation

$$\nabla^2\hat{\Phi}=\frac{1}{1+\nu}\hat{E}(\nabla^2\Phi-\nabla\cdot b)+\frac{3}{1+\nu}\operatorname{def} b-\frac{2-\nu}{1-\nu}\nabla\nabla\Phi. \tag{1.27}$$

Now eliminating $\nabla^2\hat{\Phi}$ from expressions (1.14), (1.22) and (1.27), we obtain the representation of the stress vector $\hat{T}$ in an elastic isotropic medium:

$$\hat{T}=\frac{\nu}{1+\nu}\hat{E}(\nabla^2\Phi-\nabla\cdot b)-\frac{1-2\nu}{1+\nu}(\operatorname{def} b-\nabla\nabla\Phi). \tag{1.28}$$

Now, according to (1.19), the deformation tensor $\hat{\mathscr{E}}$ is determined and from it the displacement vector u (the displacement of the solid is discarded):

$$2\mu\hat{\mathscr{E}} = \frac{1-2\nu}{1+\nu}\operatorname{def}(\nabla\Phi - \boldsymbol{b}), \quad 2\mu\boldsymbol{u} = \frac{1-2\nu}{1+\nu}\nabla\Phi - \boldsymbol{b}. \tag{1.29}$$

The formulas of Yu. A. Krutkov (1.28) and (1.29) are one form of the general solution of the equations of linear elasticity theory. They determine on the basis of the tensor of the functions of the stresses $\ddot{\Phi}$ satisfying the differential equation (1.27), the stress tensor T and the displacement vector u. The latter depends only on $I_1(\ddot{\Phi}) = \Phi$ and $b = \nabla\cdot\ddot{\Phi}$. Therefore, it suffices, using (1.27), to obtain only a relation between these quantities. This relation can be obtained by setting up the divergence in the left and right members of (1.27):

$$\nabla^2\boldsymbol{b} + \frac{\nabla\nabla\cdot\boldsymbol{b}}{1-2\nu} = \frac{2(1-\nu)}{1-2\nu}\nabla\nabla^2\Phi. \tag{1.30}$$

Letting

$$\boldsymbol{b} = \nabla^2 G, \quad \Phi = \frac{\nabla\cdot G}{2(1-\nu)}. \tag{1.31}$$

we satisfy equation (1.30) if the vector G is biharmonic. In accordance with (1.29) this leads to the Galerkin-Bussinesk solution (1.2).

A particular solution of equation (1.30) is $b = \nabla\Phi$ and the corresponding homogeneous equation (the right member equal to zero) differs from equation (1.3) only in the values of the constants for the displacement vector. Therefore, the vector b can be constructed on the basis of a solution of the Papkovich type (1.4):

$$\boldsymbol{b} = \frac{1-\nu}{1-2\nu}[4(1-\nu)B - \nabla(R\cdot B + B_0)] + \nabla\Phi, \tag{1.32}$$

and substitution in (1.29) leads to the representation (1.4) of the vector u.

Yu. A. Krutkov obtained on the basis of formulas (1.30), (1.28) and (1.29) many other "general solutions." For example, if the vector K with divergence equal to zero and rotor defined in terms of the vectors b and $\nabla\Phi$

$$\nabla \cdot K = 0, \quad \nabla \times K = \nabla\Phi - \frac{b}{2(1-\nu)},$$

is introduced into the discussion, according to (1.29) and (1.30) we obtain, discarding the inessential constant factor the Korn solution

$$u = \nabla \times K - (1 - 2\nu)\, b \quad (\nabla^2 K = \nabla \times b, \ \nabla \cdot K = 0).$$

1.3. Integral Equations for the Three-Dimensional Problem

The setting up of integral equations for three dimensional boundary value problems and overcoming the difficulties connected with their study, existence proofs and effective methods for constructing their solutions are the results of many years of work of V. D. Kupradze (1963) and his collaborators. A presentation of these methods and the results of these studies with a detailed bibliography can also be found in the monograph of V. D. Kupradze, T. G. Gegeliya, M. O. Basheleyshvili and T. V. Burchuladze published in 1968.

Next, we will consider in this survey only the first and second boundary value problems in three-dimensional elasticity theory for an isotropic homogeneous medium. We will restrict ourselves to the interior problem (i) for a simply connected finite volume (V_i) and the exterior problem (e) for an infinite medium (V_e) with a cavity. It is assumed that the surface which bounds V_i from the outside (V_e from the inside) is smooth.

Potentials in the Theory of Elasticity. The Kelvin-Somilini tensor $\hat{U}(M, Q)$, which determines the displacement $u(M, Q)$ of the point M in the unbounded elastic medium caused by the action of the unit concentrated force e at the point Q is introduced into the discussion

$$u(M, Q) = \hat{U}(M, Q) \cdot e, \quad \hat{U}(M, Q) = \frac{1}{4\pi\mu}\left[\frac{\hat{E}}{R} - \frac{\nabla\nabla R}{4(1-\nu)}\right] \tag{1.33}$$

($\hat{E}$ is a unit tensor, $R = \vec{QM} = r_M - r_Q$, $R = |R|$). The stress tensor $n_M \cdot \hat{T}$ on the small area at the point M with the normal n_M is determined from the expression

$$n_M \cdot \hat{T} = \hat{\Phi}(M, Q) \cdot e, \quad \hat{\Phi}(M, Q) = \\ = \frac{1}{8\pi(1-\nu) R^3}\left[(1-2\nu)(n_M R - R n_M) - 2(1-\nu)\hat{E} n_M \cdot R - R^3 n_M \cdot R \nabla\nabla \frac{1}{R}\right]. \tag{1.34}$$

Let O be a closed surface ($M \subset O$). Then

$$\iint_O R \times \hat{\Phi}(M, Q)\, do_M = 0 \tag{1.35}$$

and the generalized Gauss theorem is valid

$$\iint_O \hat{\Phi}(M, Q)\, do_M = -\hat{E}\delta(Q), \qquad \delta(Q) = \begin{cases} 1 & Q \subset V_i, \\ 1/2 & Q \subset O, \\ 0 & Q \subset V_e \end{cases} \tag{1.36}$$

(V_i is the volume inside O and V_e outside O).

From among the vector potentials of the theory of elasticity introduced by V. D. Kupradze, we will subsequently consider two: the first, which is similar to the potential for the simple layer A(Q) on the surface O, and the second which is similar to the potential for a double layer B(Q):

$$A(Q) = \iint_O a(M) \cdot \hat{U}(M, Q)\, do_M, \tag{1.37}$$

$$B(Q) = \iint_O b(M) \cdot \hat{\Phi}(M, Q)\, do_M. \tag{1.38}$$

Clearly, A(Q) and B(Q) for $Q \not\subset O$ are solutions of the equation of elasticity theory in the displacements in the absence of body forces.

The limiting values from the inside and outside of the first potential on O denoted by

$$A_i(Q_0) = \lim_{V_i \supset Q \to Q_0} A(Q), \quad A_e(Q_0) = \lim_{V_e \supset Q \to Q_0} A(Q),$$

are equal to its direct value which is determined by the improper converging integral

$$A(Q_0) = \iint_O a(M)_0 \cdot \hat{U}(M, Q_0)\, do_M. \tag{1.39}$$

For the limiting values of the second potential, the relations

$$B_i(Q_0) = B(Q_0) - \frac{1}{2} b(Q_0), \quad B_e(Q_0) = B(Q_0) + \frac{1}{2} b(Q_0), \tag{1.40}$$

which are analogous to the Plemeli formulas whose direct value is determined by an integral which only converges in the sense of the principal value:

$$B(Q_0) = \iint_O b(M) \cdot \hat{\Phi}(M, Q_0)\, do_M = \lim_{\varepsilon \to 0} \iint_{O - O(Q_0, \varepsilon)} b(M) \cdot \hat{\Phi}(M, Q_0)\, do_M$$

hold ($O(Q_0, \varepsilon)$ is a 2ε neighborhood of the point Q_0 on O).

When the point $Q \subset V_e$ is sufficiently far from the surface O, $R \to -r_Q$, according to (1.33) and (1.37), we have ($e_Q = r_Q/r_Q$)

$$\lim_{Q \to Q_\infty} A(Q) = \frac{1}{16\pi\mu(1-\nu) r_Q} [(3-4\nu)\hat{E} - e_Q e_Q] \cdot \iint_O a(M)\, do_M;$$

which is the displacement vector on Q under the action of the force applied at the coordinate origin which is given by the integral of the density a(M) on O.

The second potential vanishes not slower than r_Q^{-2}, as $Q \to Q_\infty$, and it may be treated as the displacement formed by a system of forces distributed over the surface O with the principal vector equal to zero.

Integral Equations. In the first boundary value problem the displacement vector u(Q) which takes on a given value $v(Q_0)$ on the surface O (the volume V_i in the interior problem and the "cavity" in the exterior problem) is sought in the form of the second potential in the theory of elasticity with the unknown density b (M):

$$u(Q) = B(Q) = \iint_O b(M) \cdot \hat{\Phi}(M, Q)\, do_M. \tag{1.41}$$

In the case of the exterior problem, this representation presupposes that $u(Q_\infty)$ has the order r_Q^{-2}. The principal vector of forces which must be distributed on the surface O of the "cavity" in order that it have points is the displacement vector $v(Q_0)$ which must be equal to zero. Therefore, the solution of the first external boundary value problem in the form (1.41) may only exist in the special case when $v(Q_0)$ is given. In the general case, the solution will be represented by the sum of (1.41) and the potential of the simple layer (the solution of the Robin elastoplastic problem).

The integral equations for the interior (i) and exterior (e) problem are obtained from specifying (1.41) by a transition to the limit $\lim\limits_{Q \to Q_O} u(Q) = v(Q_0)$ with the aid of the Plemeli formulas (1.40):

$$\mathrm{I}^{(i)} \ldots \frac{1}{2} b(Q_0) - \iint_O b(M) \cdot \hat{\Phi}(M, Q_0)\, do_M = -v(Q_0), \tag{1.42}$$

$$\mathrm{I}^{(e)} \ldots \frac{1}{2} b(Q_0) + \iint_O b(M) \cdot \hat{\Phi}(M, Q_0)\, do_M = v(Q_0). \tag{1.43}$$

It is easily verified, on the basis of (1.30) and (1.36) for $\delta(Q) = 1/2$ that when b(M) is given in the form of a displacement of a solid

$$b^* (M) = r_0 + \omega \times r_M = r_0 + \omega \times r_{Q_0} + \omega \times R \tag{1.44}$$

it is the solution of the homogeneous equation corresponding to (1.43)

$$\mathrm{I}_0^{(e)} \ldots \frac{1}{2} b^* (Q_0) = \iint_O b^* (M) \cdot \hat{\Phi} (M, Q_0) \, do_M = 0. \tag{1.45}$$

Also $b(M) = - b^*(M)$ is a solution of equation (1.42) when the surface O is displaced as a solid . The entire volume V_i is also displaced as a solid which follows from (1.41) and (1.36) for $\delta(Q) = 1$.

In the second boundary value problem the forces $F = (n \cdot \hat{T})_0$ are given on O and the displacement vector is sought in the form of the first potential

$$u (Q) = \iint_O a (M) \cdot \hat{U} (M, Q) \, do_M. \tag{1.46}$$

Using (1.34) after the (nontrivial) transformations which were omitted here, we obtain the integral equations

$$\mathrm{II}^{(i)} \ldots \frac{1}{2} a (Q_0) + \iint_O \hat{\Phi} (Q_0, M) \cdot a (M) \, do_M = F (Q_0) = (n_Q \cdot \hat{T})_0, \tag{1.47}$$

$$\mathrm{II}^{(e)} \ldots \frac{1}{2} a (Q_0) - \iint_O \hat{\Phi} (Q_0, M) \cdot a (M) \, do_M = - F (Q_0) = (n_Q \cdot \hat{T})_0, \tag{1.48}$$

where n_Q is the outer normal to V_i.

It was shown above that the integrals in the vector equations (1.42)-(1.43), (1.47)-(1.48) are discussed in the sense of their principal values. These are singular systems of equations. The difficulty connected with the subsequent discussion consists of proving the applicability to them of the Fredholm theorems and alternatives (for μ and ν which ensure a positive potential energy of the deformations), see B. D. Kupradze (1963, 1968) and also S. G. Mikhlin (1962).

We will rewrite the equations that were obtained in the sequence[1]:

$$\left.\begin{aligned} &\mathrm{I}^{(i)} \ldots \tfrac{1}{2} \boldsymbol{b}(Q_0) - \iint_O \boldsymbol{b}(M) \cdot \hat{\Phi}(M, Q_0)\, do_M = -\boldsymbol{v}(Q_0), \\ &\mathrm{II}^{(e)} \ldots \tfrac{1}{2} \boldsymbol{a}(Q_0) - \iint_O \hat{\Phi}(Q_0, M) \cdot \boldsymbol{a}(M)\, do_M = -\boldsymbol{F}(Q_0); \end{aligned}\right\} \tag{1.49}$$

$$\left.\begin{aligned} &\mathrm{I}^{(e)} \ldots \tfrac{1}{2} \boldsymbol{b}(Q_0) + \iint_O \boldsymbol{b}(M) \cdot \hat{\Phi}(M, Q_0)\, do_M = \boldsymbol{v}(Q_0), \\ &\mathrm{II}^{(i)} \ldots \tfrac{1}{2} \boldsymbol{a}(Q_0) - \iint_O \hat{\Phi}(Q_0, M) \cdot \boldsymbol{a}(M)\, do_M = \boldsymbol{F}(Q_0); \end{aligned}\right\} \tag{1.50}$$

($\mathrm{I}^{(i)}$, $\mathrm{II}^{(e)}$) and ($\mathrm{I}^{(e)}$, $\mathrm{II}^{(i)}$) are coupled pairs.

The corresponding homogeneous equations are written in the form

$$\mathrm{I}_0^{(i)},\ \mathrm{I}_0^{(e)} \ldots \tfrac{1}{2} \boldsymbol{b}(Q_0) - \lambda \iint_O \boldsymbol{b}(M) \cdot \hat{\Phi}(M, Q_0)\, do_M = 0, \tag{1.51}$$

$$\mathrm{II}_0^{(i)},\ \mathrm{II}_0^{(e)} \ldots \tfrac{1}{2} \boldsymbol{a}(Q_0) - \lambda \iint_O \hat{\Phi}(Q_0, M) \cdot \boldsymbol{a}(M)\, do_M = 0, \tag{1.52}$$

1. N. Kinosita and T. Mura (1956) also obtained the integral equations for the first and second boundary value problems in the form given here, but they did not pay attention to the difficulty that was pointed out.

where $\lambda = 1$ for $I_0^{(i)}$, $II_0^{(e)}$ and $\lambda = -1$ for $I_0^{(e)}$, $II_0^{(i)}$.

Existence and Uniqueness of the solution of Problems $I^{(i)}$ and $II^{(e)}$. It is sufficient to verify that $\lambda = 1$ is not an eigenvalue of the homogeneous equation $II_0^{(e)}$ (hence, also, the coupled equation $I_0^{(i)}$). It is proved that the assumption that a solution of $II_0^{(e)}$ exists which is different from the trivial solution ($a(M) \neq 0$) is inconsistent with the requirement that the specific potential energy of the deformation be positive. According to the Fredholm theorem, the existence and uniqueness of the solutions of the nonhomogeneous equations $II^{(e)}$ and $I^{(i)}$ follow for an arbitrarily given $F(Q_0)$ in the first set of equations and $v(Q_0)$ in the second set.

Second Interior Boundary Value Problem $II^{(i)}$. The equation $I_0^{(e)}$ which is coupled with $II_0^{(i)}$ has the nontrivial solution (1.44). Therefore, also $II_0^{(i)}$ has a nontrivial solution and, according to a theorem of Fredholm, the nonhomogeneous equation $II^{(i)}$ can only have a solution when its free term $F(Q_0)$ is orthogonal to (1.44):

$$\iint_O (\boldsymbol{v}_0 + \boldsymbol{\omega} \times \boldsymbol{r}_Q) \cdot \boldsymbol{F}(Q)\, do_Q = \boldsymbol{v}_0 \cdot \iint_O \boldsymbol{F}(Q)\, do_Q + \boldsymbol{\omega} \cdot \iint_O \boldsymbol{r}_Q \times \boldsymbol{F}(Q)\, do_Q = 0.$$

Since v_0 and ω are arbitrary, the principal vector and the principal moment of the surface forces vanish. When this is satisfied, the solution of problem $II^{(i)}$ is determined with an accuracy up to the term involving the displacement of the solid , the solution of the coupled equation $I_0^{(e)}$.

Roben's Electrostatic Problem. The determination of the potential in the field surrounding the closed conductive surface from a given charge on it is known as the Roben electrostatic problem. In the theory of elasticity, the term was introduced by V. D. Kupradze (1963), a stressed state in an unbounded elastic medium is sought when the displacement

$$\boldsymbol{v}_i(Q) = \boldsymbol{v}_0 + \boldsymbol{\omega} \times \boldsymbol{r}_Q. \tag{1.53}$$

is imparted to the solid sealed in it. The solution of the problem is sought in the form of a potential for a simple layer

$$w(Q) = \iint_O a^0(M) \cdot \hat{U}(M, Q)\, do_M, \tag{1.54}$$

and it is proved that in this case the solution of the problem $II_0^{(i)}$ must be taken as the density vector $a^0(M)$ where $F(Q_0) = a^0(Q_0)$ (this follows immediately from the integral equation $II^{(e)}$). Thus, $a^0(Q_0)$ determines the distribution of the displaced solid on the surface O through the reaction of the medium on it. We will denote by $\overset{k}{a}(Q_0)$ and $\overset{K+3}{a}(Q_0)_k$ the distribution on O of these forces, caused by the unit force $\underset{K+3}{V} = i_k$ acting on the body along the axis i_k, and, the corresponding unit moment $n = i_k$. Suppose next that $\overset{r}{u} = i_r$. $\overset{r+3}{u} = i_r \times \omega$ is a system of singular solutions of $I_0^{(e)}$. With this notation

$$\iint_O \overset{k}{a} \cdot \overset{r}{u}\, do_Q = \delta_{kr} \quad (k, r = 1, 2, \ldots, 6). \tag{1.55}$$

This defines the system of singular solutions of the integral equation $II_0^{(i)}$ which are orthonormal to the system of singular solutions $I_0^{(e)}$.

First Exterior Boundary Value Problem ($I^{(e)}$). The problem has a solution if the free term in the equation $I^{(e)}$ is orthogonal to the singular solution $a^0(M^0)$ of the problem $II_0^{(i)}$:

$$\iint_O v(Q_0) \cdot a^0(Q_0)\, do_Q = 0. \tag{1.56}$$

This condition is due not to the problem but to the representation adopted for u(Q) in the form of the second potential. The vector

$$v^*(Q_0) = v(Q_0) - \sum_{r=1}^{6} D_r \overset{r}{u}, \tag{1.57}$$

is introduced in the discussion and the constants D_r are determined in such a way that the orthogonality condition (1.56) is satisfied for this vector. According to (1.55), we have

$$\iint_O v^*(Q_0) \cdot \overset{k}{a}(Q_0)\, do_{Q_0} = \iint_O v(Q_0) \cdot \overset{k}{a}(Q_0)\, do_{Q_0} - D_k = 0 \quad (k = 1, \ldots, 6).$$

Now, taking

$$\left.\begin{aligned} u_0 &= \sum_{k=1}^{3} i_k \iint_O v(Q_0) \cdot \overset{k}{a}(Q_0)\, do_{Q_0}, \\ \omega &= \sum_{k=1}^{3} i_k \iint_O v(Q_0) \cdot \overset{k+3}{a}(Q_0)\, do_{Q_0}, \end{aligned}\right| \tag{1.58}$$

we obtain

$$v^*(Q_0) = v(Q_0) - (u_0 + \omega \times r_{Q_0}). \tag{1.59}$$

Problem $I^{(e)}$ with the three terms equal to $v^*(Q_0)$ has a solution which is determined for $Q \subset V_e$ by the vector $u^*(Q)$. The Roben problem in the form (1.54) is solved from the displacement $v_0 + \omega \times r_{Q_0}$ determined in accordance with (1.58) and the unknown solution for problem $I^{(e)}$ is represented as the sum

$$u(Q) = u^*(Q) + w_e(Q) \quad (Q \subset V_e). \tag{1.60}$$

In the books of B. D. Kuparadze (1963, 1968) integral equations and existence problems for the solutions are considered not only for problems of statics but also for the steady state oscillations of an elastic medium. A number of other boundary value problems are also considered, for anisotropic and inhomogeneous media, thermal elastic problems, problems for a bounded volume and an infinite medium with several "cavities" are also discussed. A number of difficulties connected with the singularity of the integral equations considered are overcome and conceptually simple (but not simple to apply) numerical methods are proposed for the solution of these equations (B. D. Kupradze, 1964, 1967).

The integral equation for problem $I^{(i)}$ was considered in 1907 by G. Laurichella and D. I. Sherman (1962) generalized the solution to the case of an elastic body of finite volume with several "holes".

This survey does not include the approximate solution methods based on the application of variational principals (the Ritz-Timoshenko method and the Galerkin and Kantorovich method). The manner in which they are applied in practice is described in the monograph of L. S. Lebenzon (1951). A large number of studies is devoted to the study of the convergence of variational methods and to error estimates (in a number of cases two sided) of the approximate solutions (S. G. Mikhlin, M. G. Sloboyanskiy).

§2. Three-Dimensional Problems in the Theory of Elasticity

A systemmatic study of three-dimensional problems in the theory of elasticity was undertaken by B. G. Galerkin. Using the representation for the general integral of the equations of elasticity theory found by him in terms of three biharmonic functions (1930) and using series, he developed, starting in the early 30's a method for calculating thick plates which assume that the conditions for arbitrary loads at the ends and the integral conditions on the side surface were satisfied. It was he who studied rectangular, circular, sectional, and triangular plates (1931, 1932). In 1931 Galerkin constructed a solution for the equilibrium problem of a layer subjected to the action of a normal load. With the aid of series, containing Bessel and Hankel functions, Gal erkin considered the problem of the equilibrium of a hollow cylinder and its parts (1933),

and later obtained particular solutions for the problem of an axisymmetric deformation of a hollow sphere (1942).

After these studies, the work of G. N. Maslov (1938) appeared in which the thermoelastic equilibrium of a thick plate, a hollow cylinder and sphere under the action of a stationary thermal flux are considered.

The extension of the Bussinesk problem to the halfspace is given by V. G. Korotkin (1938) who investigated the case when a load is appliedto a rectangle, according to a constant and variable linear law. Problems for the halfspace when the displacements are given on the boundary, and also the case of adjoint halfspaces were considered by D. I. Sherman (1943, 1945). A solution with a singularity of the "center" type at some point of the halfspace was obtained by V. K. Fedyan (1965).

Lately studies have been published which consider the torsion of a halfspace (N. A. Rostovtsev, 1955; B. L. Mintsberg, 1957) and an elastic layer (Ya. S. Ufland , 1959). The case of the torsion of a multilayer medium (base) was discussed by V. I. Petrishin (1965), and the torsion of a two-layer medium was studied by D. V. Grilitskiy (1961).

Problems of the Bussinesk type for an anisotropic medium were considered by V. A. Sveklo (1964). Studies appeared which discussed the behavior of a halfspace consisting of the nonhomogeneous medium: S. G. Lekhnitskiy (1962) studied a halfplane and wedges with variable elasticity moduli, L. N. Ter-Mkrtich'yan (1961) considered three-dimensional problems for a nonhomogeneous medium (the Bussinesk problem for a symmetrically loaded cylinder). A more general form of the nonhomogeneous halfspace and halfplane were studied by N. A. Rostovtsev (1964), the Bussinesk problem for a special type of linearly deformed continuous medium was formulated and solved by A. I. Vinogradov (1966).

The thermoelastic problem for a halfspace bordering on a medium whose temperature is given by a Gaussian distribution was considered by I. B. Kill (1966).

Using Fourier integrals, G. S. Shapiro (1942, 1944) studied the equilibrium of an elastic layer and solved the problem of the transfer of pressure distributed over the area of a circle through the layer on a rock foundation. He studied, together with D. Yu. Eisenberg (1950) the transfer of pressure through a layer with a circular opening. The transfer of pressure through a layer on an elastic foundation with complete adhesion of the layer and foundation was studied by R. M. Rappoport (1948).

The flexure of a thick plate due to a harmonic load on the surface was studied by S. G. Gutman (1940). He also obtained the solution of the problem of the flexure of a thick plate under a natural weight (1941). Later problems of the flexure of thick plates were studied by many authors (S. A. Alekseyev, 1946, B. R. Bloch, 1954, M. I. Guseyan-Zade, 1956, V. K. Prokopov, 1963).

In 1942 A. R. Lur'e proposed a new symbolic method the solution of the equilibrium problem of an elastic layer and a thick plate based on representing the solution of the equations of the three-dimensional elasticity theory problem in the form of entire transcendental functions of the two-dimensional Laplace operator. This representation made it possible to simplify operations on power series which were written compactly with the aid of symbolic operators, and in addition it led naturally to a new class of solutions which made it possible to satisfy more accurately the boundary conditions on the lateral surface of the plate. These solutions were called by Lur'e "homogeneous," since they satisfy the condition for the absence of a load at the ends of the plate.

Lur'e's method as applied to the theory of plates was later used by Ye. M. Krug (1955), R. G. Teregulov (1961), T. T. Khachaturyan (1963), U. K. Nigul (1963). In the monograph of V. A. Agarev (1963) the domain of applicability of the symbolic method is extended to the theory of plates, and a further application of the symbolic method to the theory of plates in combination with the minimum potential energy principle is given by B. K. Prokopov (1965). In the work of S. G. Lekhnitskiy (1959, 1962), the symbolic method is used in the discussion of the equilibrium of a transverse isotropic layer and a thick plate. The same author also obtained the corresponding homogeneous solution. P. F. Nedorezov (1964) solved, using the symbolic method, the problem of the torsion of a hollow multilayer cylinder.

Using the symbolic representation of the solutions, it was easily established (A. R. Lur'e 1955) that in an unbounded plate ($|z| < h$) the components σ_x, σ_y, τ_{xy} of the tensor of thermal stresses are different from zero. They are expressed in terms of the function M (x, y, z), which plays a role similar to that of the Eyre function in the plane problem. The function M is determined from quadratures according to the given law for the stationary temperature distribution.

S. S. Dymakov (1966) solved with the aid of the Fourier transform the problem of the equilibrium of an elastic layer. This approach also enabled the author to obtain asymptotic formulas for the solution. For a layer for which the displacements on its boundaries are given (the second fundamental problem) a solution in a series was obtained by M. D. Martynenko (1964). The action of a concentrated force inside the layer was considered by O. Ya. Shekhter and O. Ye. Prikhodchenko (1964). In particular, these authors obtained the solution of the problem of the action of a vertical force inside a layer on a rock base. The case of a layer of variable thickness and a circular plate of variable thickness under an axisymmetric load was analyzed by I. I. Semenova (1965).

The equilibrium of a circular thick plate under a uniformly distributed load was studied with the aid of homogeneous solutions by G. N. Bukharinov (1952), who applied P. F. Papkovich's generalizedorthogonality relation (1940). This relation was derived by Papkovich for the boundary conditions for functions of the homogeneous solutions which corresponded to the vanishing of the functions themselves and their first derivatives on the parallel side of the strip. A rigorous foundation for the Papkovich method was later given by G. A. Grinberg (1953). The equilibrium of a circular plate under the action of an arbitrary axisymmetric load was investigated with the aid of homogeneous solutions by V. K. Prokopov (1958). The axisymmetric flexure of a circular plate in a very general formulation was studied by B. L. Abramyan and A. A. Babloyan (1958). An exact solution of the problem of equilibrium of a plate fixed on the lateral surface was obtained by V. T. Grinchenko and A. F. Ulitko (1963) with the aid of an infinite system of equations. Analogous results were obtained by G. N. Valov (1962). Certain special cases of the axisymmetric torsion of thick plates were studied by N. D. Glazunova (1963). A. A. Babloyan (1964) studied a non-axisymmetric load on a circular plate when the displacements are given on the lateral surface (the solution was represented in double series, whose coefficients were found from infinite systems).

An infinite thick plate with a circular opening was considered in the study of O. K. Aksentyan (1965). Using homogeneous solutions it was possible to solve the problem of the concentration of stresses near the opening by reducing the problem to an infinite system of equations for the coefficients of the homogeneous solutions. M. Abenova (1965) reduced a similar problem to integral equations of the Fredholm type.

The nonstationary problem of thermoelastic (quasi-elastic) equilibrium of a thick plate was discussed by A. A. Sheveled (1965). R. M. Rappoport (1962) obtained approximate homogeneous solutions for a thick plate which were constructed on the assumption of the absence of a transverse deformation. The last assumption leads to orthogonal eigenfunctions.

The elastic equilibrium of an infinite cylinder was studied by many authors. The axisymmetric problem of the action of normal pressure on a hollow cylinder applied on a sector of the lateral surface was considered in 1943 by G. S. Shapiro. He obtained a solution for this problem with the aid of Fourier-Bessel integrals (this solution was also obtained later by V. N. Popov, 1956). Homogeneous solutions for a solid and hollow cylinder with an axisymmetric deformation were studied by V. K. Prokopov (1949, 1950). The axisymmetric problem for an infinite solid cylinder under normal loads on the lateral surface was studied in 1953 by A. I. Lur'e. The solution of this problem represented in the form of Fourier integrals is expressed, using contour integration, in terms of functions which correspond to the homogeneous solutions of the problem of the cylinder. The solution of a girdled cylinder is obtained by passing to the limit. The case of a tangential load, and also the case of the flexure of an infinite cylinder by surface forces were studied using the same method in the articles of P. Z. Livschits (1960, 1963, 1964).

The complex loading of an infinite cylinder on its lateral surface, when the load can be represented by a Fourier integral with respect to the axial coordinate and a Fourier series in terms of the angle was studied by K. V. Solyanik-Krass (1960). He also considered the more general problem of the equilibrium of a body of revolution, when the trigonometric functions of the meridional angle can be isolated in the form of individual factors in the solution (1958). For a hollow cylinder, he investigated (1965) the effect of a load distributed on the lateral surfaces in the direction of the angle φ in an arbitrary manner, which represented a polynomial in the coordinate of the z axis (at the ends the integral conditions were satisfied).

The mixed axisymmetric problem for an infinite solid or hollow cylinder was considered in the articles of B. I. Kogan, A. F. Khrustalev, F. A. Vaynshteyn (1958, 1959, 1963). The Loew stress function was constructed by them in the form of a contour integral containing appropriately selected functions depending on the parameters of the homogeneous solutions for the cylinder. The study of P. I. Kogan, A. F. Khrustalev, (1959) used the method of coupled integral equations.

The equilibrium of a solid and hollow finite cylinder in the axisymmetric case was studied with the aid of homogeneous solutions by V. K. Prokopov (1950, 1958). G. N. Bukharinov (1956) reduced the solution of the problem of an axisymmetric deformation of a solid cylinder of finite length to finding an additional function for which an integr - differential equation is set up. In recent years, many studies appeared which are devoted to the axisymmetric equilibrium problem of a solid cylinder of finite length, in which the solution of the problem is reduced to infinite systems of linear algebraic equations (B. L. Abramyan, 1954; G. M. Valov, 1962; V. A. Likhachev, 1965). The compression of a circular cylinder was studied by G. M. Valov (1961) and Ye. P. Miroshnichenko (1957). The equilibrium of a revolving cylinder was studied by V. T. Grinchenko (1964), who also gave a very comprehensive analysis of all aspects under which the boundary conditions in the axisymmetric problem for a semi-infinite cylinder are satisfied (1965). The axisymmetric deformation of a cylinder of finite length made from a transversal-isotropic material was studied by A. A. Babloyan (1961).

In some cases, it is possible to satisfy all boundary value conditions in the equilibrium problem of a cylinder of finite length without having to solve infinite systems (see B. L. Abramyan, 1958; G. M. Valov, 1957, 1958).

The complexity of satisfying simultaneously all boundary conditions on the surfaces of the cylinder made it necessary to seek approximate methods for the solution of the problem. Thus, S. I. Trenin (1952) represented the stressed state in terms of two tensors: the principal and correction tensor, where the latter does not yield stresses on the lateral surface (homogeneous solutions), and his parameters are determined energetically. The more general (not axisymmetric) problem of a hollow cylinder was studied in an analogous manner by V. I. Ionov (1957). Ya. S. Shein (1962) gave the construction of the correction tensor in first approximation.

The nonsymmetric deformation of a thick-walled cylinder was studied with the aid of series containing Bessel and McDonald functions in the work of I. I. Smolovik and A. N. Shchepetev (1961) and in a number of studies of V. S. Sumtsov (1957-1959). A rigorous satisfaction of the boundary conditions in the general case of a hollow cylinder under a load, leading to infinite systems was obtained by E. N. Bayda (1959, 1960).

The articles of A. L. Kvitki (1959) are devoted to the development of techniques which can be used to reduce the study of the axisymmetric deformation of a thick-walled cylinder to computers (1959).

The symbolic method of A. I. Lur'e as applied to solid and hollow cylinders mainly under an axisymmetric load, was used by F. A. Gokhbaum (1964)

An approximate method for calculating hollow (and solid cylinders) under an axisymmetric load) was proposed by V. L. Biderman (1946, 1950), who represented the tangential stress in the form of a sum of the products of the axial and radial functions. Biderman, using appropriate functions of the radius, for the axial functions derived ordinary differential equations which followed from minimum potential energy theory, and contained in the right members functions of the normal loads applied along the level surfaces of the cylinder. The method was subsequently extended to the case when tangential forces are present by V. G. Gorskiy (1963).

Another approximate method for calculating hollow cylinders, under a normal load to the lateral surface was proposed by S. V. Boyarshinov (1953), who proposed to use for the displacements expressions which are a generalization of those used in the theory of thin elastic shells. An original method of successive approximations as applied to the equilibrium problem of a cylinder was developed by F. M. Detinko (1953), who constructed a solution in a power series in powers of a small parameter (Poisson ratio).

The stationary thermoelastic equilibrium problem of a hollow cylinder (in the axisymmetric case) was first studied by P. M. Ogibalov (1954), and then by Yu. N. Shevchenko (1958) who took into account the change in the elasticity modulus of the material along the axis of the cylinder. A. N. Podgornyy (1965) took into account the effect of the end of the cylinder and also of centrifugal forces. An approximate solution was obtained for the problem using the Lagrange variational principle. P. I. Yermakov (1961) and V. A. Shachnev (1962) considered the stationary thermoelastic problem for a solid cylinder of finite length during its axisymmetric deformation. In the first study the conditions at the ends were satisfied approximately, in accordance with the Biderman method, and in the second study the solution of the problem was reduced to the solution of an integrodifferential equation. The stationary thermoelastic problem for an infinite cylinder with several "holes" was formulated by A. S. Kosmodamianskiy (1962). The temperature field and the thermoelastic state are determined by the Bubnov-Galerkin method.

The nonstationary thermoelastic problem for a hollow rotating cylinder was studied by Yu. N. Shevchenko (1961) who satisfied approximately the conditions at the end with the aid of the Castigliano variational method. A. A. Shevelev (1966) solved the thermoelastic problem for an infinite cylinder, where the temperature of the surrounding medium varies according to an exponential law which is a function of time. He determined the relation for the maximum thermal stresses as functions of the heating parameters, which makes it possible to formulate the optimal problem. A. I. Uzdalev (1962) studied the nonstationary plane axisymmetric thermal elastic problem for solid and hollow cylinders from an anisotropic material.

Homogeneous solutions for a hollow sphere in the case of an axisymmetric deformation were obtained in 1943 by A. I. Lur'e. Using these solutions, it was possible to solve the problem for a hollow sphere cut by a conical surface with vertex at the center of the sphere at one or both of its poles. Lur'e also estimated the accuracy of the solutions which were based on applying the kinematic Kirchoff-Loew hypotheses to a spherical shell.

The equilibrium problem of a hollow sphere for an arbitrary deformation was solved by A. I. Lur'e (1953) with the aid of the general P. F. Papkovich solution. Selecting appropriately the fourth function and applying harmonic vectors, the author was able to reduce considerably the number of computations both in the case of the second fundamental problem and in the case of the first fundamental problem for a hollow sphere. The results of the studies of Lur'e in three-dimensional problems of the theory of elasticity are collected in his monograph (1955), which also contains solutions of the problem of a heavy and rotating sphere with a spherical cavity in an infinite medium, and other problems.[1]

1. The solution of the stressed state problem in an unbounded elastic medium near an ellipsoidal cavity for given stresses at infinity published in the monograph of A. I. Lur'e (1952) is incorrect. A solution for more general conditions at infinity is given by Yu. N. Podil'chuk (1964). Later, A. R. Lur'e (1967) considered the stressed state formed in an elastic medium when the rigid ellipsoid embedded in it receives successive displacements and rotation (Robbins elastostatic problem).

Another method of solving the problem of a sphere based on the connection between the plane and axisymmetric problems in the theory of elasticity, using the theory of analytic functions was proposed by A. Ya. Aleksandrov and Yu. I. Solov'ev (1962).

Compression, pure flexure, and the flexing of a hollow sphere cut at the poles by conical surfaces by a force, were considered by K. V. Solyanik-Krass (1962). The stressed state in a spherical strip under the action of internal pressure was studied by A. F. Ulitko (1962).

The problem of the stressed state of a heavy elastic block near a vertical cylindrical cavity was first formulated by A. N. Dinnik (1925) in connection with the problem of the pressure of rocks. Subsequently, this problem was studied in greater detail by S. G. Lekhnitskiy (1938, 1940) including a transversal-isotropic halfspace. The effect of a cylindrical cavity on the stress concentration for a volumetric stressed state was studied by S. G. Gutman (1960). G. G. Chankvetadze (1956, 1959) took into account the action of external forces applied on a sector of the surface of the cylindrical cavity in an elastic halfspace. In other studies he considered an elastic halfspace with spherical (1955) and cylindrical (1956) cavities. His method is based on introducing in the axisymmetric problem complex variables and applying the methods of N. I. Muskheleshvili. The concentration of stresses near a spherical cavity in a heavy halfspace was studied by N. P. Fleyshman and V. N. Gnatykiv (1954).

R. N. Kaufman (1958) considered the problem of an elastic layer containing a spherical cavity. Her method of solution consists of translating the coordinate origin of the spherical system and of introducing translation formulas for the spherical functions. In another article, Kaufman (1964) solved, using the same methods, the equilibrium problem of a sphere with a spherical cavity which was not concentric. P. I. Perlin (1964) constructed a solution of the second fundamental equilibrium problem for a hollow ellipsoid of rotation whose internal surface is a sphere. Yu. N. Podil'chuk (1965) studied in spherical coordinates the interior and exterior problem for an ellipsoid of rotation. In the three studies that were mentioned here, the solutions are constructed in series, whose coefficients must be determined from an infinite system of equations.

V. N. Zharkov (1963) formulated the important problem of thermoelastic stresses in a gravitating sphere with an arbitrary temperature distribution. The stationary thermoelastic problem for a hollow sphere whose modulus is a power function of the radius was solved by I. N. Danilova (1962).

The equilibrium problem of a cone (solid and hollow) under the action of an axisymmetric load was considered in 1944 by G. S. Shapiro. He obtained polynomial solutions for the problem for certain types of surface loads and for the effect of the gravitational force. This problem was investigated, using a different method, by A. Ya. Aleksandrov (1962). The action of a concentrated moment applied to the vertex of a cone was studied by A. F. Ulitko (1960). In another study (1960), the general equilibrium problem of an elastic cone is solved with the aid of the Mellin transformation. The elastic equilibrium of an axisymmetric loaded cone was also considered by K. B. Solyanik-Krass (1955, 1962) and the solution is represented by him in the form of a Fourier integral. V. N. Ionov (1963) gave the solution of the problem of the axisymmetric deformation of a conical body where the satisfaction of the boundary conditions leads to an infinite system of equations for the constants of the correction tensor. The torsion of a cone through a surface load was considered by K. V. Solyanik -Krass (1965) and P. F. Nedorezov (1965).

The problem of the equilibrium of a heavy paraboloid of revolution was solved by G. S. Shapiro (1950). The expansion and flexure of a paraboloid and also the expansion and flexure of a body containing a paraboloid cavity were considered by K. V. Solyanik-Krass (1958), and in another study (1958) he investigated the compression of an ellipsoid and a hyperboloid with a single cavity. N. N. Lebedev and I. P. Skal'skaya (1966) investigated the torsion of a hyperboloid.

A. F. Zakharevich (1952) studied the equilibrium of a hollow torus with the aid of toroidal coordinates. V. A. Levshin (1962) constructed the solution of the problem of a hollow torus subjected to external and internal pressure. The torsion of a torus of a circular cross section in connection with the calculation of helical springs with small windings was studied in detail by K. V. Solyanik-Krass (1950). The solution obtained by him with the aid of bipolar coordinates contains series including hyperbolic, trigonometric functions and associated Legendre functions.

The expansion of a circular beam containing a small ellipsoidal cavity was investigated by K. V. Solyanik-Krass (1958) using ellipsoidal coordinates. N. A. Forsman (1958) solved the problem of concentrated stresses in an expanded beam with a circular cross section at the spot where the thickness varied. The solution was obtained in the form of definite integrals which were then evaluated approximately.

The studies of I. I. Vorovich and his students devoted to the construction of asymptotic solutions for plates and shells began to appear in 1963. The basis for the construction were the homogeneous solutions corresponding to the three dimensional elasticity theory problem. Infinite systems of equations were set up using the variational Lagrange method for the contour values of the unknown functions. The solutions of the systems were constructed in power series in powers of the thickness of the plate or shell. The problem of the flexure of the plate was investigated using this method (O. K. Aksentiyan and I. I. Vorovich 1963, 1964) and also the axisymmetric problem of the equilibrium of a cylindrical and spherical shell (N. A. Bazarenko and I. I. Vorovich, 1965, T. V. Vilenskaya and I. I. Vorovich, 1966).

The classical Lame problem for the equilibrium of a rectangular parallelopiped, loaded on all edges by given forces attracted the attention of many investigators, starting with the work of M. M. Filonenko-Borodich. In the first article along these lines, published in 1946, M. M. Filonenko-Borodich introduced into the discussion cosine binomials, a sequence of complete non-orthogonal functions on the interval on which they are defined, which vanish together with their first derivatives at the endpoints of the interval.

In the subsequent studies of M. M. Filonenko-Borodich the cosine binomials were used for the approximate solution of the elastic equilibrium problem of a rectangular parallelopiped. The idea of solving the problem consisted of decomposing the stress tensor into two parts: the principal tensor satisfying the equilibrium equations and the conditions on the edges of the parallelopiped and the correction tensor constructed with the aid of the cosine binomials and their derivatives. The latter tensor, which satisfies the equilibrium conditions and zero boundary conditions contains arbitrary constants which are determined by the variational methods of Castigliano. M. M. Filonenko-Borodich (1951) studied the problem of the compression of a parallelopiped under equal loads oriented in the opposite directions and he considered the thermoelastic equilibrium of a parallelopiped. Later (1953) he extended the method to the case of cylindrical coordinates. The concept of the selection of the principal tensor for a parallelopiped under an arbitrary load is due to him (1957).

Ye. S. Kononenko applied the method of M. M. Filonenko-Borotich to the study of the problem of the flexure of a thick plate (1953) and the compression of a parallelopiped between rigid plates (1954). The case of an oblique parallelopiped was studied by A. I. Meshkov (1961). V. N. Spikhtarenko (1959) used this method in calculations of a plate on an elastic parallelopiped.

Another approach to the solution of the equilibrium problems of an elastic parallelopiped was developed in the studies of B. A. Bondarenko (1961, 1963) who used polynomial solutions of the equations of elasticity theory in the displacements in which the arbitrary coefficients in these solutions were determined using the method of least squares.

Certain special problems for a rectangle whose solutions could be obtained in series were considered by G. M. Valov (1959), A. P. Melkonyan (1960), A. A. Babloyan and S. M. Saakyan (1964).

The two articles of E. N. Baydy (1958, 1959) are devoted to the study of the equilibrium problem of a parallelopiped, using infinite systems. More detailed studies of the solution of the equilibrium problem using infinite systems for various types of loads and various boundary conditions were carried out in the work of G. M. Valov (1957-1959, 1966), S. M. Saakyan (1965), A. A. Baloyan and S. M. Saakyan (1964). This series of studies also considers the first and second fundamental problems as well as certain mixed and contact problems, and special attention is given to the proof of regularity (or quasi-regularity) of the infinite systems that are obtained.

§3. The St.-Venant and Almanzi Problems

It is known that the problem of the free torsion of a prismatic rod reduces to the harmonic problem for which solution methods have been developed. The early studies on the theory of the torsion of rods are devoted to the solution of the problem in closed form or with the aid of trigonometric series. These studies include the articles of B. G. Galerkin in which the torsion of a prism with a cross section in the form of an equilateral triangle with equal edges is studied (1919) and a prism of a parabolic cross section (1924). A number of problems of the torsion of cross sections bounded by algebraic curves were solved in the studies of D. Yu. Panov (1935, 1937) and D. L. Gavry (1939) and later V. I. Bloch (1959) studied the torsion of parabolic prisms. The effect of a radial crack during the torsion of a solid and hollow rod was studied in the articles of A. Sh. Lokshin (1928) and V. N. Lyskov (1930). The monograph of A. N. Dinnik, published in 1938 is devoted to various methods for the solution of the problem of the theory of torsion.

In 1925 G. V. Kolosov and D. L. Gavra applied for the first time in the solution of the torsion problem complex variables. They considered the problem of the torsion of a non-circular sector with a small central angle. The fundamental results along these lines were obtained by M. I. Muskhelishvili (1929) who has shown that the problem of the torsion of simply- and doubly-connected regions reduces to finding a function of a complex variable which maps the given region, respectively, onto a circle or onto a circular ring. The methods of the theory of functions of a complex variable were applied in the solution of the problem of torsion of prismatic bars of various cross sections in the articles of D. V. Avazashvili (1940), A. V. Batyrev (1953), Kh. M. Mushtari (1938), A. G. Ugodchikov (1956), et al.

R. O. Kuz'min (1946) used conformal mapping in a different form. He obtained a convenient formula for the direct calculation of the rigidity of a twisted beam. This formula made it possible to calculate the rigidity for cross sections whose contour contains the corner points. Another study in which the complex variable method is extended to the case of a countour goes back to P. P. Kufarev (1937). The method of Kufarev was used by O. I. Babakova (1954) in the study of the torsion of a Z-shaped cross section.

Using the method of conformal mapping, Ye. A. Shiryayev considered the torsion of a shaft with a radial and also with a longitudinal arc-like crack (1956). In another study Shiryayev investigated the torsion of a circular shaft with two cuts of different depths, along the diameter of the cross section (1958). The torsion of shafts with circular grooves was studied by A. A. Skorobogat'ko (1958, 1962). The torsion of hollow airfoils with the aid of the theory of functions of a complex variable was studied by G. A. Tirskiy (1959).

The approximate solution of the problem of the torsion of angular, cross-shaped and T-shaped cross sections with the aid of conformal mapping was obtained by B. I. Makhovikov (1957). A. G. Ugodchikov (1956) who developed approximate conformal mapping methods, studied the torsion of a circular shaft with teeth and a shaft in the form of a pipe with internal teeth (slotted coupling).

A new method for solving the problem of the torsion and flexure of hollow rods was proposed in 1948 by D. I. Sherman. The method consists of introducing an auxiliary function, which is related on one of the edges of the doubly-connected region to the complex torsion function by a certain relation. This auxiliary function satisfies the Fredholm integral

equation, whose solution reduces to the solution of a quasi-regular (and sometime regular) infinite system of linear algebraic equations. Sherman solved, using this method, a number of concrete problems for the torsion of doubly-connected cross sections bounded by circles and ellipses (1950, 1951, 1953).

Additional theoretical studies along these lines, which led to the solution of a number of problems of the torsion of hollow rods were carried out by D. I. Sherman (1953, 1955, 1959), R. D. Stepanov and D. I. Sherman (1952), Yu. A. Amenzade (1958). The method of Sherman was used in the studies of L. K. Kapanyan (1952, 1957), V. I. Yakovyeva (1956) and also by I. A. Bakhtiyarov (1959) for the torsion of a box-shaped rod, by M. U. Ismailov (1959) in the problem of the torsion of a circular shaft with a triangular prismatic cavity and by M. I. Nayman (1958) in the problem of the torsion of a circular shaft with a polygonal coaxial cavity.

The exact solution of the problem of the torsion and flexure of prismatic rods with a cross section bounded by the two arcs of intersecting circles (moonshaped) was obtained in 1949 using bipolar coordinates by Ya. S. Uflyand. A detailed presentation of the solutions of the flexure and torsions problems for regions in which the solution can be obtained in bipolar coordinates is given in his monograph (1950). Later, V. I. Bloch (1956) published an article in which he applied bipolar coordinates to the problem of the torsion of a rectangle formed by the arcs of orthogonal circles. The torsion of a rod with a lenticular cross section was considered by Ya. I. Burak and M. Ya. Leonov (1960). S. A. Gridnev applied polar coordinates to the study of the torsion of a doubly connected cross section (1963) and reduced the solution of this problem to a infinite system of equations.

K. A. Kitover (1954) obtained the solution of the problem for a sector of a ring. For a number of regions, formed by the arcs of ellipses and hyperbolas, the exact solution of the problem of the torsion in elliptic coordinates were obtained by V. I. Bloch (1964).

Approximate methods for the solution of the problem of the torsion and flexure of beams were developed by D. Yu. Panov (1934, 1936, 1938), who developed the method of a small parameter and the graphical method, and studied the torsion of nearly-prismatic rocks and the torsion and flexure of a helical profile. He also studied, using finite differences, the problem of a double-T-shaped beam and shaft with a key joint.

In the studies of M. G. Slobodyanskiy on the theory of torsion (1939, 1940, 1951) the method of finite differences is applied only to one variable, and the solution of the problem is reduced to a system of ordinary differential equations. This method enabled Slobodyanskiy, and then also A. M. Pivovarov (1953) to calculate the coefficients of the concentrations in the angles of inlet of the polygonal cross sections. An analogous technique was used by V. M. Fadeyeva (1949) in the solution of the problem of the torsion of a rod with a trapezoidal cross section. The problem of the torsion of a standard section was studied by B. N. Lopovok (1952). B. A. Rozovskaya (1940) studied, using the method of finite differences, the torsion of rolled section (angle, channel, and H-beam). In another study of this author (1956) and also in the study of Ye. P. Obolenskiy (1959), this method was also used to solve the problem of the torsion of a shaft with slits.

Among other approximate methods for solving problems of the flexure and torsion of prismatic beams, the most important ones are variational methods which became very popular primarily due to the work of L. S. Leybenzon and L. V. Kantorovich. In the first study of L. S. Leybenzon on the theory of torsion which was published in 1924, the problem of the torsion of a helical profile was studied. In this study an approximate expression was obtained for the torsional rigidity of the profile of the helix. V. P. Vepchinkin (1926) and D. Yu. Panovich (1937) made this formula more precise.

The study of L. S. Leybenzon (1935) on the theory of the flexure of prismatic rods in which he developed in detail an effective variational method for the solution of this problem and investigated the problem of determining the flexural center of the profile and in which he also obtained for the first time the theorem on the circulation of the tangential stress during bending is of great importance. A further extension of the problem of finding the flexural center was obtained in the studies of N. V. Zvolinskiy (1936), D. Yu. Panov (1934) and G. Z. Proktor (1936).

The results of the studies of L. S. Leybenzon on the theory of the flexure and torsion of beams over many years and also on the development of effective techniques for the solution of the problems are summarized in his monograph (1943).

In 1933 L. V. Kantorovich proposed a new approximate method for the solution of the problem of finding the minimum of a double integral, according to which the problem reduces to ordinary differential equations (the convergence of the method was studied by him later, 1941). In another joint article with P. V. Frumkin (1937), Kantorovich applied successfully his method to the solution of the problem of the torsion of a rectangular and standard symmetric and asymmetric cross section. T. K. Chepova (1937) studied the torsion of an equilateral trapezoid, and also of straight and oblique symmetric angles, V. L. Biderman (1950) studied the torsion of a trapezoid and an equilateral triangle, A. P. Karpov (1955) developed the solution for the problem of the torsion of a rhombus.

A. I. Lur'e (1939) applied the Kantorovich method to the problem of the flexure and torsion of a symmetric profile bounded by parallel and algebraic curves described by two-term equations. The problems of the torsion of triangles, right angled and equilateral triangles, were studied in detail by N. O. Gulkanyan (1953). By introducing a special type of non-orthogonal coordinates, N. Kh. Arutyunyan was able to solve the problem of the torsion of an angle and a channel (1942) and in another study to obtain the solution of the torsion problem for an elliptic annular section which was isotropic or had an anisotropy of a special type (1947).

Another approximate method for solving the problem of the torsion of a prismatic rod based on point interpolation was developed by L. A. Galin (1939). An approximate solution of the problem of the torsion of a rod with a T-shaped cross section was obtained by B. A. Bondarenko with the aid of the alternating Schwarz method (1956).

M. Ya. Leonov proposed an approximate method for determining the rigidity of thin-walled profiles based on the introduction of "mean lines" of equal tangential stresses (1956, 1957). While developing this method, M. Ya. Leonov (1957, 1960), G. S. Kit (1958, 1960) and others, obtained approximate solutions for a number of simply and doubly connected regions.

G. K. Galimkhanov (1955, 1956) developed an approximate solution for the problem of the torsion of flat keyway shafts whose cross section consists of arcs of the principal circle and chords. The constants in his solution are determined from the condition that the integrals of the functions of the stresses along rectilinear and arched. sectors of the contour vanish. Approximate methods were also used to study torsion problems by G. M. Sarkisov and Yu. A. Amenzade

(1952) for regular polygonal profiles, by L. M. Mitel'man (1955, 1959) for a square, semicircle, equilateral triangle and an airfoil, and by L. V. Mikhaylov (1962) for a rod with a semicircular cross section weakened by a circular cylindrical cavity.

The problem of the constrained torsion of a prismatic rod of arbitrary cross section was considered by V. K. Prokopov (1959) and for a symmetric profile by G. P. Geondzhyan (1959). In both studies, it was assumed that the normal stresses in the constrained cross section were proportional to the plane displacement of the free torsion and applying the variational method for solving the problem, elliptical and rectangular cross sections were studied as examples. The constrained torsion of a rod with a rectangular cross section was also studied by V. P. Netrebko (1956), who used the M. M. Filonenko-Borodich method which he combined with the Castigliano principle. In another study, Netrebko (1954), using the same method, studied the problem of the torsion of a right-angled prism for a given distribution of tangential stresses at its ends. The constrained torsion of a hollow elliptical cylinder was studied by S. A. Gridnev (1963).

An exact solution of the problem of the flexing of a prismatic rod with a cross-section in the form of an annular sector was given in 1927 by B. G. Galerkin, who expressed the function of the stresses in the form of a series. In this study Galerkin studied, with the aid of curvilinear coordinates, the symmetric flexure of a floating core whose profile was bounded by parabolic arcs, parabolas and a line, arcs of an ellipse and of a hyperbola. The last case was also studied in the article of V. S. Tonoyan (1961).

D. Z. Avazashvili (1940) constructed the solution of the problem of the flexure of a cantiliver prismatic rod with the aid of functions of a complex variable. Through a conformal mapping onto an annular region, B. A. Obodovskiy obtained the solution of the problem of the flexing of a hollow beam with an elliptical cross section by a force (1960). L. K. Kapanyan (1956) used an approximate conformal mapping in the solution of the problem of the flexure for a circle with "a curvilinear square" cutout. V. N. Rakivnenko (1962) studied the bending of a circular cylinder with two cavities with cross sections in the form of a square.

The symmetric bending of a rod whose cross section consisted of rectangular regions was studied by A. S. Bozhenko (1948). In another article (1954) he studied the asymmetric bending of rolled sections (channel, I-beam, T-shaped beam) and he determined the position of the flexing center. N. O. Gulkanyan (1955) determined the coordinates of the center of flexure of an equilateral trapezoid and an equilateral triangle using an approximate method. The solution of the problem of the flexure of a prism with a cross section in the form of an equilateral triangle in closed form was obtained by N. I. Popov (1954).

D. I. Sherman extended his auxiliary function method to the problem of the flexure of hollow prismatic rods, and, in particular, he studied the case of an elliptical bar weakened by a circular cylindrical cavity (1953). A number of problems of the flexure of hollow rods were studied, using the Sherman method by Yu. A. Amenzade, a circle with elliptic (1955) and curvilinear (1956) holes, a circle with a non-coaxial elliptical hole (1958) and others. A cross section in the form of an ellipse with two circular holes was studied by A. S. Kosmodamianskiy (1960).

In 1948 N. Kh. Arutyunyan proposed a new method for solving the problem of the torsion of rods with polygonal cross sections, which is based essentially on the introduction of auxiliary functions that are used to obtain the stress functions and a subsequent reduction of the solution of the problem to complete regular infinite systems of linear algebraic equations. Later, he studied the problem of the torsion of an angle (1949). Using the Arutyunyan method, problems of the torsion of rods with various types of cross sections were studied. A cross section in the form of a trapezoid was studied by B. L. Abramyan and N. Kh. Arutyunyan (1951), a channel and a T-shape by Ye. A. Aleksandryan and N. O. Gulkanyan (1953), a cross-shaped section and a cylinder with wedge grooves by B. L. Abramyan (1949, 1959), a box-shaped profile with a crack by A. A. Babloyan (1958). Ye. A. Aleksandryan (1952) studied the cases of an H-bar, a square and rectangle with a cut-off angle and of a parallelogram with a 45° angle. A triangular cross section and a rectangle with cracks was studied by N. O. Gulkanyan (1952, 1953), a section with teeth was studied by B. L. Abramyan and V. S. Tonoyan (1959).

The torsion (and flexure) of prismatic beams with a hollow rectangular cross section was studied in 1950 by B. L. Abramyan. In another article he studied the case of a circular shaft with longitudinal cavities (1959). The torsion of a circular rod with longitudinal recesses or teeth with a central

circular cavity were studied in the article of B. L. Abramyan and A. A. Babloyan (1960). Using the same method of auxiliary functions and reduction to infinite systems, N. O. Gulkanyan (1960) studied the torsion of a rectangular prism with two symmetric rectangular cavities. V. S. Tonoyan (1961) obtained the solution of the problem of the torsion of a hollow elliptical bar with longitudinal grooves. A detailed presentation of the method of auxiliary functions as applied to the torsion of prismatic solid and hollow bars as well as the problem of the torsion of composite bars and bodies of rotation can be found in the monograph of N. Kh. Arutyunyan and B. L. Abramyan (1963).

The application of the method of auxiliary functions to the problem of the flexure of bars with a polygonal profile and the reduction of the problem to infinite systems was given in the article of N. Kh. Arutyunyan and N. O. Gulkanyan (1954). Exact values of the coordinates of the flexural center for a T-shape, a channel and an angle were obtained in this article. N. O. Gulkanyan (1959) also obtained the coordinates of the flexural center for a rectangular section with an asymmetric rectangular cutout.

Using the Arutyunyan method, M. S. Sarkisyan (1956) studied the problem of the flexure of an H-bar, Ye. Ya. Kirin (1963) studied a cross-shaped cross section, V. S. Tonoyan (1961) a cross section in the shape of an ellipse with recesses. The studies of A. A. Babloyan (1960, 1961) are devoted to the problem of the flexure of a circular shaft with longitudinal lateral recesses, a sectional prism with a tooth and a shaft with teeth.

N. I. Muskhelishvili (1932) developed the theory of the torsion and flexure of beams consisting of various materials welded along the lateral surfaces. The solution of this problem for the case of torsion of two welded beams from different materials is presented in his well-known monograph (second issue, 1935). I. N. Vekua and A. K. Rukhadze (1933) studied the torsion of a circular cylinder reinforced with a circular bar, and also the torsion and flexure of a composite beam whose cross section had the shape of confocal ellipses. A. K. Rukhadze (1935) studied the flexure and torsion of a composite profile formed by epitrochoids. The case of demarcation by hypotrochoids was studied by G. A. Kutateladze (1956). The torsion of a composite rod with cross section in the shape of two circular segments welded along a chord was studied using bipolar coordinates by V. M. Dzyuba and A. Sh. Asaturyan (1965).

The general problem of the torsion of a composite rod was studied in the article of K. S. Chobanyan (1955), in which he presents the theorem on the circulation of the tangential stress and studies the problem of the torsion of a composite rod with a T-shaped cross section. In other studies of K. S. Chobanyan, the flexure of a composite rod is discussed (1956), and the coordinates of the flexural center and the torsion of a composite shaft with a variable diameter are determined (1958). The torsion of a multiply-connected composite rod was investigated by I. V. Sukharevskiy (1954). A. G. Ugodchikov (1964) considered the torsion and flexure of composite rods inserted one in another. The solution of the problem is obtained with the aid of conformal mapping and a reduction to infinite systems of linear equations.

The problems of the torsion and flexure of prismatic anisotropic beams were formulated in the studies of S. G. Lekhnitskiy (1938, 1942, 1956). The results of these studies and the solutions of a number of other problems in the theory of elasticity of anisotropic media are summarized in his monograph (1950). The torsion of anisotropic prisms with the aid of the generalized membrane analogy was studied even earlier by A. Sh. Lokshin (1927), who studied sections in the shape of a circle, ellipse, rectangle and parallelogram. Certain problem in the flexure and torsion of anisotropic prisms using the variational method were investigated by L. S. Leybenzon (1940). The article of V. D. Vantorin (1939) is devoted to the approximate solution of the torsion of an anisotropic beam in an airfoil. Certain problems in the torsion of an anisotropic beam were studied using an approximate method by N. Kh. Arutyunyan (1947, 1948). The torsion of an anisotropic cylinder was studied by B. L. Abramyan and A. A. Babloyan (1958).

The flexure and torsion of an anisotropic beam with a cross section in the shape of a parallelogram was studied by R. S. Minasyan (1938). A number of problems on the flexure of anisotropic beams were studied by V. S. Sarkisyan (1961, 1962), using the method of power series expansion in powers of a small parameter. Solving the problem of the flexure of an anisotropic beam with the aid of conformal mapping, Ye. Ye. Antonov (1964) expressed the coordinates of the flexural center in terms of the coefficients of the mapping function. A. S. Kosmodamianskiy (1962) presented an approximate solution for the problem of the torsion and flexure of orthotropic beams with an elliptic profile with cavities with an elliptical cross section.

Problems of the torsion of a nonhomogeneous prismatic beam were solved by B. L. Abramyan (1951) and A. Kh. Manukyan (1952). V. S. Sarkisyan and V. V. Mikayelyan (1965) developed formulas for the coordinates of the flexural center for a composite anisotropic beam. Recently solutions of flexing problems (P. O. Galfayan, 1960, 1961) and the torsion (A. A. Babloyan, 1959, P. O. Galfayan and K. S. Chobanyan, 1959) appeared for bodies with thin reinforcing coatings. S. G. Lekhnitskiy studied certain problems in the torsion of bodies with a variable modulus of elasticity (1964, 1965).

In 1950 M. E. Berman derived formulas for the coordinates of the flexural center that were expressed in terms of functions that solved the torsion problem for a beam with the same cross section. Later, V. V. Novozhilov (1957) obtained an analogous result, and V. K. Prokopov (1960) generalized these formulas to the case of a multiply-connected cross section of the flexed beam. The further study of the problem mentioned above is due to G. Yu. Dzhanelidze (1963). In the case of an anisotropic beam, analogous results were obtained by V. S. Sarkisyan (1961, 1966). K. S. Chobanyan and V. V. Mikayelyan (1963) derived formulas for the coordinates of the flexural center of a beam with a cross section consisting of different materials.

The torsion of bodies of rotation was studied using various methods. A. Sh. Lokshin (1923) studied, with the aid of curvilinear coordinates, the torsion of a cone, an ellipsoid, a hyperboloid and a paraboloid of rotation. In a more general formulation. the problem of the torsion of bodies of revolution in curvilinear coordinates was studied by B. A. Sokolov (1944). This author also studied the problem of applying the Ritz method to the problem of the torsion of a stepped shaft (1939). The torsion of a hollow truncated cone was studied by N. Ya. Panarin (1937).

K. V. Solyanik-Krassa used curvilinear coordinates to solve the problem of the torsion of shafts with cavities (1947) or circular recesses (1948, 1955). The results of these studies are also available in his monograph "The Torsion of Shafts with a Variable Cross Section" (1949). Using the same method, he studied a number of problems in the flexure of a beam with variable cross sections, in particular, he investigated the stress concentration near a spherical cavity in a cylindrical beam (1955).

An estimate of the stress concentration during the torsion of a circular shaft with a circular recess based on the application of the theory of functions of a complex variable combined with the variational method was obtained by G. N. Polozhiy (1957). The problem of the concentration of stresses during torsion where the diameter of the shaft changes sharply using the method of grids was studied by B. A. Rozovskaya (1956, 1958). The torsion of a pipe with a variable cross section was discussed by Yu. A. Amenzade and G. M. Sarkisov (1959).

The torsion of anisotropic bodies of rotation was investigated in the studies of S. G. Lekhnitskiy (1940), D. V. Grilitskiy (1957), B. L. Abramyan and A. A. Babloyan (1958).

The action of forces distributed along the lateral surface of a circular shaft leading to its twisting was studied by N. V. Zvolinskiy and P. M. Riz (1939), who studied a uniformly and linearly distributed load. A more general case of a prismatic beam was studied by L. S. Gil'man and S. S. Golushkevich (1943) and P. M. Riz (1940). The problem of the torsion of an elastic ring by couples uniformly distributed along its axis was studied in the article of L. S. Gil'man (1937). The case of uniformly distributed twisting tangential forces along the generatrices of the cylinder was studied by S. A. Bakanov (1959). The torsion of solid and hollow circular cylinders with axisymmetric distributed surface loads were studied with the aid of Fourier-Bessel series by V. I. Bloch (1954, 1956). P. Z. Livshitz (1962) returned to the same problem for a solid cylinder. The problem of the torsion of an anisotropic beam by forces distributed along its lateral surface was solved by S. G. Lekhnitskiy (1961).

The torsion of a stepped shaft with axisymmetric loads applied to its lateral and end surfaces was studied by B. L. Abramyan and M. M. Dzhrbashyan (1951), who reduced the solution of the problem to an infinite system of linear equations. Using the same method, B. A. Kostandyan solved the problem of the torsion of a hollow stepped shaft (1956). He also studied the torsion of a shaft with a circular rectangular-shaped recess (1954) and the torsion of a shaft with a disc slipped over it (1958). The torsion of a conical beam and a cylindrical beam with a conical part were studied by B. L. Abramyan (1958, 1960) and the torsion of a hollow composite halfsphere was studied by him jointly with N. O. Gulkanyan (1961).

The problem of the equilibrium of an elastic prismatic beam under the action of forces applied to its ends which is free from loads on the lateral surfaces is known as the St.-Venant problem. In linear elasticity theory, this problem is broken up naturally into two simple problem (expansion and pure flexure by couples), which are solved in an elementary way, and two more complex problems (torsion and bending by a force) which were discussed in detail above. In nonlinear elasticity theory, the mutual effect exerted by various loads is essential. It is necessary to take into account the secondary effects, whose study began in 1938, 1939 in the joint studies of N. V. Zvolinskiy and P. M. Riz. In the last study in this series of studies (1939) the torsion of an expanded beam was considered. N. V. Zvolinskiy also studied the torsion of a beam expanded by body forces (1939). The problem of the torsion of an expanded beam is the subject of the studies of P. M. Riz (1939) and A. K. Rukhadze (1941), who also considered the effect on the flexure of a beam by a couple, the bending from a transverse force (1947). The secondary effects which occur during the expansion and flexure of composite bars were discussed in the study of A. Ya. Gorgidze and A. K. Rukhadze (1943). These studies were made more precise and developed in the subsequent work of A. Ya. Gorgidze (1955, 1956), R. S. Minsyan (1957), A. K. Rukhadze (1954), and his joint studies with D. N. Dolidze (1957), V. Kh. Metsugov (1954, 1956). The method of a small parameter was used extensively in this series of studies.

Problems of the deformations of slightly conical and naturally twisted beams occupied a considerable place in the studies of Soviet scientists. Here the method of a small parameter also turned out to be very useful. This method was applied for the first time by D. Yu. Panov to the solution of the problem of the torsion of a slightly conical beam (1938). Problems dealing with the expansion, torsion and flexure by couples of naturally twisted beams were studied by P. M. Riz (1939). In a more general formulation, using a special system of non-orthogonal coordinates, the St.-Venant problem for a naturally twisted beam was solved by A. I. Lur'e and G. Yu. Dzhanelidze (1940). Later, G. Yu. Dzhanelidze extended this method to slightly conical beams (1947). In Cartesian coordinates the flexure of a twisted beam by couples was investigated by A. Ya. Gorgidze and A. K. Rukhadze (1944) and the flexure by a transverse force by A. K. Rukhadze (1947). Subsequent studies complement these fundamental results, studv secondary effects in greater detail, complicate the load diagrams (A. Ya. Gorgidze, 1958, 1963), and examine the torsion of naturally twisted components (A. Ya. Gorgidze and V. Kh. Metsugov, 1957, A. K. Rukhdze, 1956, A. F. Sharangiya, 1955) and composite slightly conical beams (S. V. Berdzenishvili, 1957).

The method of a small parameter was also successfully applied to the solution of the equilibrium problem of a beam with a slightly bent axis. Problems of this type were solved for the first time by P. M. Riz (1940, 1947) and by A. K. Rukhadze (1942). Subsequently, expansion (R. S. Minasyan, 1954), flexure by couples (A. K. Rukhadze, 1953) and flexure by a force (A. Ya. Gorgidze, 1956) were considered.

The expansion and flexure of an anisotropic beam were studied in 1949 by S. G. Lekhnitskiy. Later G. M. Khatiashvili studied a more complex problem, an anisotropic beam with a slightly bent axis (1965). He also studied the St.-Venant problem for composite nearly prismatic anisotropic bodies (1963).

The problem of the elastic equilibrium of a beam whose lateral surface is subjected to loads which are polynomial functions of the axial coordinate is known as the Almanzi problem. A special case of this problem, when the lateral load is independent of the axial coordinate has already been studied by G. G. Mitchell. In 1960 G. Yu. Dzhanelize published a general method for the solution of the Almanzi problem in the stresses which reduces to the solution of a series of two dimensional problems which are related to one another by recurrence relations. This method yielded a general method for solving the Mitchell-Almanzi problem and opened up the way for the application of methods from the theory of functions of a complex vairable. A special case of the Mitchell problem when a uniformly distributed normal load is acting on the lateral surface of the beam was studied by A. L. Khasis (1960). He showed that a line of flexural centers exists which can be found by determining the harmonic torsion function for the St.-Venant problem. For composite beams, the solutions of the St.-Venant problems were found. G. M. Khatiashvili (1953, 1955) obtained the solutions of the Mitchell and Almanzi problems for composite beams. (A classification of and sequence in which the boundary value problems are solved which arise in connection with the Mitchell problem was given by A. I. Lur'e (1966).

The action of a lateral polynomial load on a transverse-isotropic cylinder leading to its torsion and to an axi-symmetric deformation were studied by S. G. Lekhnitskiy (1961). A. S. Kosmodamianskiy (1956, 1961) studied the Mitchell and Almanzi problems for an anisotropic rod. G. Yu. Dzhanelidze (1961) extended the method proposed by him for the solution of the Almanzi problem to the case of an ansiotropic beam. This problem was studied in greater detail by G. M. Khatiashvili, who investigated the Mitchell problem for composite orthotropic and anisotropic beams (1962) and also generalized the Dzhanelidze problem to the case of the Almanzi problem for a composite orthotropic beam (1964).

§4. Mixed Three-Dimensional Problems in the Statics of an Elastic Body

By mixed boundary value problems in the mathematical theory of elasticity we usually mean elastic equilibrium problems when the lines which divide the boundary conditions of various types lie on the surface of the body. If the surface of the elastic body under consideration consists of several smooth faces, two qualitatively fundamentally different variants of the mixed problems may occur.

1) Within each face the type of boundary condition does not change. The simplest examples of such mixed problems are the equilibrium of an elastic layer for which the stresses are given on one face and the displacements on the other face and analogous problems for a wedge, a hollow cylinder, a cone, etc. The solution of the above-mentioned concrete problems is obtained using integral Fourier transforms, Hankel transforms, etc. As shown by G. Ya. Popov and N. A. Rostovtsev (1966), general problems of this type reduce in principle to infinite systems of equations. These problems are not touched on in this survey.

2) When at least on one of the faces of the body, there is a dividing line for boundary conditions of various types. Problems of this type reduce, generally, to integral equations which we will analyze here in greater detail, since these served as the impetus for the development, mainly in the USSR, of various methods used in the solution of many important mixed problems in potential theory and the theory of elasticity. A number of applied problems, in particular contact problems and certain problems dealing with stress concentrations, are mixed problems of this type.

At the present time contact problems for an elastic half-space deformed by a rigid die, a circular or elliptical die in the plane have been studied in greatest detail. Such a problem was discussed for the first time already by Zh. Bussineskiy for the case of the axial indentation of a circular cylinder without friction. This category of problems includes the classical problem of G. Hertz of the compression of elastic bodies when the contact area is an ellipse. Soviet scientists contributed considerably to the further development of this class of problems. A. N. Dinnik (1909) and N. M. Belyaev (1924) calculated stresses in bodies making contact along a circular or elliptical area (see also M. S. Krolevets, 1966). A considerable number of important studies on contact problems were made in the 30's and 40's. V. A. Abramov (1939 and A. I. Lur'e (1940) obtained the solution for contact problems

for a non-centrally loaded circular and elliptical die. Important results along these lines were obtained by I. Ya. Shtayerman (1939, 1941, 1943), who studied various cases of the contact of bodies of revolution without assuming a small contact surface, and who also investigated for the first time the problem of a closely fitting die. In 1941 A. I. Lur'e, using the Lame function studied in detail certain contact problems and then developed a natural unique approach to the Hertz problem and the problem of a closely fitting die. In the study of M. Ya. Leonov (1939, 1940) and L. A. Galin (1946, 1947), a number of contact problems for the halfspace are further generalized. A great deal of information both of an original and survey character dealing with the problems that were discussed is contained in the monographs of I. Ya. Shtayerman (1949), L. A. Galin (1953), A. I. Lur'e (1955), as well as in the survey articles of D. I. Sherman (1950) and G. S. Shapiro (1950), which contain many references not included in this survey.

In recent years, the development of methods based on the use of the general equations of the theory of elasticity, in particular the Papkovich-Neuber function, made it possible to reduce many general mixed problems in elastic equilibrium of the halfspace to certain classes of mixed problems in the theory of the potential. An important example in this class of problem is the case when on the entire boundary of the halfspace the tangential stresses are given in some finite region S and on the boundary plane $z = 0$, the normal displacement $u_z = f(x, y)$ is known and in the interior of S (in the domain S') the normal stress $\sigma_z = \sigma(x, y)$ is given. Thus, for the contact problem without friction and additional loads, we have $\sigma \equiv 0$ and the function f is determined by the shape of the base of the die. It is essential that mixed problems of the type that was mentioned can ultimately be reduced to finding a single harmonic function defined on S, whose normal derivative is known in the domain S'. Soviet scientists developed efficient methods for approaching such problems in the theory of the potential which can be used, in particular, to obtain exact solutions of certain contact problems and similar mixed problems. Some of the fundamental methods are: the use of spherical and ellipsoidal coordinates (A. I. Lur'e), the construction and use of Green's function (L. A. Galin, M. Ya. Leonov, 1953), the method of integral equations (I. Ya. Shtayerman, V. I. Mossakovskiy, 1953), the use of toroidal coordinates and

integral transformations (Ya. S. Uflyand, 1956, 1967), the method of complex potentials (N. A. Rostovtsev, 1953, 1957). Here, we deliberately refrain from mentioning the method of coupled integral equations developed successfully by J. N. Sneddon,[1] since its effectiveness is essentially verified by solving more complex mixed problems which we will discuss later.

The authors that were mentioned, as well as many other authors, developed in the last decades exhaustive solutions for a number of new mixed problems in three-dimensional elasticity theory, including contact problems. Thus, L. A. Galin (1947) and V. L. Rvachev (1959) considered the problem of the indentation of a wedge-shaped die in the halfspace, the studies of N. A. Kil'chevskiy (1958, 1960) generalized the Hertz problem and pointed out the connection between the elastic contact problem and a certain extremum problem, V. L. Rvachev (1956, 1957) solved the problem of a strip and polygonal die and also discussed the case of a die whose base was bounded by a second-order curve. The studies of G. Ya. Popov (1961, 1963) deal with mixed problems for a circular contact region, and a die in the shape of a halfplane and a quadrant. N. M. Borodachev (1962, 1964, 1966) and A. F. Khrustalev (1965) studied a number of thermoelastic problems for the halfspace. In particular, the complex problem of the action of a hollow circular cylinder on the halfspace, which is known in the literature as the annular die problem should be mentioned. The exact solution of this problem is connected with functions of an annulus with an oval cross section which have not been tabulated (see N. N. Lebedev, 1937). Various approximate methods for solving this problem were proposed in the studies of A. Ya. Aleksandrov (1955), Yu. O. Arkad'eva (1962), V. S. Gubenko and V. I. Mossakovski (1960), K. I. Yegorov (1963), G. Ya. Popov (1967). In recent years, still another approach to this and similar problems based on the use of coupled integral equations using the Meler-Fok transformation has been proposed (V. T. Grinchenko and A. F. Ulitko, 1963, A. A. Babloyan, 1964, A. N. Rukhovets and Ya. S. Uflyand, 1965-1967), and also on the use of triple integral equations[2] (N. N. Borodachev and F. N. Borodacheva, 1966). The methods that were mentioned can be used to obtain good approximations based on the numerical solution of Fredholm integral equations.

1. See, for example, his "Fourier Transforms" (1951, Russian translation: Moscow, 1955), "Mixed Boundary Value Problems in Potential Theory."
2. Triple integral equations were investigated in the studies of K. G. Tranter (Quart. J. Mech. & Appl. Math., 1961, 14:3, 283-293) and G. K. Cook (Quart. J. Mech. & Appl. Math., 1963, 16:2, 193-203; 1965, 18:1, 57-72).

A large number of studies dealing with mixed problems connected with problems in the flexure of beams and plates on an elastic base have been published in the Soviet literature. Here we will only mention the studies of A. G. Ishkova (1947), M. Ya. Leonov (1939) and V. A. Pal'mov (1960), dealing with the flexure of a circular plate on an elastic halfspace, and also the monographs of M. I. Gorbunov-Posadov (1953) and B. G. Korenev (1954, 1960). The results of the many studies along these lines and a large bibliography can be obtained by the reader in the survey article of A. G. Ishkova and B. G. Korenev (1966).

Along with contact problems, the mixed problems in the theory of the potential for the halfspace that were considered above can be treated as problems in the deformation of an unbounded elastic body weakened by a plane crack occupying the region S (or S'). In fact, in the case when the edges of the crack are loaded symmetrically with respect to its plane, it suffices to consider the halfspace on the boundary of which in the region S (or S') the stresses are given and in its exterior there are no tangential stresses and no normal displacement. In the case of an antisymmetric load, even for a circular crack, certain additional difficulties arise which were solved in the articles of V. I. Mossakovskiy (1955) and Ya. S. Uflyand (1967). In the last article, this problem was considered as a special case of the general mixed problem when the normal stress is given on the entire boundary of the halfspace, and in the region S (S') the tangential displacement is known, and the tangential stresses are given in the domain S' (S). An interesting problem on the contact of two different media whose common boundary has a circular crack was solved by V. I. Mossakovskiy and M. T. Rybka (1964), which generalizes the well-known Griffith-Sneddon criterion to the case of a nonhomogeneous body (see also the article of the same authors, 1965). Among the studies dealing with the deformation of bodies with cracks, we also point out the interesting articles of V. T. Grinchenko and A. F. Ulitko (1965), V. M. Aleksandrov and B. I. Smetanin (1965), and also the study of Ya. S. Uflyand (1958) dealing with the problem of the equilibrium of a body with a plane semi-infinite cross section.

In the majority of studies that were discussed connected with contact problems, it was assumed that there was no friction between the die and the elastic body. The second limiting case, when the die and the base adhere (this problem is a special case in the basic mixed problem of elasticity theory) is mathematically much more difficult. In contrast to the simpler

mixed problems, in this case the problem reduces to finding two harmonic functions in the halfspace with boundary conditions of the first and second type which are not separated. Such a problem was solved for the first time for a circular die by V. I. Mossakovskiy (1954) by reducing it to a plane problem for the linear conjugation of two analytic functions. Subsequently Ya. S. Uflyand (1954, 1967) obtained a direct solution for this problem using toroidal coordinates and the Meler-Fok integral transformation. In the article of B. L. Abramyan, N. Kh. Arutyunyan and A. A. Babloyan (1966), another approach is taken to this problem which is based on using coupled integral equations. The study of V. I. Mossakovskiy (1963) also deals with contact problems in the presence of adhesion. The solution of the fundamental mixed problem in the theory of elasticity for a halfspace with a rectangular boundary which separates the boundary conditions was obtained by Ya. S. Uflyand (1957) with the aid of the Kontorovich-Lebedev integral transformation.

The behavior of the stresses near the boundary line of the die under adhesion conditions was studied in the article of G. N. Savin and V. L. Rvachev (1963).

A natural generalization of the classical problem of the indentation of a rigid die in an elastic halfspace is the contact problem for an unbounded elastic layer. These problems were studied intensely in the USSR in the 50's and in contrast to the case of the halfspace, here it was not possible to obtain exact solutions. It was only possible to reduce the corresponding problems to integral equations. Here the first study was the article of B. I. Kogan (1954) in which an integral equation of the first kind was set up and solved numerically for the contact pressure between a circular die and a layer on the halfspace. A more efficient solution of a similar problem was obtained by N. N. Lebedev and Ya. S. Uflyand (1958) who studied the axial indentation of a circular rigid die in the plane in an elastic layer on a rigid base in the absence of friction. This problem was reduced to coupled integral equations of the form

$$\left.\begin{aligned}&\int_0^\infty \Phi(\lambda) J_0(\lambda r)\,d\lambda = f(r) \qquad (0 \leqslant r < a),\\ &\int_0^\infty \lambda \Phi(\lambda) J_0(\lambda r)\frac{d\lambda}{1-g(\lambda)} = 0 \quad (a < r < \infty),\end{aligned}\right| \quad g(\lambda) = \frac{\lambda h - e^{-\lambda h}}{\lambda h + \underset{a}{\mathrm{ch}}\,\lambda h\,\underset{b}{\mathrm{sh}}\,\lambda h},$$

Key: a. cosh
b. sinh

where a is the radius of the die, h is the thickness, of the layer, $f(r)$ is a given function related to the shape of the base of the die, and $\Phi(\lambda)$ is the unknown quantity. By representing the solution as a quadrature of a new unknown function

$$\Phi(\lambda) = [1 - g(\lambda)] \int_0^a \varphi(t) \cos \lambda t \, dt$$

the second equation is satisfied identically, and the first equation reduces to the Fredholm equation with a continuous symmetric kernel. This method of solution can be used to carry out a number of numerical calculations, in particular, to find the relation between the displacements of the die and the axial force P with the aid of the simple formula

$$P = 2\pi \int_0^a \varphi(t) \, dt.$$

K. Ye. Yegorov (1960) applied a similar method tothe case of the non-axial identation of a die. In the article of V. A. Pupyrev and Ya. S. Uflyand (1960) and in the monograph of the latter (1967), a solution of the general mixed problem for an elastic layer is obtained and the case of the adhesion of a layer and the base is also discussed. It is important to point out that the method of coupled integral equations made it possible to study effectively the more complex axisymmetric problem when the layer is compressed by two dies with different radii (Yu. N. Kuz'min and Ya. S. Uflyand, (1967)). I. I. Vorovich and Yu. A. Ustinov (1959) obtained a singular integral equation directly for the function $\Phi(\lambda)$ and developed an approximate method for its solution in a series expansion in powers of a/h. An analogous method was used by D. V. Grilitskiy in the problem of the torsion of a multi-layer medium with the aid of a die adhering to it and also in a number of similar contact problems. The method of coupled integral equations enabled a number of authors (see, for example, G. M. Valov, 1964, S. M. Kotlyar, 1964, V. I. Dovnorovich, 1964) to solve various contact problems for an elastic layer, including thermoelastic problems. Contact and mixed problems for anisotropic bodies were discussed by S. G. Lekhnitskiy (1950), D. V. Grilitskiy and Ya. M. Kizyma (1962, 1964), and R. Ya. Suncheleyev (1964, 1966).

A special efficient method for approaching contact problems where a die acts on an elastic layer based on a direct investigation of the integral equation for the pressure below the die was proposed by V. M. Aleksandrov and I. I. Vorovich (1960, 1964). The solution of the problem was obtained in the form of an expansion in the small parameter, the ratio of the characteristic dimension of the die to the thickness of the layer. It is essential that effective results were obtained not only for a circular but also for an elliptical die in the plane and also for certain other differently shaped bases. The method that was mentioned was further developed in the studies of V. M. Aleksandrov (1963, 1964, 1967) and other students of I. I. Vorovich (see, for example, the dissertation of V. A. Babeshko, 1966). At the present time it can be assumed to be one of the most effective methods for the solution of the class of contact problems under consideration for an arbitrary value of the ratio of the thickness of the layer to the characteristic dimension of the die.

From among the studies dealing with more complex compact problems, we will mention the article of V. S. Gubenko (1960) in which the problem of the action of annular dies on an elastic layer is studied, and also the article of I. I. Vorovich and V. V. Kopasenko (1966) for the contact problem for a halfstrip.

Problems of the stress concentration in an elastic layer weakened by coaxial circular cracks which are parallel to the boundaries of the layer can be successfully solved with the aid of coupled integral equations. The simplest problem of this type (Ya. S. Uflyand, 1959) is the equilibrium of an elastic layer which has in the middle plane one symmetrically loaded circular crack. I. A. Markuzon (1963) studied this problem in connection with the problem of finding the dimensions of the equilibrium crack using the G. I. Barenblatt method.

Among other studies dealing with the equilibrium of bodies with cracks and holes, we mention the articles of V. V. Panasyuk (1960), N. N. Lebedev and Ya. S. Uflyand (1960), Yu. N. Kuz'min and Ya. S. Uflyand (1965), Yu. N. Kuz'min (1966) and N. V. Pal'tsun (1967), and also the survey article of G. N. Savin, A. S. Kosmodamianskiy and A. N. Guz' (1967).

We will now discuss briefly contact problems dealing with the equilibrium of an infinite cylinder. In the study of these problems, the most effective method is the method of coupled integral equations which are related to the Fourier transform along the axial coordinate. A characteristic feature of this method is the fact that in the case of a semi-infinite contact region, these equations can be solved exactly using methods of the theory of functions of a complex variable which are based on the possibility of factoring

the analytic function which is defined on the strip. The first study along these lines was the article of B. I. Kogan (1956) which studied the axisymmetric stressed state of an infinite cylinder pressed without friction into a semi-infinite rigid ring. Assuming that in the contact region the constant radial displacement is given, the problem reduces to coupled equations of the form

$$\left.\begin{array}{ll} \int_0^\infty f(\lambda)\, e^{i\lambda\zeta}\, d\lambda = 0 & (\zeta > 0), \\ \int_0^\infty \dfrac{f(\lambda)\, I_1^2(\lambda)\, e^{i\lambda\zeta}\, d\lambda}{(\lambda^2 - 2 + 2\nu)\, I_1^2(\lambda) - \lambda^2 I_0^2(\lambda)} = u_0 & (\zeta < 0), \end{array}\right|$$

whose exact solution is obtained by constructing certain merophoric functions in the form of an infinite product. In the later studies of B. I. Kogan, A. F. Khrustalev and F. A. Vayanshteyn (1958-1965) this method was applied to various mixed problems, both for a solid and a hollow cylinder, and also to the case of transverse anisotropy. The method for the solution of such problems, which is based on reducing them to a Wiener-Hopf integral equation for the contact stresses, was developed by G. Ya. Popov (1964). He also obtained a solution of the contact problem for an infinite cylinder with two symmetric contact sectors. We also point out the article of G. M. Valov (1966), in which the problem of the torsion of a hollow infinite cylinder is studied with the aid of coupled integral equations and trigonometric kernels.

Very recently the domain of solvable contact problems was expanded considerably due to the development of the new apparatus of coupled series which is applied to mixed problems for an elastic sphere.[1] By coupled series (or coupled equations with summations) is meant the system of equations

1. The solution of certain mixed problems in the theory of the potential with the aid of the method of coupled series is presented in the second book by Ya. N. Sneddon that was mentioned above (see p. 49).

$$\left.\begin{aligned}\sum_{n=0}^{\infty} \alpha_n A_n K_n(x) = f_1(x) \quad (a \leqslant x < c),\\ \sum_{n=0}^{\infty} \gamma_n A_n K_n(x) = f_2(x) \quad (c < x < b),\end{aligned}\right\}$$

in which the coefficients A_n must be determined, where it is assumed that the kernels $K_n(x)$ form on the interval (a, b) a closed system and that the numbers α_n and γ_n are given. Using these coupled series, which contain series in Legendre polynomials, several interesting problems related to the deformation of an elastic sphere and also an ellipsoid of rotation with mixed boundary conditions were solved in the articles of N. Kh. Arutyunyan, B. L. Abramyan and A. A. Babloyan (1964, 1966). He also considered the axisymmetric compression of a sphere by two symmetrically spaced equal rigid dies on the assumption that no friction was present. It was possible to reduce this problem to coupled series of the type given above where $K_n(x) = P_n(x)$, $\alpha_n = n + 1/2$, $\gamma_n = 1 + \beta_n$ (the quantities β_n as $n \to \infty$ have the order $1/n$), $a = -1$, $b = 1$. If $V(x)$ denotes the value of the sum of the first paired series for $x > c$, the solution reduces to the integral equation

$$V(x) + \frac{1}{\pi}\frac{d}{dx}\int_c^x \frac{dy}{\sqrt{x-y}} \int_c^1 V(\xi) S(\xi, y)\, d\xi = \Phi(x),$$

where

$$S(\xi, y) = \sqrt{2}\sum_{n=0}^{\infty} \beta_n P_n(\xi) \cos\left[\left(n + \frac{1}{2}\right)\arccos y\right],$$

and $\Phi(x)$ is a known function. Using a similar technique, the solution of the problem of the torsion of an elastic sphere by two symmetrically spaced equal dies adhering to it was obtained. Using the method of coupled series in Legendre polynomials, the solutions of certain mixed problems connected with the compression and torsion of an elastic sphere and an elongated ellipsoid of rotation have also been solved. Finally, the

contact problem for the indentation of a rigid die in an elastic medium was studied in which the coupled series in Legendre polynomials were reduced to an infinite system of linear algebraic equations. As an example, a sphere at rest was considered without friction on the semispherical groove which was loaded on the remaining part of the surface.

Coupled series in Legendre polynomials can also be applied effectively with the aid of bispherical coordinates to the solution of the mixed problem of the torsion of a half space with a spherical inclusion (see A. N. Rukhovets and Ya. S. Uflyand, 1967).

We also mention the interesting article of N. M. Borodachev (1967) in which coupled series in Bessel functions are used in the axisymmetric problem on the indentation of a circular die into the end of a semi-infinite cylinder.

Another class of three-dimensional mixed problems in the theory of elasticity which underwent considerable development in the studies of Soviet scientists in recent years must also be mentioned. These are contact problems for a linearly deformed base and related problems connected with the action of a die on an inhomogeneous elastic halfspace. The fundamental studies here go back to B. G. Korenev (1954, 1957, 1960). Subsequently, these problems were studied by V. I. Mossakovskiy (1958), G. Ya. Popov (1959), A. F. Rakov and V. L. Rvachev (1961), N. A. Rostovtsev (1961, 1964) and many other authors. The most detailed information on these problems is available in the survey article of A. G. Ishkova and B. G. Korenev (1966).

In conclusion we note that a considerable amount of information and a large bibliography on mixed three-dimensional problems in the theory of elasticity that were studied in recent years can be found in the surveys of D. I. Sherman (1962), B. L. Abramyan and A. Ya. Aleksandrov (1966), G. Ya. Popov and N. A. Rostovtsev (1966), N. A. Kil'chevskiy and E. N. Kostyuk (1966) and V. L. Rvachev (1967).

§5. Formulation and Methods for the Solution of Problems in Two-Dimensional Elasticity Theory

One of the most important and best developed branches in the theory of elasticity at the present time in which the achievements of Soviet science are especially impressive is the so-called plane problem in the theory of elasticity. The success in the development of plane problems is explained by using in the discussion the theory of analytic functions of a complex variable. The first basic results along these lines which are responsible for the contemporary form of plane theory as a whole were obtained in the fundamental studies of G. V. Kolosov and N. I. Muskhelishvili.

By a plane problem in the theory of elasticy is meant a plane deformation of an elastic medium which is parallel to a given plane (the deformation of a long cylinder with free bases), or its plane stressed state (deformation of the thin plate by forces in its plane). The determination of the elastic equilibrium in these cases reduces to the solution of boundary value problems for the biharmonic equation. Equilibrium problems of elastic plates subjected to a normal load are also reduced to the biharmonic equation. Plane problems and problems in the bending of plates and their mathematical formulation are very similar and the methods used for their solution are also similar. Therefore, it is useful to consider together these two types of problems.

5.1. General Complex Representation of the Solution of the Plane Problem

The fundamental relations for the plane problem in the notation of N. I. Muskhelishvili are assumed to be known. The domain S, occupied by the elastic medium, is a connected region in the Oxy plane which is bounded by one or several closed contours which do not have common points L_1, L_2, . . . L_m, L_{m+1}, where the last includes all previous ones. When there is no contour L_{m+1} we have an infinite region in the plane with "holes." Cases are also considered when among the contours L_k are open contours of finite length or infinite ones (plane with cracks, halfplane with "holes," etc.). It is assumed that no body forces are present.

The stresses and displacements are expressed in terms of the complex Kolosov-Muskhelishvili potentials $\varphi(z)$, $\psi(z)$, according to the formulas

$$\left.\begin{aligned} X_x+Y_y &= 2\,[\varphi'(z)+\overline{\varphi'(z)}],\\ Y_y-X_x+2iX_y &= 2\,[\bar z\varphi''(z)+\psi'(z)],\\ 2\mu\,(u+iv) &= \varkappa\varphi(z)-\overline{z\varphi'(z)}-\overline{\psi(z)}. \end{aligned}\right\} \qquad (5.1)$$

These formulas were derived for the first time by G. V. Kolosov in 1909 in his fundamental study, "An Application of the Theory of Functions of a Complex Variable to a Plane Problem in the Mathematical Theory of Elasticity." They were derived rigorously by N. I. Muskhelishvili (see his monograph "Certain Basic Problems in the Mathematical Theory of Elasticity," 1933, 5th ed., 1966).

The potentials $\varphi(z)$ and $\psi(z)$ are holomorphic in a simply connected and finite domain S in the absence of concentrated forces and moments. In the case of a multiply connected region, the requirement on the uniqueness and finiteness of the stresses and displacement in S leads to the representation

$$\left.\begin{aligned}\varphi(z) &= -\frac{1}{2\pi(1+\varkappa)}\sum_{k=1}^{m}(X_k+iY_k)\ln(z-z_k)+\varphi^*(z),\\ \psi(z) &= \frac{\varkappa}{2\pi(1+\varkappa)}\sum_{k=1}^{m}(X_k-iY_k)\ln(z-z_k)+\psi^*(z),\end{aligned}\right\} \quad (5.2)$$

where $\varphi^*(z)$ and $\psi^*(z)$ are holomorphic on S, z_k are points in the interior of L_k, $X_k + iY_k$ is the principal vector of external forces on L_k. For an infinite domain S, in the absence of L_{m+1}, when the stress field in parts of the body at an infinite distance is finite, φ and ψ are represented near a point at infinity in the form

$$\left.\begin{aligned}\varphi(z) &= -\frac{X+iY}{2\pi(1+\varkappa)}\ln z+\varphi_0(z)+\Gamma z,\\ \psi(z) &= \varkappa\frac{X-iY}{2\pi(1+\varkappa)}\ln z+\psi_0(z)+\Gamma' z.\end{aligned}\right\} \quad (5.3)$$

The complex constants Γ, Γ' determine the stresses and rotation at infinity, $X + iY$ is the principal vector of external forces on the boundary L of the region, and $\varphi_0(z)$ and $\psi_0(z)$ are holomorphic in the neighborhood of $z = \infty$. The displacement vector at infinity is bounded for the conditions $\Gamma = \Gamma' = 0$, $X + iY = 0$.

5.2 Formulation of Fundamental Problems in Plane Elasticity Theory

By the basic problems in plane elasticity theory are usually meant the following three problems:

The first fundamental problem requires that the elastic equilibrium of a body be determined when the external forces are given on its boundary. This problem leads to the following limiting problem in the theory of analytic functions:

$$\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)} = f(t) + C(t) \text{ на } L, \tag{5.4}$$
a

Key: a. on

where f(t) is a given function on L determined by the external forces from the formula

$$f(t) = i\int_0^s (X_n + iY_n)\,ds,$$

where s is the arc of the contour L_k measured on each L_k from some fixed point on it in the positive direction, and $C(t) = C_k$ on L_k, and C_k is a complex constant.

The second fundamental problem consists of determining the elastic equilibrium of a body from the given displacements of points on its boundary. To find the functions φ and ψ, which are analytic on the domain C, we have in this case, the boundary condition

$$\varkappa\varphi(t) - t\overline{\varphi'(t)} - \overline{\psi(t)} = g(t) \text{ on } L, \tag{5.5}$$

where g(t) is a given function and $g(t) = 2\mu(u + iv)$ on L.

For the sake of simplicity, we will formulate the fundamental mixed problem for a finite simply connected domain bounded by a single closed contour. In this problem, on a part of the boundary $L' = a_1b_1 + a_2b_2 \ldots + a_nb_n$, where a_kb_k $(k = 1, \ldots, n)$ are nonintersecting arcs of the contour L which occur in a certain order the external stresses are given, and on the second part $L'' = b_1a_2 + b_2a_3 + \ldots + b_na_{n+1}$ $(a_{n+1} = a_1)$ the displacements are given. The corresponding problem in the theory of analytic functions has the form

$$k\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)} = h(t) + C(t), \tag{5.6}$$

where h(t) is a given function $k = 1$ for $t \in L'$, $k = -\varkappa$ for $t \in L''$, $C(t) = C_k$ = const for $t \in L'$, $C(t) = 0$ for $t \in L''$.

Conditions (5.4) or (5.5) must be satisfied on each contour L_k. Generally the constant C(t) in the right member of (5.4) may take on different values on different contours. Only on one of them its value can be selected (usually $C_{m+1} = 0$) but on the other contours the values remain completely arbitrary and must be determined during the solution of the problem. In precisely the same way the constants C_k in the right member of (5.6) (except one which is selected arbitrarily) are not specified in advance but must be determined together with the functions φ and ψ..

In plane elasticity theory, the so-called third fundamental problem when the normal component of the displacement vector and the tangential component of the external stress vector are given on the boundary is also considered. This corresponds to the contact of an elastic body with a rigid profile of a given form when the contact between the elastic and rigid bodies occurs along their entire boundary.

If the arbitrary constants in the right members of (5.4) and (5.6) are fixed, the additional conditions, as shown above, for φ and ψ will have the following form:

in the first problem

$$\varphi(0) = 0, \quad \mathrm{Im}\, \varphi'(0) = 0;$$

in the second and in the mixed problem

$$\varphi(0) = 0 \ \overset{a}{\text{или}} \ \psi(0) = 0.$$

Key: a. or

This exhausts any indeterminacy in the selection of the functions φ and ψ.

It can be proved that in the case of the first problem when the rigid displacement of the body as a whole is ignored and in the third problem for a circle when the rigid rotation about its center is ignored, each of the formulated problems has only one solution. The necessary conditions for the existence of a solution for the first fundamental problem is that the principal vector and the principal moment of the external forces applied to the boundary of the region be zero. When the function f(t), in the right member of (5.4) is singlevalued and continuous, these two conditions reduce to the condition (N. I. Muskhelishvili, 1966):

$$\operatorname{Re} \int_L f(t)\, \overline{dt} = 0.$$

In the theory of the bending of plates, it is proved that the flexure w(x, y) of the middle surface of a thin homogeneous elastic plate subjected to a normal load uniformly distributed on its surface satisfies the nonhomogeneous biharmonic equation

$$\Delta\Delta w = \frac{q}{D}, \tag{5.7}$$

where q is the intensity of the load, and D is the cylindrical rigidity.

After a particular solution of (5.7) is found, we can represent the general solution, using the well-know Goursat formula in terms of two analytical functions φ and χ, where $\chi'(z) = \psi(z)$. The basic quantities which determine the stressed state of the plate are expressed in terms of these functions. The following formulas are valid (S. G. Lekhnitskiy, 1938), which are analogous to the Kolosov-Muskhelishvili formulas:

$$\left.\begin{aligned} M_y - M_x + 2iH_{xy} &= 4D(1-\nu)[\bar{z}\varphi''(z) + \psi'(z)] + M_y^0 - M_x^0 + 2iH_{xy}^0, \\ M_x + M_y &= -8D(1-\nu)[\varphi'(z) + \overline{\varphi'(z)}] + M_x^0 + M_y^0, \\ N_x - iN_y &= -8D\varphi''(z) + N_x^0 - iN_y^0. \end{aligned}\right\} \tag{5.8}$$

Here M_x, M_y are the bending moments, H_{xy} is the torque, N_x, D_y are the shearing forces per unit length, M_x^0 . . ., N_y^0 are the same magnitudes referring to the selected particular solution of equation (5.7). The degree to which the functions φ and ψ are defined is the same as in the plane problem.

To determine the flexures from equation (5.7), the boundary conditions corresponding to the particular character with which the boundary is fixed must be added to it.

Here we have the following three basic problems. We will formulate them keeping in mind the case of the simply connected domain bounded by a closed contour.

I. THE EDGE OF THE PLATE IS FIXED. This means that on the boundary of the region S occupied by the middle surface of the plate, the relations

$$w = 0, \quad \frac{dw}{dn} = 0, \tag{5.9}$$

must be satisfied where n is the outer normal to the contour.

II. THE EDGE OF THE PLATE IS FREE. The boundary conditions have the form

$$\left.\begin{aligned} &\nu \Delta w + (1-\nu)\left[\frac{\partial^2 w}{\partial x^2}\cos^2\theta + \frac{\partial^2 w}{\partial y^2}\sin^2\theta + \frac{\partial^2 w}{\partial x\,\partial y}\sin 2\theta\right] = 0, \\ &\frac{\partial \Delta w}{\partial n} + \frac{1-\nu}{2}\frac{d}{ds}\left[\left(\frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 w}{\partial x^2}\right)\sin 2\theta + 2\frac{\partial^2 w}{\partial x\,\partial y}\cos 2\theta\right] = 0, \end{aligned}\right\} \tag{5.10}$$

where θ is the angle subtended by the outer normal and the Ox axis. The left members in the equations are respectively the bending moment and the generalized shearing force reduced to unit length acting on an element of the plate with the normal n.

III. THE EDGE OF THE PLATE IS SUPPORTED. The following conditions correspond to the free support of the edge:

$$\left.\begin{aligned} &w = 0, \\ &\nu \Delta w + (1-\nu)\left[\frac{\partial^2 w}{\partial x^2}\cos^2\theta + \frac{\partial^2 w}{\partial y^2}\sin^2\theta + \frac{\partial^2 w}{\partial x\,\partial y}\sin 2\theta\right] = 0. \end{aligned}\right\} \tag{5.11}$$

In addition to these basic types of boundary conditions, particularly interesting mixed conditions are often encountered in applications, for example, when one part of the boundary is fixed, another supported and the remaining one free.

Since the boundary values of the partial derivatives of this function with respect to x and y can always be found from the boundary values of the function w and its normal derivative, problem I for the bending of the plate is equivalent to the first fundamental problem in plane elasticity theory. The boundary conditions of problem I coincide exactly with condition (5.4) with nothing arbitrary in the right member of the latter.

The conditions on the free boundary in (5.10), as noted by S. G. Lekhnitskiy (1938) and I. N. Vekua (1942) lead, after their appropriate transformation, to a boundary value problem in the theory of functions which is completely analogous to (5.5). The only difference is that the constant $\varkappa$ in the left member of (5.5) is replaced by another constant, $\varkappa^* = (3 + \nu)/(1 - \nu)$, and the right member is given with an accuracy up to a constant having the form $iCt + C_1$, where C is a real and C_1 a complex constant. Incidentally, in the case of the simply connected domain under consideration, these constants may be set equal to zero.

Finally, the conditions for the free support of the edges (5.11) can be written in terms of the functions φ and ψ in the form (A. I. Kalandiya, 1953)

$$\left.\begin{aligned} &\operatorname{Re}\left\{\lambda_0\varphi'(t)-\left(\frac{dt}{ds}\right)^2[\bar{t}\,\varphi''(t)+\psi'(t)]\right\}=g_1(t),\\ &\operatorname{Re}\left\{\frac{dt}{ds}[\overline{\varphi(t)}+\bar{t}\,\varphi'(t)+\psi(t)]\right\}=g_2(t)\end{aligned}\right\}\text{на}_a\ L, \qquad (5.12)$$

Key: a. on

where g_1, g_2 are given functions defined on L and $\lambda_0 = 2(1 + \nu)/(1 - \nu)$.

It is easily verified that problem (5.12) and the third problem in plane elasticity theory are equivalent.

It is clear from what was said above that the methods for the solution of plane problems can be sometimes applied without any problems in the bending of thin plates. This possibility was investigated for the first time by A. I. Lur'e (1928).

5.3 Methods for the Solution of Plane Problems

Below we will give a brief characterization of the methods used in the solution of plane problems which are based on the application on the theory of functions of a complex variable.[1]

1. Section 5.3.9 also discusses the method of integral transforms in plane problems in the theory of elasticity

We will mainly restrict ourselves to the consideration of the case when the elastic medium occupies a finite simply connected domain bounded by a closed contour. The principal domain S, inside L, will here be denoted by S^+, and outside (the complement of S^+) by S^-.

5.3.1. We will recall certain elementary concepts and propositions in the theory of analytic functions which will be used in the presentation below.

By the Gauchy integral formula is meant the expression

$$F(z)=\frac{1}{2\pi i}\int_L \frac{f(t)\,dt}{t-z}, \tag{5.13}$$

where t is a point on the contour L, and z is a point in the plane. If z coincides with the point t_0 in the interior of the contour L, we shall mean by the integral (5.13) its principal value according to Cauchy.

The function F(z) defined by formula (5.13) is holomorphic both in the region S^+ and in S^-, and when the density f(t) is sufficiently smooth (for example, if it satisfies the Hölder condition on L), it is continuous in the corresponding closed regions $S^+ + L$ and $S^- + L$. The limiting value of this function from the left and right of L at some point $t_0 \in L$, usually denoted by $F^+(t_0)$ and $F^-(t_0)$, respectively, is given by the well-known Sokhotskiy-Plemeli formulas.

A function which is holomorphic both in S^+ and in S^- with continuous values F^+ and F^- in the limit is called, following N. I. Muskhelishvili, piecewise-holomorphic. An example of a piecewise-holomorphic function is given, when the conditions for the function f(t) are known, by the integral (5.13).

A necessary and sufficient condition that a given function f(t) on L be the limit of a function f(z), which is holomorphic on S^+, is

$$\frac{1}{2\pi i}\int_L \frac{f(t)\,dt}{t-z}=0 \text{ для всех}_{a}\ z\in S^-. \tag{5.14}$$

Key: a. for all

Analogously, a condition that the function f(t) be the boundary value of the function f(z), which is holomorphic on S^-, is the equality

$$\frac{1}{2\pi i}\int_L \frac{f(t)\,dt}{t-z}\,dt = \text{const} \quad \text{для всех}^{a} \; z \in S^+. \tag{5.15}$$

Key: a. for all

In the case when S^+ is the unit circle, the previous conditions can be expressed in a more convenient form for the purposes below (N. I. Muskhelishvili, 1966). From the given function f(z), which is holomorphic on S^+, we will determine another function of a complex argument according to the equality

$$f_*(z) = \overline{f\left(\frac{1}{\bar{z}}\right)}. \tag{5.16}$$

For this function we will sometimes use the notation

$$f_*(z) = \bar{f}\left(\frac{1}{z}\right). \tag{5.17}$$

Through direct verification of the Cauchy-Rieman conditions, it is easily verified that the function $f_*(z)$ is holomorphic in the domain which includes the point at infinity. Conversely, if the function f(z) is holomorphic on S^-, $f_*(z)$ will be holomorphic in z in the domain S^+.

The notation (5.16) can also be used in the more general case, when, for example, f(z) has in the interior of S^+ a finite number of poles. The function $f_*(z)$ will then have poles of the same orders at the points which are the images of the poles f(z) in the unit circle.

For the boundary values of the function (5.16), we will have

$$f_*^-(t) = \overline{f^+(t)}, \quad f_*^+(t) = \overline{f^-(t)}. \tag{5.18}$$

Applying (5.14) to the function $f_*(z)$, we obtain the condition

$$\frac{1}{2\pi i}\int_L \frac{\overline{f(t)}\,dt}{t-z} = \text{const} \quad \text{для всех} \; z \in S^+. \tag{5.19}$$

Key: a. for all

which is a necessary and sufficient condition that the function f(t) which is continuous on the circle be the limiting value of some function f(z) which is holomorphic on S^+. The constant in the right member of the equality has the value $\overline{f(0)}$. Analogously, as before, condition (5.15) takes on the form

$$\frac{1}{2\pi i}\int_L \frac{\overline{f(t)}\,dt}{t-z} = 0 \quad \text{для всех} \; z \in S^-. \tag{5.20}$$

Key: a. for all

The operation (5.16) is one possible way of constructing on S^- a holomorphic function from a given holomorphic function f(z) on S^+. Clearly, the extension of a function which is holomorphic in the circle to its exterior can be obtained in an infinite number of ways. However, the method that was indicated is one of the few methods that are useful in applications.

The function f(z), which is defined both on S^+ and on S^- by the formula

$$f(z) = \begin{cases} f(z) & \text{при} \; |z| < 1, \\ f_*(z) & \text{при} \; |z| > 1, \end{cases}$$

Key: a. when

is clearly piecewise-holomorphic. In addition to this, f(z) has an analytic continuation on those arcs of the circle $|t| = 1$ on which $\operatorname{Im} f(t) = 0$. The last property of f(z) follows directly from (5.18).

This type of extension of holomorphic functions is often used in applications even in the case when S^+ is a halfplane. Then, instead of (5.16),

$$\bar{F}(z) = \overline{F(\bar{z})}. \tag{5.21}$$

is used (N. I. Muskhelishvili, 1966).

From among the various methods used to solve plane problems which are known in the scientific literature at the present time, we will mainly touch here only on those that are directly connected with the names of Soviet scientists and are most effective both in general studies of boundary value problems and in their study in special cases. Primarily we will deal with the following four methods:

1. The method of power series using conformal mapping,

2. Reduction to functional (in particular, integral) equations using conformal mapping (the case of simply connected domains),

3. General methods leading to integral equations without conformal mapping,

4. Reduction to linear conjugate problems.

In a number of special cases, especially in the study of multiply connected media, it is useful to use in the study a particular combination of these methods.

Below we will describe briefly the methods that were mentioned.

5.3.2. In the solution of the plane problem it is often useful to map conformally the given region occupied by the elastic medium onto another region in the plane using the auxiliary variable ζ. In the case of a finite simply connected domain S, bounded by a closed contour, the mapping is usually onto the unit circle, and in the case of a finite doubly connected domain onto a circular concentric ring and in the case of a semi-infinite domain with the boundary at infinity on both sides, onto the halfplane, etc.

We will show here one of the variants of the conformal mapping applied in the first case above (N. I. Muskhelishvili, 1966). Let

$$z = \omega(\zeta)$$

be the relation used for the conformal mapping of the unit circle $|\zeta| < 1$, whose contour we denote by γ onto the region S. The functions $\varphi(z)$ and $\psi(z)$ expressed in terms of the new variable ζ will be denoted by $\varphi(\zeta)$, $\psi(\zeta)$.

The boundary conditions (5.4) for the first problem will take on the form

$$\varphi(\sigma) + \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\overline{\varphi'(\sigma)} + \overline{\psi(\sigma)} = f(\sigma) \underset{a}{\text{на}} \gamma, \tag{5.22}$$

Key: a. on

where σ is a point on the contour γ, $\sigma = e^{i\theta}$, and f is a given function on γ.

We will assume that the following Fourier series expansions can be obtained:

$$\frac{\omega(\sigma)}{\overline{\omega'(\sigma)}} = \sum_{-\infty}^{\infty} b_k\sigma^k, \quad f(\sigma) = \sum_{-\infty}^{\infty} A_k\sigma^k, \tag{5.23}$$

and we will assume that in the unit circle (for $|\zeta| < 1$)

$$\left.\begin{aligned} &\varphi(\zeta) = \sum_{0}^{\infty} a_k\zeta^k, \quad \psi(\zeta) = \sum_{0}^{\infty} a'_k\zeta^k, \\ &\varphi'(\zeta) = \sum_{1}^{\infty} k a_k\zeta^{k-1}. \end{aligned}\right\} \tag{5.24}$$

Then, on the basis of (5.22), when the conditions for the convergence of the above series are known, we obtain the following systems of equations for the unknown coefficients a_k, a'_k which must be determined:

$$a_m + \sum_{k=1}^{\infty} k\bar{a}_k b_{m+k-1} = A_m \quad (m = 1, 2, \ldots), \tag{5.25}$$

$$a'_m + \sum_{k=1}^{\infty} k\bar{a}_k b_{-m+k-1} = A_{-m} \quad (m = 0, 1, 2, \ldots). \tag{5.26}$$

It can be proved that the infinite system of linear equations (5.25) is solvable if the static conditions are satisfied, and that its solution, together with (5.26), is the solution of the plane problem under consideration when the given function f(t) is sufficiently smooth.

The following fact is of considerable importance in practice. In the case when the mapping function is a polynomial

$$\omega(\zeta) = c_1\zeta + c_2\zeta^2 + \ldots + c_n\zeta^n \quad (c_1 \neq 0,\ c_n \neq 0), \tag{5.27}$$

the infinite system (5.25) degenerates to the following finite system:

$$\left.\begin{array}{l} a_m = A_m \qquad (m \geqslant n+1), \\ a_1 + \bar{a}_1 b_1 + 2\bar{a}_2 b_2 + \ldots + n\bar{a}_n b_n = A_1, \\ a_2 + \bar{a}_1 b_2 + 2\bar{a}_2 b_3 + \ldots + (n-1)\,\bar{a}_{n-1} b_n = A_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_n + \bar{a}_1 b_n = A_n. \end{array}\right| \tag{5.28}$$

and formula (5.26) gives

$$a'_m + \sum_{k=1}^{m+n+1} k\bar{a}_k b_{-m+k-1} = A_{-m} \quad (m = 0, 1, 2, \ldots). \tag{5.29}$$

The problem reduces to the solution of the finite system (5.28).

The technique that was presented can clearly also be applied in the case of a mapping onto a circular ring.

The method of power series combined with conformal mapping is used extensively to this day in the solution of particular concrete problems. It is sometimes applied in a slightly modified form (see, for example, D. I. Sherman, 1951, K. Grey, Quart. J. Mech. and Appl. Math., 1951, 4:4, 444-448, M. Kikykawa, Proc. Japan. Nat. Congr. Appl. Mech., 1953 and 1954).

5.3.3. An especially useful method for the effective solution of the problem turned out to be the method presented below which combines conformal mapping with the application of the apparatus of Gauchy integral formulas (N. I. Muskhelishvili, 1966, pp 78-85). It consists of the following.

Starting with the boundary condition (5.22) and expressing the condition that $\psi(\sigma)$ is the boundary value on the circle of the function $\psi(\zeta)$ which is holomorphic in the interior of the circle and vanishes at $\zeta = 0$, we obtain on the basis of (5.19) the functional equation

$$\varphi(\zeta) + \frac{1}{2\pi i}\int_{\gamma} \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}} \frac{\overline{\varphi'(\sigma)}}{\sigma - \zeta}\, d\sigma = A(\zeta) \quad (\zeta \in S),$$
$$A(\zeta) = \frac{1}{2\pi i}\int_{\gamma} \frac{f(\sigma)\, d\sigma}{\sigma - \zeta}. \tag{5.30}$$

It can be proved that for the fixed constant $\mathrm{Im}[\varphi'(0)/\omega'(0)]$ equation (5.30) defines the function $\varphi(\zeta)$ uniquely. After it is defined, the function $\psi(\zeta)$ is found directly from (5.22) using the Cauchy integral formula.

The functional equation (5.30) can be used to construct by elementary means the exact solution of the problem for a large class of regions. In principle an approximate solution can be obtained for the more general case of a simply connected domain.

As an illustration of what was discussed above, we will consider the case when the mapping function $\omega(\zeta)$ is rational. In this case the expression

$$\frac{\omega(\sigma)}{\overline{\omega'(\sigma)}} \overline{\varphi'(\sigma)},$$

under the integral in (5.30) will be the boundary value of the function

$$\frac{\omega(\zeta)}{\overline{\omega}'\left(\frac{1}{\zeta}\right)}\overline{\varphi}'\left(\frac{1}{\zeta}\right), \tag{5.31}$$

which is holomorphic outside γ except at a finite number of poles, the singular points of the function $\omega(\zeta)$.

Since the point ζ in the integral (5.30) is in the interior of γ, this integral is evaluated in closed form and it will represent a rational function with a number of unknown coefficients from the expansion of $\varphi(\zeta)$. A finite system of linear equations is set up for these coefficients and they can always be determined uniquely.

From this follows the well-known proposition of N. I. Muskhelishvili, according to which the solution of the plane problem for the class of regions under consideration can be obtained in quadratures with an accuracy up to the solution of the finite system of linear algebraic equations. In the special case of a polynomial mapping of the form (5.27), the function $\varphi(\zeta)$ in (5.30) will be represented as the sum of the Cauchy integral A (ζ) and the polynomial of degree n in ζ, which contains as the unknowns the first n coefficients of the function $\varphi(\zeta)$. The linear system of equations obtained for determining the latter coincides exactly with system (5.28). Both unknown functions, $\varphi(\zeta)$ and $\psi(\zeta)$ are determined in closed form by solving this system with the given function f.

If the function $\omega(\zeta)$ is not rational but its expansion on the circle is known, the method leads to an infinite system of linear equations, which can be used to construct an approximate solution of the problem with an arbitrary preassigned accuracy.

5.3.4. The complex representation of elastic fields combined with various integral representations of analytic functions is convenient apparatus for the reduction of the plane problem to integral equations. At the present time several variants for constructing such equations are known. We will point out some of these.

The Fredholm integral equation in $\varphi'(\sigma)$ can be obtained directly from the functional equation (5.30) by first writing it in a slightly different form and then letting the point ζ tend from the inside to the point γ on the circle (N. I. Muskhelishvili, 1966, §79). An elementary analysis of this integral equation can be used to prove the existence of its

solution (hence, also the existence of the solution of the corresponding plane problem), provided that in the case of the first problem for a finite medium, the static conditions are satisfied. A more detailed analysis of this equation was carried out by D. I. Sherman (1938). He studied the distribution of the characteristic numbers of the integral equation, and proved that it can be solved for both fundamental problems by the method of successive approximations.

A more general method which includes the case of multiply connected domains is the reduction to integral equations without a preliminary conformal mapping. One such method was proposed by N. I. Muskhelishvili (1966, §98). We will explain the essence of the method by assuming that the medium is finite and simply connected.

In equality (5.4) which expressed the boundary conditions for the problem, we will use the conjugate values, and, according to (5.14), write down the conditions that the function $\psi(t)$ is the boundary value of a function of z, which is holomorphic on S^+. We obtain the functional equation

$$\frac{1}{2\pi i}\int_L \frac{\overline{\varphi(t)}\,dt}{t-z} + \frac{1}{2\pi i}\int_L \frac{\bar{t}\varphi'(t)\,dt}{t-z} = A(z) \quad \text{для всех}^{a}\ z \in S^-,$$

$$A(z) = \frac{1}{2\pi i}\int_L \frac{\overline{f(t)}\,dt}{t-z}.$$

Key: a. for all

Now if we write down the same condition for $\varphi(t)$ and $\overline{\varphi'(t)}$, we will have two additional equalities which are analogous to the previous ones. Combining the three equalities after passing to the limit to z from the right, we obtain the Fredholm equation for $\varphi(t)$ derived by N. I. Muskhelishvili:

$$\overline{\varphi(t_0)} - \frac{1}{2\pi i}\int_L \overline{\varphi(t)}\, d\ln\frac{\bar{t}-\bar{t}_0}{t-t_0} + \frac{1}{2\pi i}\int_L \varphi(t)\, d\,\frac{\bar{t}-\bar{t}_0}{t-t_0} = -A(t_0). \qquad (5.32)$$

A very similar but essentially different equation for the plane problem was constructed in another way by D. I. Sherman (1940), which we will discuss in greater detail below.

According to the studies of D. I. Sherman (1935-1937) equation (5.32) is useful for any multiply connected domain. It always has a solution which yields the solution for the corresponding plane problem. In addition, the method of successive approximations can be applied in preliminary fashion to equation (5.32) in a slightly and easily modified form (D. I. Sherman, 1940).

An integral equation for the plane problem which is also useful for any multiply connected domain was constructed earlier by S. G. Mikhlin (1934, 1935). The so-called complex Green's function is introduced for this purpose into this discussion and then, using this function, the generalized Schwarz kernel, which is analytic in the region but not single-valued. In a multiply connected domain the generalized kernel has a property which is analogous to the property of the ordinary Schwarz kernel for the circle. The Mikhlin equation for a simply connected region coincides with equation (5.32). S. G. Mikhlin analyzed the equations that were constructed and proved that they can be solved, and also that the method of successive approximations can be applied to obtain their solution. The results are presented in his monograph (1949) which also includes applications of the Schwarz kernel to the solution of the plane problem in a number of special cases.

The studies of L. G. Magnaradze (1937, 1938) have shown that the Muskhelishvili equations remain also valid when the boundary has corners, provided that the integrals in the equations are interpreted in a certain generalized sense.

A simple and in many respects convenient form of the integral equation in the general case of a multiply connected region was obtained in 1940 by D. I. Sherman. We will derive the Sherman equation, restricting ourselves as before, to the case of a finite simply connected domain. The first and second fundamental problems will this time be discussed simultaneously and we will combine their boundary conditions in the following equality:

$$k\varphi(t)+t\overline{\varphi'(t)}+\overline{\psi(t)}=f(t) \quad \underset{a}{\text{на}}\ L, \tag{5.33}$$

Key: a. on

where k = 1 in the first problem, and k = - $\varkappa$ in the second case. Following Sherman, we let in the domain S

$$\left.\begin{aligned} \varphi(z) &= \frac{1}{2\pi i}\int_L \frac{\omega(t)\,dt}{t-z}, \\ \psi(z) &= \frac{k}{2\pi i}\int_L \frac{\overline{\omega(t)}\,dt}{t-z} - \frac{1}{2\pi i}\int_L \frac{\bar{t}\,\omega'(t)\,dt}{t-z}, \end{aligned}\right| \tag{5.34}$$

where $\omega(t)$ is some function at a point of the contour L, which must be defined. Passing in these formulas to the limit when the point z tends from the inside to the point $t_0 \in L$, and substituting the boundary values that were found in (5.33), we obtain after several simple transformations the relation

$$k\omega(t_0) + \frac{1}{2\pi i}\int_L \omega(t)\,d\ln\frac{t-t_0}{\bar{t}-\bar{t}_0} - \frac{1}{2\pi i}\int_L \overline{\omega(t)}\,d\frac{t-t_0}{\bar{t}-\bar{t}_0} = f(t_0). \tag{5.35}$$

This is the Fredholm integral equation for $\omega(t)$ that was already mentioned above. It is known as the Lauricelli-Sherman equation.

In the case of a multiply connected medium, it is useful, following Sherman, to change slightly the representation (5.34), as a result of which equation (5.35) is also modified.

An analysis shows that the homogeneous Lauricelli-Sherman equation does not have nontrivial solutions and that its unique solution gives, according to formulas (5.34), the solution of the original boundary value problem.

The representation (5.34) can also be applied to the solution of the fundamental mixed problem. However, in this case we will work with integral equations with kernels of the Cauchy type, whose theory, at the present time, has not been developed to the same extent as for Fredholm equations (N. I. Mushelishvili, 1946, 1952, N. P. Vekua, 1950).

Integral equations undoubtedly are a convenient means for a general analysis of boundary value problems, in particular, for proving the existence of their solutions. However, the method of integral equations is often criticised as not being sufficiently effective, not entirely without justification. Attempts to solve in practice problems on the basis of this method, using the usual scheme for calculating the discrete analogue of the integral equations are not very promising even with contemporary computer technology. Therefore, due to the absence of a more accepted algorithm for the solution

in the general case of a multiply connected domain it is necessary to develop special methods for an efficient solution which are adapted to a particular class of boundary value problems.

In this sense, various combinations of the methods that were enumerated above are important. Primarily we have in mind the combination of functional equations with the method of power series, the linea· combinations of functions with conformal mapping and also more general schemes using the apparatus of integral equations. Some of these special techniques will be discussed below.

5.3.5. In a number of studies of D. I. Sherman (see, for example, 1947, 1951) an efficient method for solving the plane problem was developed for a particular class (of finite and infinite) doubly connected domains bounded by two closed curves. The basic feature of the method which determines the class of the admissible domains is the requrement that the plane problem for a simply connected domain (exterior or interior relative to one of the closed contours which bound the region) have a solution in closed form. Thus, the boundary of the domain can be circles, ellipses, regular polygons with rounded vertices, etc. An example of an infinite domain is a plane with two "holes" with the desired shape. A halfplane with two "holes" (triply connected domain) can also be included in the discussion if the "holes" lie far from the rectilinear boundary and if the boundary conditions on the latter can be satisfied only approximately. Problems of this type are very important in applications in mining. To present the substance of the method, we will assume for definiteness that the domain S is finite and bounded by the curves L_1 (interior) and L_2 (exterior).

We introduce into the discussion the auxiliary function $\omega(t)$ which is defined on L_2 by the equality

$$\varphi(t) - t\,\overline{\varphi'(t)} - \overline{\psi(t)} = 2\,\omega(t) \qquad (t \text{ on } L_2). \tag{5.36}$$

Adding and subtracting term by term equality (5.36) and (5.4) on L_2 and assuming $C_2 = 0$, we obtain

$$\left.\begin{aligned} \varphi(t) &= \omega(t) + f(t), \\ \psi(t) &= -[\overline{\omega(t)} + \bar{t}\,\omega'(t)] + [f(t) - \bar{t} f'(t)] \end{aligned}\right\} \tag{5.37}$$

Using $\omega(t)$, we introduce the two new functions $\varphi_0(z)$ and $\psi_0(z)$ of the following form:

$$\left.\begin{aligned}\varphi_0(z) &= \varphi(z) - \frac{1}{2\pi i}\int_{L_2}\frac{\omega(t)\,dt}{t-z} - F(z),\\ \psi_0(z) &= \psi(z) + \frac{1}{2\pi i}\int_{L_2}\frac{\overline{\omega(t)} + \bar{t}\,\omega'(t)}{t-z}\,dt - G(z),\end{aligned}\right\} \tag{5.38}$$

where

$$F(z) = \frac{1}{2\pi i}\int_{L_2}\frac{f(t)\,dt}{t-z}, \quad G(z) = \frac{1}{2\pi i}\int_{L_2}\frac{f(t) - \bar{t}f'(t)}{t-z}\,dt.$$

Now if we define completely the unknown functions φ and ψ by setting them equal to zero outside L_2, equalities (5.37) will express the condition for the analytic continuation of the newly introduced functions through the contour L_2 which is easily verified. For these functions, φ_0 and ψ_0 which are holomorphic everywhere outside L_1, we obtain on the basis of equality (5.4) on L_1 the boundary condition

$$\varphi_0(t) + t\,\overline{\varphi'(t)} + \overline{\psi_0(t)} = \Omega\,[t;\,\omega(t)], \tag{5.39}$$

where Ω is a linear operator.

According to the fundamental requirement of the method, we further assume that the auxiliary plane problem (5.39) can be solved in finite form. Clearly, this will always be the case if the function which maps the region outside L_1 onto the circle is rational.

In the right part Ω, which is considered as a given function of time t, using the Mushkelishvili method of functional equations (see above, Section 3.3) the solution of problem (5.39) is found in closed form and the functions φ_0, ψ_0 that were found are substituted in condition (5.36). This gives a relation in the form of a Fredholm integral equation of the second kind for $\omega(t)$ which must be determined. Then, expanding $\omega(t)$ in a complex Fourier series, the integral equation is reduced to an infinite system of linear algebraic equations.

5.3.6. In a number of cases the integral equations can be applied directly to the effective solutions of the problems. We will discuss one possible application of the Laurichelli-Sherman equation.

We will assume that the function $\omega(\zeta)$, which maps conformally the circle onto the region (exterior or interior with respect to the contour L) is known. Making a change of variable in equation (5.35) according to $t = \omega(\sigma)$ we obtain an integral equation on the unit circle. The kernel of this equation is expressed in elementary fashion in terms of $\omega(\sigma)$ and it preserves its simple structure in many cases, for example, in the case of an arbitrary transformation of the type (5.27). In all these cases, the method of Fourier series can be applied to the integral equation that was obtained, which leads to an effective solution of the problem.

5.3.7. By a boundary value problem for the linear conjugate equation we will mean the following problem. To find a function F(z) which is holomorphic on the line L of the complex plane from the boundary condition

$$F^{+}(t) = a(t) F^{-}(t) + b(t). \qquad (5.40)$$

where a(t) and b(t) are functions defined on L, $F^{+}(t)$ and $F^{-}(t)$ are the boundary values on L of the unknown function F(z) from the left and right with respect to the positive direction selected along the line L. It is assumed that these boundary values exist everywhere, except, possibly a finite number of points $C_1, C_2, \ldots, C_m$ on the line L, in the neighborhood of which F(z) satisfies the bound

$$|F(z)| \leqslant \frac{A}{|z - C_k|^{\alpha}} \quad \text{(A and } \alpha \text{ are constants, } \alpha < 1\text{)}.$$

Sometimes a solution for the boundary value problem (5.40) is sought, which has a pole at some point of the plane not on L. Usually a point at infinity is selected as such a point.

We will consider problem (5.40) under the following assumptions: L consists of a finite number of smooth contours which are closed or not closed, the functions a(t) and b(t) satisfy the Hölder condition on L except for a finite number of points where they have a discontinuity of the first type, and $a(t) \neq 0$.

Under these assumptions problem (5.40) is solved in explicit form (in quadratures). The solution (which has a pole at infinity) has the form

$$F(z) = \frac{X(z)}{2\pi i} \int_L \frac{b(t)\,dt}{X^+(t)(t-z)} + X(z)P(z), \tag{5.41}$$

where P(z) is an arbitrary polynomial and X(z) is the so-called canonical solution of the homogeneous problem $F^+(t) = a(t)F^-(t)$, which is constructed in explicit form (in quadratures).

To construct a solution with a definite order at infinity, certain constraints must be imposed on the polynomial P(z) and also on the function b(t) (see N. I. Mushelishvili, 1966).

The reduction of problems in plane elasticity theory to linear conjugate problems is one effective method of solving these problems (especially mixed problems).

As an illustration we will present the solution of the fundamental mixed problem for a halfplane, by reducing it to the linear conjugate problem (N. I. Muskhelishvili, 1966).

Suppose that the isotropic body occupies the lower half-plane $y < 0$, which we will denote by S^-. The upper halfplane will be denoted by S^+, the real axis by L and we will take as the positive direction on L the direction from $-\infty$ to $+\infty$.

We will use the formulas for the general complex representation of the stresses and displacements, in particular, we will use the formulas

$$Y_y - iX_y = \Phi(z) + \overline{\Phi(z)} + z\,\overline{\Phi'(z)} + \overline{\Psi(z)}, \tag{5.42}$$

$$2\mu\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) = \varkappa\Phi(z) - \overline{\Phi(z)} - z\,\overline{\Phi'(z)} - \overline{\Psi(z)}, \tag{5.43}$$

where $\Phi(z)$ and $\Psi(z)$ are the unknown functions which are holomorphic in the region S^- which, for large $|z|$ have the form

$$\Phi(z) = -\frac{X+iY}{2\pi z} + O\left(\frac{1}{z^2}\right),$$

$$\Psi(z) = \frac{X-iY}{2\pi z} + O\left(\frac{1}{z^2}\right),$$

where (X, Y) is the principal vector of external forces applied to L.

Instead of two holomorphic functions $\Phi(z)$ and $\Psi(z)$ on the region S^-, we will introduce one piecewise-holomorphic function $\Phi(z)$ which is defined both on S^- and on S^+, which will be defined in the upper halfplane S^+ in such a way that its values will be the analytic continuation of the values of $\Phi(z)$ in the lower halfplane S^- through sectors that are not loaded (provided these exist). We define $\Phi(z)$ on S^+ by the following formula:

$$\Phi(z) = -\overline{\Phi}(z) - z\overline{\Phi'}(z) - \overline{\Psi}(z).$$

This formula gives an expression for the function $\Psi(z)$ in terms of the function $\Phi(z)$ extended also to S^+:

$$\Psi(z) = -\Phi(z) - \overline{\Phi}(z) - z\Phi'(z);$$

Hence the stress components are expressed only in terms of one function $\Phi(z)$ which is defined both on S^+ and on S^-.

In particular, we have the formula

$$Y_y - iX_y = \Phi(z) - \Phi(\bar{z}) + (z - \bar{z})\overline{\Phi'(z)}, \tag{5.44}$$

and from formula (5.43) we obtain

$$2\mu\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) = \varkappa\Phi(z) + \Phi(\bar{z}) - (z - \bar{z})\overline{\Phi'(z)}. \tag{5.45}$$

Suppose

$$L' = \sum_{k=1}^{n} a_k b_k$$

is a set of segments $a_k b_k$ on the real axis and suppose that the displacement components are given on L' and the external forces on the remaining part L" = L - L'. Without loss of generality, we can assume that the external forces which are given on L" are zero (the general case is easily reduced to this).

We will assume that **the function** $\Phi(z)$ can be extended continuously from the left and right on L except possibly at the points a_k, b_k, and that in the neighborhood of these points

$$|\Phi(z)| \leqslant \frac{A}{|z-a_k|^{\alpha}}, \quad |\Phi(z)| \leqslant \frac{A}{|z-b_k|^{\alpha}} \quad (\alpha < 1).$$

We will also assume that

$$\lim (z - \bar{z})\, \Phi'(z) = 0,$$

when z tends to the point t on the real axis, which is different from the points a_k and b_k.

Under these assumptions it follows from formula (5.44) that

$$\Phi^+(t) = \Phi^-(t) \ \text{на}\ L'',$$

Key: a. on

i.e., the function $\Phi(z)$ is holomorphic on the entire plane, along L, and it vanishes at infinity.

From formula (5.45) we have

$$\Phi^+(t) = -\varkappa\, \Phi^-(t) - 2\mu\, g'(t) \qquad (t \in L'), \tag{5.46}$$

where g(t) is a given function, g(t) = u(t) + i v(t), and u(t) and v(t) denote the known limiting values of the displacement components on L'. We will assume that the derivative g'(t) satisfies the Hölder condition.

Applying formula (5.41) to the solution of the conjugate problem (5.46), we obtain

$$\Phi(z) = \frac{\mu X(z)}{\pi i} \int\limits_{L'} \frac{g'(t)\, dt}{X^+(t)(t-z)} + X(z)\, P(z).$$

where X(z) (the canonical solution) has the form

$$X(z)=\prod_{k=1}^{n}(z-a_k)^{-\frac{1}{2}+i\beta}(z-b_k)^{-\frac{1}{2}-i\beta}\quad\left(\beta=\frac{\ln\varkappa}{2\pi}\right).$$

Since $\Phi(z)$ vanishes at infinity and the order of $X(z)$ at infinity is n, the degree of the polynomial $P(z)$ must not exceed $n - 1$:

$$P(z)=C_0z^{n-1}+C_1z^{n-2}+\ldots+C_{n-1}.$$

When we reduced the problem under consideration to the linear conjugate problem, we differentiated the boundary condition along the segments a_kb_k. Hence, so far we were able to satisfy the boundary conditions along a_kb_k with an accuracy up to the constant terms c_k. What remains to be taken into account are the conditions $c_1 = c_2 = \ldots = c_n = 0$. However, it is easily seen, that it suffices to satisfy the conditions $c_1 = c_2 = \ldots = c_n$.

These conditions reduce to the following:

$$\int_{b_k}^{a_{k+1}}(u'-iv')\,dt=g(a_{k+1})-g(b_k)\quad(k=1,2,\ldots,n-1). \tag{5.47}$$

Substituting in equations (5.47) instead of $u(t) + iv(t)$ its expression in terms of the function Φ, we obtain a system $n - 1$ linear equations.

The coefficient C_0 of the highest order term in the polynomial $P(z)$ is determined from the given vector (X, Y) of external forces

$$C_0=-\frac{X-iY}{2\pi}.$$

Substituting the value of C_0 in system (5.47), we obtain a system of $n - 1$ linear equations for the coefficients $C_1, C_2, \ldots, C_{n-1}$. By the uniqueness theorem for the solution of the fundamental mixed problem, this system has a unique solution.

5.3.8. Methods of the theory of functions of a complex variable that were discussed above in connection with the plane problem in the theory of elasticity have been developed considerably in the studies of I. N. Vekua and applied to more general problems in the theory of partial differential equations. A large class of elliptic equations in the case of two independent variables is studied in the monograph of I. N. Vekua (1948) from this point of view. Applications of the apparatus developed by the author to various problems in the theory of elasticity (stationary oscillations of an elastic cylinder, the bending of thin plates, etc.) are presented by the author.

Here we will mention the many applications of the same methods to the theory of elastic shells (I. N. Vekua, A. L. Gol'denveyzer, G. N. Savin).

5.3.9. Along with the methods of the theory of functions of a complex variable which can be used to solve the plane problem for regions of a comparatively general form effective solutions for some regions with a concrete form can be found using special techniques, for example, using the integral Fourier and Mellin transforms.

The Fourier transform is a very useful tool for studying various elastic equilibrium problems in an infinite strip. The simplest solutions of this kind were already obtained by L. N. G. Failon. This method which was developed to a great extent in the work of Soviet authors was generalized at the end of the 30's and summarized in the well-known monographs of P. F. Papkovich (1939, 1941). Subsequently, various authors studied many new problems, dealing with the deformation of a strip, a halfstrip of the corresponding layer media and anisotropic bodies by thermal stresses, etc. Not being able to enumerate them, we refer the reader to the survey articles of D. I. Sherman (1962), G. Ya. Popov and N. A. Rostovtsev (1966), the monographs of S. G. Lekhnitskiy (1957) and M. P. Sheremet'ev (1968).

We also point out the articles of I. G. Al'perin (1930), M. Ya. Balen'ko (1952) and S. Ye. Birman (1954), while discussing mixed problems for an infinite strip, and also the articles of I. A. Markuzon (1963), V. S. Tonoyan (1963, 1964) in which certain classes of mixed problems are solved with the aid of coupled or three equations related to a Fourier transform for a halfplane, a strip and the quadrant. Similar problems dealing mainly with circular crescents were considered by Ya. S. Uflyand (1950, 1963), G. N. Savin (1951), M. A. Savruk (1957), V. V. Yeganyan (1959, 1964) and by other authors.

Certain plane problems in the theory of elasticity for an infinite wedge can be solved exactly with the aid of the Mellin integral transformation. The first studies in this class of problems go back to I. G. Brats and V. M. Abramov (1937). The problem of the action of a concentrated force on a wedge was studied for the first time by A. I. Lur'e and B. Z. Brachkovskiy (1941). An anisotropic wedge was studied by P. P. Kufarev (1941). A bibliography on the problems that were mentioned is available in the book of Ya. S. Uflyand (1963).

The development of the method of integral Fourier and Mellin transforms combined with Cauchy formulas is presented in the studies of S. M. Belonosov (1962) dealing with regions with corner points, and, in particular a strip and wedge (see below, Section 6.1.4.).

§6. Fundamental Results in the Study of Problems in Plane Elasticity Theory

In this section we will discuss certain concrete results in the theory of plane problems that were obtained in the USSR in the last 50 years. The studies that we will touch on are mainly closely related to complex variable methods, and, in this sense, they will serve as an illustration of their application and further development.

6.1. Solution of Fundamental Problems for a Homogeneous Medium

The first concrete results dealing with the equilibrium of plane profiles were obtained by G. V. Kolosov and N. I. Muskhelishvili.

6.1.1. Using the method presented in Section 5.3.2., N. I. Mushkhelishvili obtained a simple solution for the first and second fundamental problems for a circle, a circular ring, and an infinite plane with a circular "hole," He analyzed a set of particular examples for various types of external forces. For regions of this type, of course, a preliminary conformal mapping is not needed. Applying conformal mapping Mushkhelishvili solved, at that time, the difficult problem of the equilibrium of a solid ellipse. Later this problem was solved, using a different technique by D. I. Sherman (see Section 5.3.6).

Using power series, the problem of a confocal elliptical ring was investigated in an effective form (A. I. Kalandiya, 1953). An algorithm for an efficient solution of this problem was outlined earlier by M. P. Sheremet'ev, who used the method of functional equations in combination with conformal mapping (see Section 5.3.3).

The method that was just mentioned turned out to be most convenient for simply connected regions. As mentioned above, it always leads to an effective solution provided the regions are mapped by a rational function. The first applications of the method were pointed out by N. I. Muskhelishvili himself, who obtained closed solutions for the fundamental problems in several concrete cases. From this set of problems, we will select the equilibrium of a circular disc under the action of concentrated loads on the contour and an infinite plate with an elliptic hole. The results of Muskhelishvili that were mentioned were obtained by the author in his studies during the 20's and 30's (in particular, his memoir published in 1922 should be mentioned). All these results, together with other results by the same author are presented in detail in the monograph of N. I. Muskhelishvili that was cited on a number of occasions above.

We will mention here one important application of this method which is due to G. N. Savin. We will consider the problem of concentrated stresses in an infinite plate weakened by some hole. Assuming that the contour of the hole is a rectilinear polygon, we will map the interior of the circle onto the region exterior to the hole with the aid of the Schwarz-Christoffel integral. Expanding this integral in a series in powers of ζ and retaining in the series a finite number of terms, we obtain an approximate mapping which transforms the circle into a curve which is close to the original contour which has the form

$$z = \omega(\zeta) = C\left(\frac{1}{\zeta} + \sum_{k=1}^{n} C_k \zeta^k\right) \tag{6.1}$$

or in a special case

$$z = C\left(\frac{1}{\zeta} - m\zeta^n\right), \tag{6.2}$$

where C, C_k, m are some constants. By changing in (6.1) the constants C, C_k, n, we can obtain holes in the form of a circle, an ellipse, an oval shape, a curvilinear triangle and a quadrangle, etc. When (6.1) is mapped, the method leads directly to a solution in closed form which makes it possible to obtain an approximate solution for problems of the type that were mentioned.

G. N. Savin and his students studied in this manner many concrete problems dealing with the concentration of stresses with "holes" of various shapes and configurations in a homogeneous field. The solutions of these problems were carried through all the way to numerical results represented in the form of tables and diagrams. In addition to this in cases which are particularly important for applications, graphs were constructed for the distribution of the stresses on the contours. Savin solved, in a similar manner, the problem of the bending of a thin plate with a hole subjected to the action of moments and normal stresses at infinity. A detailed presentation of these results is given in the book of G. N. Savin (1951) which played an important role in the subsequent development of this type of problems.

At the same time as G. N. Savin M. I. Nayman (1937, 1958) who applied an original approach to the choice of the approximate mapping studied stress problems in plates with "holes" in the shape of curved polygons. He studied mainly the torsion of shafts weakened by longitudinal grooves.

6.1.2. The method presented in Section 5.3.3 can also be applied in a certain modification to the case of semi-infinite regions, when the boundaries of the medium are a curve receding to infinity in both directions. In this case, it is more convenient to use a mapping onto the halfplane. The application of the method in a general formulation is discussed in the monograph of N. I. Mushelishvili (1966) which also gives the solution of certain special problems of a similar kind.

Of particular interest for applications is the problem of the concentration of stresses in a halfplane weakened by a cutout or with recesses near the rectilinear boundary. A great deal of attention has been given recently to problems of this type, especially abroad (F. Neyber, M. Seika, S. Shioya).

The most successful approach to these problems is the approach of N. S. Kurdin. He was able to work out in detail, using the Muskhelishvili method, certain interesting problems of the type that was mentioned (1962).

The possibility of applying the method of the theory of functions to problems of the bending of plates was illustrated for the first time in the work of A. I. Lur'e (1928), which studied a plate with supported edges whose mean surface was mapped conformally onto a circle using a rational function. This problem was later studied in greater detail by A. I. Kalandiya (1953). In another study, A. I. Lur'e (1940) obtained by the same method solutions in closed form of the three fundamental problems of bending theory for the case of a circle. Here, as in the preceding work by the same author, the Muskhelishvili method was used (Section 5.3.3).

The study of S. G. Lekhnitskiy (1938), which was cited above, applied systematically the methods of complex variables to problems of the bending of plates. It derived general complex representations for the basic magnitudes for the isotropic and anisotropic cases and formulated the fundamental problems in final form in terms of complex variables and gave the solutions in certain special cases.

The studies of A. I. Lur'e and S. G. Lekhnitskiy were the beginning of intensive studies in the theory of the bending of plates.

Using the method of Section 5.3.3, M. M. Fridman (1945) obtained the solution of certain concrete problems in the bending of plates with a curvilinear "hole," flexed by moments and forces applied to its edge.

Particular attention was given to the equilibrium problem of a plate with supported edges. This problem was studied in the work of Z. I. Khalilov (1950), M. M. Fridman (1952), D. I. Sherman (1959), A. I. Kalandiya (1953).

6.1.3. The method of linear conjugate functions (see Section 5.3.7) is a very convenient means for the general study of the problems and also for their effective solution in special cases. It has clear advantages over other methods in the study of mixed and contact problems in which it is important to detect special properties of the solution. Problems of this type will be considered below in a separate section.

The application of the method of linear conjugate functions to plane problems was first developed in the work of N. I. Muskhelishvili (1941), which considered the case of an elastic halfplane. The solutions of the fundamental problems in this case were found in a simple and very elegant form. The subsequent important generalization of the method was proposed by I. N. Kartsivadze (1943) who extended the method to the case of a circular region and also to the more general case where the region is mapped onto a circle by means of a rational function. The first results in which the method is applied to the solution of concrete problems in the regions that were mentioned go back to this author. Kartsivadze's results are presented in detail in the book of N. I. Muskhelishvili (1966). The mixed plane problem with a circular "hole" was studied by B. L. Mintsberg (1948).

Using the same method, N. I. Muskhelishvili obtained a solution in closed form for the third fundamental problem of plane elasticity theory (see pages 53-55).

The conjugate problem with a rigid profile, using other methods was studied by G. N. Polozhiy. The boundary conditions of the problem were subjected to certain preliminary transformations which simplified the form of these solutions on rectilinear sectors of the boundary. This enabled the author to obtain a solution of the problem in explicit form, first for convex polygons (1948, 1950), and then, after rather sophisticated investigations of the behavior of the stresses at the corner points with the condition that the displacement vector be continuous, for general polygons and also for an infinite plane with an arbitrary polygonal "hole" (1957).

6.1.4. S. M. Belonosov (1954, 1962) who studied fundamental plane problems for simply connected domains proposed a method for their solution which became the theoretical basis for the practical application of an approximate solution based on rounding the corners. The conformal mapping of a given region onto the halfplane Re $\zeta > 0$ makes it possible to find the complex potentials φ and ψ by applying the apparatus of Laplace transforms. As a result, using a method which is analogous to that developed by N. I. Muskhelishvili (1966, Section 78, 79), integral equations whose structure is relatively simple are constructed which are applied in a certain sense to domains with corner points. If the contour L does not contain the corner points and generally is sufficiently smooth, the kernel of the equation is a Fredholm kernel and in the general case of a piece-wise smooth contour, it is a Carleman type of kernel.

The integral equations of S. M. Belonosov are solvable for every fundamental problem which was shown (1962). In the special cases of an infinite wedge or strip, the integral equations are solved in quadratures, which leads in these cases to the solution of the problem in finite form. In the book of S. M. Belonosov, which was cited, to which we refer the reader for details, the class of domains for which the fundamental problems are solvable in quadratures using the method that was mentioned is determined. This class of domains which are similar in form to a wedge, strip, and the outer region of a hyperbola, includes also a circular concentric ring.

6.1.5. The method of D. I. Sherman which was presented above (see Section 5.3.5) was first proposed by him (1947) for the solution of problems in the torsion and bending of a certain class of doubly connected profiles. When applied to a plane deformation, it was subsequently illustrated (1951) on the example of a halfplane weakened by two different circular "holes." In later studies Sherman's method was subjected to a basic revision, which resulted in the elimination of a large volume of intermediate computations. As a result of this the solution process became more tractable, and the main part is based on recurrence relations.

In the many studies of D. I. Sherman and his students which were published in the last few years, the method is applied to concrete plane deformation problems. Problems of a ponderable halfplane with two openings (circular and elliptical) at a considerable distance from the rectilinear boundary of the medium were considered, an elastic circle with a "hole" with a sufficiently general outline was considered, problems of a halfplane with a "hole" on whose edge a ring from another material was welded as well as analogous problems were considered. A thorough review of the results of the application of the method of integral equations with a complete bibliography is available in the survey of D. I. Sherman (1962), to which the reader is referred to acquire thorough familiarity with this class of problems.

In certain special cases of a multiply connected medium the generalized Schwarz algorithm which was developed in general form by S. G. Mikhlin (1949) was applied to the fundamental biharmonic problem. The first illustration of the method was given by the same author (1934) on the example of a ponderable halfplane with an elliptical "hole," when the stresses at infinity were distributed according to the hydrostatic law.

The convergence of the successive approximations, according to Schwarz, was studied with certain constraints on the region in the work of S. G. Mikhlin and A. Ya. Gorgidze. The convergence of the method in a general case was established by S. L. Sobolev (1936).

The Schwarz algorithm does not converge fast, which must be kept in mind when the method is applied in practice. Nevertheless in a number of cases it may give fairly good results. Examples of this are the studies of A. S. Kosmodamianskiy (1961, 1964) which study the case of two different "holes" in an infinite medium.

In the study of stresses in a plate with many "holes," one of the fundamental problems is determining the degree to which the medium is weakened around a given "hole" due to the presence of neighborning "holes." This problem, which is of great practical interest in mining was studied in the work of D. I. Sherman and his followers that were mentioned above. We will point out certain generalizations along these lines.

In the case when the medium is weakened by any finite number of "holes," A. S. Kosmodamianskiy (1961, 1962) applied the Bubnov-Galerkin method. To find the unknown complex potentials φ and ψ, he used infinite series of functions of a special form with undetermined coefficients and obtained for the approximate solution a finite system of algebraic equations. The method gives particularly good results in the case of circular "holes."

As the order of the approximation increases without limit, the algebraic systems become infinite. The studies of the same author have shown that these systems have desirable properties no matter how close the "holes" are to one another. In the case of non-circular curvilinear "holes," it is often useful to apply methods which are conceptually similar to the method of N. I. Muskhelishivili (A. S. Kosmodamianskiy, 1962). The approximate methods that were mentioned were used by Kosmodamianskiy and certain other authors to solve the problems in a number of concrete cases.

G. N. Bukharinov (1937, 1939), using an analogue of the successive approximation algorithm developed by G. M. Goluzin for the Dirichlet problem, studied the problem for a plate or disc, when the medium is weakened by any finite number of arbitrarily spaced circular "holes."

6.1.6. The periodic problem of elasticity theory is of considerable interest. Let us imagine an unbounded homogeneous medium weakened by an infinite number of equal and periodically spaced "holes." We will assume that all these "holes" are subjected to the same external forces and that their centers lie on one straight line. In the case of a halfplane, it is assumed that the center line is parallel to the boundary of the halfplane and that it lies at a distance which exceeds considerably the dimensions of the "hole."

The presence of joint symmetric geometric and force factors entails the periodicity of the displacements and stresses relative to the (real) variable which varies along the center line. This periodicity makes it possible to reduce the problem to the similar problem of finding two functions which are holomorphic in the region outside a certain closed contour. The concepts which led to the integral equations (5.32), can also be applied here, which makes it possible to construct for the problem a Fredholm integral equation which always has a unique solution. This was done by G. N. Savin (1939) (see also S. G. Mikhlin, 1949).

By using jointly the method of functional equations and power series, it is possible to construct, in a number of cases, an effective solution of the problem. We will point out certain studies along these lines.

D. I. Sherman (1961) studied the stress field in a ponderable medium weakened by periodically spaced circular and square "holes." The problem was solved by means of a reduction to an infinite system of linear algebraic equations. A quantitative analysis of the solution enabled the author to investigate the distribution of the stresses near the "holes" for a great range of the numerical parameter ε which characterized the relative dimensions, including the case of close "holes."

The periodic problem with curvilinear "holes" of general shape was already considered earlier in the work of I. I. Vorovich and A. S. Kosmodamianskiy (1959). Certain integral expressions were proposed by the authors for the unknown complex potentials, which were expressed in terms of other analytic functions that were holomorphic in a plane with one "hole." To find the latter, the method of a small parameter was used and the problem was reduced to a sequence of problems of the same type for a simply connected domain. The convergence of the method was not investigated. A detailed analysis with numerical calculations was carried out for the case of elliptic "holes" when the plate is expanded at infinity by forces applied at an arbitrary angle to the center line. A subsequent generalization of this approach was given by A. S. Kosmodamianskiy (1965).

It should be noted that the plane periodic problem in the theory of elasticity was studied for the first time by V. Ya. Natanson (1935), who studied the case of a doubly periodic system of circular "holes" in an infinite body.

The reader may find more detailed information about the periodic problem in the survey of D. I. Sherman (1962) that was cited above.

6.1.7. In the last few years a great deal of attention was given to finding effective methods for the solution of plane problems when the fundamental elasticity law is nonlinear and the assumption that the deformations are small is retained. The main interest was generated by problems connected with the determination of the stress concentrations in plates and shells with "holes."

If the nonlinearity of the elasticity law is characterized by a small numerical parameter, in this case, a nonlinear fourth order partial differential equation with a principal biharmonic term is obtained instead of the biharmonic equation for the stress function. This equation with the corresponding boundary conditions is integrated using the method of a small parameter, and the deviations of the elasticity law from the linear law and the shape of the "hole" from a circular shape are assumed to be small. Expanding the stress function, the components of the displacement vector and also the functions which occur in the boundary conditions of the problem in a series in the parameters which characterize the deviations that were mentioned above, we obtain a sequence of biharmonic problems for a plane with a circular "hole" which can be solved with the aid of approximate methods.

A number of concrete problems in nonlinear elasticity theory were solved in this way.

The numerical computations have shown that taking into account the physical nonlinearity leads to a more uniform distribution of the stresses near the "holes" in comparison with linear theory, and that the coefficient of the stress concentrations becomes smaller.

The reader can familiarize himself thoroughly with the results along these lines from the studies of G. N. Savin (1965), A. N. Guz', G. N. Savin and I. A. Tsurpal (1964).

6.2. Piecewise-Homogeneous Medium. Reinforcement and Strengthening of Plates

By a piecewise-homogeneous medium we shall mean an elastic medium consisting of a number of different homogeneous parts which differ in shape and elastic properties, which are connected into a single solid body in one way or another. The connections of the heterogeneous parts may either be natural or artificial. The latter always serve the purpose of increasing the load-bearing capacity of structures, and they are often used in engineering practice.

6.2.1. Suppose that we have a finite or infinite plate with a number of "holes" in which solid rings made from another material are inserted which, in turn, may be weakened by the "holes." When the ring is connected to the plate it can be welded into the "hole" along the circumference, pressed in, or inserted in it in the hot or cold state. Whenever the ring is not welded, it is assumed that the contours of the adjacent elastic parts touch without gaps and are maintained in a state which prevents slippage.

In this section, we will assume, in addition, that the surfaces of the bodies which make contact are never apart from one another as a result of the deformation.

Of course, not only the edges of the "holes" can be reinforced. The plate may be strengthened by rings along any edge, and also in the interior parts which are not adjacent to the boundary. In the latter case, we speak of the reinforcement of the plate by rigid edges.

The complete boundary L of the composite body consists of the external contour of the plate (of course, provided it does not extend to infinity in all directions), of the contours of the openings which are not reinforced, and finally of the interior contours of the inserted rings, provided these are present. The body may be subjected to any action, both inside and on the boundary.

The boundary conditions on the non-reinforced edge of the plate will clearly be the usual conditions which correspond to the forces acting on it which are given, or to the character with which it is reinforced. The conditions on the separation line of the media will be different, depending on the manner in which the adjacent parts are connected.

For example, in the case when all "holes" in the plate occupy a finite multiply connected domain S with the boundary $L = L_1 + L_2 + \ldots + L_{m+1}$, and are filled with solid discs from different materials welded to the plate along the circumference of the "holes," and the stressed state is caused by the external forces applied only on the exterior contour of the plate, will have the following problem in the theory of analytic functions:

$$\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)} = f(t) \quad \text{on} \quad L_{m+1}. \tag{6.3}$$

$$\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)} = \varphi_k(t) + t\overline{\varphi_k'(t)} + \overline{\psi_k(t)} \quad \text{on} \quad L_k, \tag{6.4}$$

$$\frac{\varkappa}{\mu}\varphi(t) - \frac{1}{\mu}[t\overline{\varphi'(t)} + \overline{\psi(t)}] =$$

$$= \frac{\varkappa_k}{\mu_k}\varphi_k(t) - \frac{1}{\mu_k}[t\overline{\varphi_k'(t)} + \overline{\psi_k(t)}] \quad \text{on} \quad L_k \qquad (k = 1, 2, \ldots, m). \tag{6.5}$$

where φ and ψ are holomorphic in the domain S, and φ_k and ψ_k are holomorphic in the finite region $S_{\bar{k}}$ bounded by the contour $L_k (k = 1, 2, \ldots, m)$. The meaning of boundary condition (6.3) is clear from the preceding discussion. The equations (6.4) and (6.5) express the obvious continuity conditions for the components of the displacement and stress vectors when the separation line of the media is passed. The subscript k is associated with the elastic elements of the material of the ring occupying the region $S_{\bar{k}}$.

One of the early studies dealing with nonhomogeneous elastic bodies based on complex variable methods was the study of S. G. Mikhlin (1935), which investigated, with the aid of the Schwarz kernel that was mentioned above in Section 5.3.4, the general problem of a piecewise-homogeneous medium using the method of integral equations. Certain special cases were studied in an effective way in another study by the same author (1934).

Subsequently, studies of inhomogeneous elasticity problems developed rapidly. Considerable success in this field was obtained in the Ukraine, where the corresponding problems have been studied by many authors for a long time. The results of these studies are presented in the monographs of G. N. Savin (1951), D. V. Vaynberg (1952), M. P. Sheremet'ev (1960) and G. N. Savin and N. P. Fleishman (1964). Below we will briefly touch on certain fundamental results.

6.2.2. We will start with a relatively simple case, when the basic plate and the elastic rings inserted in the "holes" are made from the same material. In this case, we must assume that the contours of the ring in the unstressed state are somewhat different from the contours of the corresponding "holes." From an applied standpoint, the case when the rings are pressed or inserted into the "hole" with a given elastic load is of interest.

The boundary conditions for this problem are obtained from (6.3)-(6.5) by adding to the right member of (6.5) the given function which expresses the presence of a jump in the [illegible] displacements and, in addition, it is taken into accou[illegible] the elastic properties of the body are the same everywhere.

A general method for solving this problem was proposed by D. I. Sherman (1940). This method is based on the analytic continuation of the function which is similar to that presented in Section 5.3.5. According to this method the problem under consideration is reduced to the usual plane problem for a complete composite region without any conditions on the separation line. However, the new problem will have a somewhat modified boundary condition (on the exterior contour). In the right member of the equation which describes this condition there will be an additional term expressing a fictitious action on the system as a whole.

In the case when the inclusions have a spherical shape, the above-mentioned correction term can be represented in explicit form. Its form is very simple and cases are often encounted in practice when the jump in the displacement is directed along the normal and its magnitude is constant.

Finally, in the case of circular inclusions, the solution is obtained completely for composite regions which are mapped onto a circle by means of a rational function. A large number of concrete problems were studied in this way. Detailed bibliographical references are available in the survey of D. I. Sherman that was cited above (1962).

6.2.3. When the inclusions have different elastic characteristics, matters are different. The study of a rigid inclusion clearly does not introduce any complications, since, in this case, we will be dealing with the usual plane problem with elastic displacements which are given on the contour (second fundamental problem). The problem of elastic inclusions from different materials is much more complicated.

This problem for one inclusion, for k = 1 in (6.3)-(6.5), was studied by a method which is similar to that outlined in Section 5.3.5 (D. I. Sherman, 1958). To obtain the auxiliary function $\omega(t)$ which this time is introduced on the entire boundary of the plate $L_1 + L_2$, the author derived a Fredholm integral equation which he studied and discussed. In the special case of an eccentric circular ring with inclusions which was discussed to illustrate the method, the integral equation is replaced as before (see Section 5.3.5) by an infinite system of linear algebraic equations which makes it possible to obtain the solution all the way to numerical results.

The case of different circular concentric rings inserted successively one in another as was mentioned above can be easily studied using the method of power series.

This method, combined with the functional equation, makes it possible to study the problem of annular reinforcements in a slightly more general case, for example, when the infinite simply connected domain occupied by the adjacent bodies is mapped onto the interior of a circle by means of a rational function, and the reinforcing ring becomes, in the process, a concentric circular ring. Under this assumption, the mapping case (6.2) was studied by M. P. Sheremet'ev (1949), who obtained a complete solution and numerical results for the reinforcement of the "hole" in the shape of a confocal elliptical ring. In the monograph of G. N. Savin that was mentioned (1951), the computational results are given also for two forms of the elastic reinforcement obtained by the mapping (6.2), and the stresses on the reinforced contour of the "holes" are compared with the same stresses in the two limiting cases, when the reinforced ring is absolutely elastic (empty) or when it is absolutely rigid.

I. G. Aramanovich (1955), who developed further the method of D. I. Sherman (see Section 5.3.5), constructed an effective solution of the problem of streses in a halfplane with a circular "hole" reinforced by an elastic ring made from a different material. Here, the medium can be loaded in various ways, for example, through expansion, normal pressure on the interior contour of the ring that was welded in, a concentrated load on the rectilinear boundary, etc. The solution is the same as

before (reduction to an infinite system of equations). It was established that the system of equations obtained is quasiregular when the "hole" is arbitrarily close to the boundary of the halfplane.

The method of linear conjugate functions was applied to the solution of problems of the type considered above. As an example, we point out the study of I. A. Prusov (1957) who studied the problem of the reinforcement of a "hole" in an infinite plate by a ring with a variable cross section bounded outside by a circle and inside by an ellipse.

6.2.4. Until now we assumed that the stressed state of the elastic ring reinforcing the edge of the "hole" in the plane is described, like the stressed state of the plate itself, by equations of plane elasticity theory or the equations for the bending of thin plates. If the reinforcing ring is sufficiently thin, or has a shaped profile, it should be considered as a circular rod, whose deformation is described by elementary equations of the theory of the strength of materials.

In this formulation, the problem of the reinforced edges was considered for the first time by M. P. Sheremet'ev (see, for example, his book, 1960). The reinforcing ring with a constant cross section was taken as a thin bar with expansion and flexual rigidity in the case of a plane stressed and flexual rigidity and torsion in the bending of thin plates.

For definiteness, we will consider infinite plates with one reinforced "hole."

The boundary conditions on the contour of the reinforced "hole" will be obtained, as before, by requiring that the corresponding forces and displacements from both sides be equal. In the previous case, these conditions were represented in the form of the equalities (6.4) and (6.5). In this case, in the right members of the above-mentioned equations, instead of the boundary values of the functions φ_k and ψ_k (it is no longer necessary to study these functions), there will be other unknowns, namely the external forces X_n^0, Y_n^0, acting on the ring from the side of the plate and the displacements u_0, v_0 of the axis of the ring.

Now, starting with the well-known equations in the theory of small deformations of curvilinear rods, expressing the displacements u_0 and v_0 in terms of the external load X_n^0, Y_n^0 and substituting the corresponding values in the above-mentioned conjugate boundary condition, to determine the functions φ and ψ which are holomorphic in the region of the plate, we obtain two complex conditions which have in the right members

the independent forces X_n^0 and Y_n^0. For the problem of the bending of a plate with a reinforcement of the type that was mentioned, in general, the unknown functions in the right-hand member can be eliminated, and we will only have one boundary condition which will be somewhat more complex than the usual condition in the fundamental plane problem.

Finally, it is possible to study effectively problems when the "holes" have special shapes. The case of a circular "hole" was analyzed in detail using the method of power series (M. P. Sheremet'ev, 1960). For noncircular "holes" the problem is more complex and the effective solution requires the method of successive approximations.

A further generalization of this approach was given by G. N. Savin and N. P. Fleishman (1961). Assuming that the reinforcing rod was very thin (i.e., assuming that the cross section of the rod was very narrow), they relaxed somewhat the boundary condition on the contour of the layer and formulated the general problem of annular reinforcements with relaxed boundary conditions in terms of complex variables. When these conditions were derived, the assumption was made that the rod in the case of a plane stressed state does not resist the bending and does not have torsional rigidity during the bending of the cross section.

The problem in the theory of analytic functions that was obtained has, like the fundamental plane problems, a solution in closed form when the region of the plate is mapped conformally onto a circle by means of a rational function. This is illustrated on the example of an elliptic "hole" in an infinite plate.

G. N. Savin and N. P. Fleishman (1964) and also M. P. Sheremet'ev (1960) considered the strengthening of a plate during its cross sectional bending by thin rings made from a different material (rigid ribs) which lie inside the plate. In the simplest case of a single rib, we have the following picture. A thin curvilinear ring (more precisely a closed elastic line) is welded to the plate in its interior part. The region occupied by the middle surface of the plate is broken up by the axial line of the ring into two connected parts (the internal and external parts relative to this axial line). In each of these regions a pair of holomorphic functions of a complex variable must be determined in accordance with certain conditions on the contour of the plate and also on the line of the ring. The conjugate conditions on the line must be set up taking into account the joint work of the plate and the reinforced ring (there are three such conditions). In the final analysis to determine the four holomorphic functions, there are four complex conditions of the type (6.3)-(6.5), which include, in addition to the given

magnitudes two complex functions of the arc of the axis of the ring that were not given before. The problem of the reinforcement of the plate by rigid ribs was studied in this way in a number of cases. For example, for a circular plate with an arbitrary number of curvilinear ribs, a solvable Fredholm integral equation was set up.

Certain special problems (for example, a circular plate with a concentric rigid rib, an elliptic plate with a central circular rib) are solved effectively using the method of series.

The monographs of G. N. Savin (1951), D. V. Vainberg (1952), M. P. Sheremet'ev (1960) and G. N. Savin and N. P. Fleishman (1964) that were mentioned above discuss also certain other problems in the plane stressed state and the bending of plates both in the isotropic and anisotropic case. For example, problems connected with the effect of the anisotropic material on the stress concentration near elliptical "holes," problems of the rational selection of the parameters of the reinforcing elements, and the effect of concentrated loads on the contour in a multilayer disc have been studied in greatest detail.

6.3. Mixed and Contact Problems

Mixed and contact problems include the most difficult problems in the theory of elasticity. When these are studied using complex variable methods, boundary value problems with discontinuous coefficients are obtained and it becomes necessary to study the behavior of the solutions in the neighborhoods of the discontinuities.

It was already mentioned above (Section 5.3.4) that D. I. Sherman (1940) constructed a singular integral equation with discontinuous coefficients for the fundamental mixed plane problem. This equation can be used to solve the problem of the bending of a thin isotropic plate under a normal load, when a part of the edge is fixed and a part is free.

A. I. Kalandiya (1952) constructed a system of singular integral equations for solving the general problem of the bending of the plate when a part of the edge is fixed, another supported, and the remaining part free. In a number of studies (see, for example, A. I. Kalandiya, 1961; D. I. Sherman, 1955) a numerical solution is given for mixed problems for the bending of plates for special regions.

One of the most effective methods for solving mixed problems in plane elasticity theory is the method of linear conjugations of functions. The solution of mixed problems using this method was discussed above (Section 5.3.7).

Problems in the indentation of rigid dies in an elastic half-plane lead to conjugate boundary value problems which are analogous to the conjugate problem constructed above (Section 5.3.7) for the fundamental mixed problem. The problem of the contact of two elastic bodies (the generalized plane Herz problem), whose shapes are nearly a halfplane, when the contact sector is small, also leads to the linear conjugate problem. The solution of these problems using the method of linear conjugation of functions is presented in the monograph of N. I. Muskhelishvili.

L. A. Galin (1953) obtained a solution for a number of contact problems by applying methods of the theory of functions of a complex variable. I. Ya. Shtayerman (1949) studied contact problems using the method of integral equations.

In the studies of V. M. Abramov (1937), N. I. Glagolev (1942, 1943), V. I. Mossakovskiy and P. A. Zagubizhenko (1954), I. G. Aramanovich (1955), V. V. Panasyuk (1953, 1954), A. I. Kalandiya (1957, 1958), M. P. Sheremet'ev (1952, 1961) a number of contact problems are investigated using different methods.

6.4. Plane Static Problem of an Anisotropic Body in the Theory of Elasticity

The methods of the theory of functions of a complex variable can be applied successfully to the plane problem of an anisotropic body as shown for the first time by S. G. Lekhnitskiy (the first studies of S. G. Lekhnitskiy along these lines were published in the 30's, see, for example, the monograph: S. G. Lekhnitskiy, 1947, 2nd ed., 1957).

Suppose that a homogeneous anisotropic body has at each point an elastic symmetry plane which is parallel to the given plane which we will take as the Oxy plane. When the body is subject to a plane deformation which is parallel to the Oxy plane, the stress function (the Eyre function) satisfies the general biharmonic equation (the case when body forces are absent)

$$a_0\frac{\partial^4 U}{\partial x^4}-a_1\frac{\partial^4 U}{\partial x^3\partial y}+a_2\frac{\partial^4 U}{\partial x^2\partial y^2}-a_3\frac{\partial^4 U}{\partial x\partial y^3}+a_4\frac{\partial^4 U}{\partial y^4}=0, \tag{6.6}$$

where $a_0, \ldots, a_4$ are real constants which depend on the elastic properties of the body under consideration (an analogous equation is also valid for the generalized plane stressed state of the plane).

Also, in this case, it is possible to construct a general solution with the aid of two analytic functions of a complex variable. This representation depends on the roots of the characteristic equation corresponding to equation (6.6):

$$a_0 + a_1 s + a_2 s^2 + a_3 s^3 + a_4 s^4 = 0. \tag{6.7}$$

S. G. Lekhnitskiy has shown that this equation has no real roots. In the case of an isotropic body, equation (6.7) reduces to the equation $1 + 2s^2 + s^4 = 0$ and, consequently, has the double roots i and $-i$. When equation (6.7) has the double roots $s = \alpha + i\beta$, $\bar{s} = \alpha - i\beta$, the general real solution of equation (6.6) is represented in the form

$$U(x, y) = \bar{z}\varphi(z) + z\overline{\varphi(z)} + \chi(z) + \overline{\chi(z)}, \tag{6.8}$$

as in the case of an isotropic body, but this time the complex variable z has the form $z = x + sy = x + \alpha y + i\beta y$ $((x,y) \in S)$, where S denotes the region occupied by the body.

Making the affine transformation

$$x' = x + \alpha y, \quad y' = \beta y, \tag{6.9}$$

we obtain the complex variable $z' = x' + iy'$, which varies over the region S', obtained from the region S by the affine transformation (6.9).

Formula (6.8) and the expression for the stress and displacement components which follows from it show that this case (i.e., the case of multiple roots of equation (6.7)) is almost completely analogous to the case of an isotropic body so that it is usually not discussed.

In the case when equation (6.7) does not have multiple roots, i.e., it has four different pairwise conjugate roots

$$s_1 = \alpha_1 + i\beta_1, \quad \bar{s}_1 = \alpha_1 - i\beta_1, \quad s_2 = \alpha_2 + i\beta_2, \quad \bar{s}_2 = \alpha_2 - i\beta_2,$$

the general real solution of equation (6.6) is written in the form

$$U(x, y) = F_1(z_1) + \overline{F_1(z_1)} + F_2(z_2) + \overline{F_2(z_2)} \tag{6.10}$$

using two analytic functions of the variables

$$z_1 = x + s_1 y = x + \alpha_1 y + i\beta_1 y, \quad z_2 = x + s_2 y = x + \alpha_2 y + i\beta_2 y,$$

which vary, respectively, over the regions S_1 and S_2 obtained from the region S by the corresponding affine transformations.

In the case under consideration, in contrast to the case of an isotropic body, we are dealing with analytic functions of two different complex variables z_1 and z_2, which vary over two different regions (it is easily seen that the variables z_1 and z_2 are related to one another by an affine non-analytic transformation). Generally, this fact complicates the solution of the boundary value problems (the class of boundary value problems that are solved effectively in the case of an anisotropic body is much smaller than in the case of an anisotropic body). However, also in the case of an anisotropic body, it is possible to obtain a solution for the boundary value problems with the aid of methods from the theory of functions of a complex variable. A number of important results along these lines were obtained by S. G. Lekhnitskiy, S. G. Mikhlin, G. N. Savin, D. I. Sherman and others.

The following complex representation of the stresses and displacements follows from the general representation of the stress function (6.10):

$$\left.\begin{aligned} X_x &= 2\,\mathrm{Re}\,[s_1^2\Phi_1'(z_1) + s_2^2\Phi_2'(z_2)], \\ Y_y &= 2\,\mathrm{Re}\,[\Phi_1'(z_1) + \Phi_2'(z_2)], \\ X_y &= -2\,\mathrm{Re}\,[s_1\Phi_1'(z_1) + s_2\Phi_2'(z_2)]; \end{aligned}\right\} \tag{6.11}$$

$$\left.\begin{aligned} u &= 2\,\mathrm{Re}\,[p_1\Phi_1(z_1) + p_2\Phi_2(z_2)] - \omega y + u_0, \\ v &= 2\,\mathrm{Re}\,[q_1\Phi_1(z_1) + q_2\Phi_2(z_2)] + \omega x + v_0. \end{aligned}\right\} \tag{6.12}$$

Here $\Phi_1(z_1) = dF_1/dz_1$, $\Phi_2(z_2) = dF_2/dz_2$, p_1, p_2, q_1, q_2 are constants which are defined and expressed in terms of the elastic constants of the body, ω, u_0, v_0 are arbitrary (real) constants, corresponding to the rigid displacement of the body.

If the domain S occupied by the body is simply connected, the analytic functions in the general complex expressions are singlevalued and, in the case of a multiply connected domain, they are generally multiple valued analytic functions. For example, if the domain S is bounded by several contours, the functions $\Phi_1(z_1)$ and $\Phi_2(z_2)$ have the form

$$\left.\begin{aligned} \Phi_1(z_1) &= \Phi_1^*(z_1) + \sum_{k=1}^{n} A_k \ln(z_1 - z_{1k}), \\ \Phi_2(z_2) &= \Phi_2^*(z_2) + \sum_{k=1}^{n} B_k \ln(z_2 - z_{2k}). \end{aligned}\right| \tag{6.13}$$

PAGE 69 OF RUSSIAN TEXT MISSING

The theory of brittle fracture dates back to the early studies of A. A. Griffith (1920), which were continued by G. R. Irwin (1948) and later) and E. O. Orovan (1950 and later). As a result of these studies, to characterize the brittle fracture strength of the material, a new constant was introduced which made it possible to study the problem of brittle cracks in the classical formulation of elasticity theory.[1]

Problems of the kinetics of the growth of cracks were considered by G. I. Barenblatt, V. M. Yentov, R. L. Salganik (1966), 1967) and also by G. I. Barenblatt, R. L. Salganik, G. P. Cherepanov (1962). L. N. Kachanov (1961) attempted to estimate the durability of a body with a crack in an elasto-viscous body.

G. I. Barenblatt and G. P. Cherepanov (1960) considered the problem of the loosening of an orthotropic elastic body by a thin rigid wedge which was displaced with a constant speed. In the problem of the loosening of an infinite body by a wedge of finite length, I. A. Markuson (1961) obtained a relationship for the length of the crack as a function of the length of the wedge. The spreading of displacement cracks was considered by G. I. Barenblatt and G. P. Cherepanov (1961). The problem of the stable development of a crack strengthened by rigid ribs was considered in the study of Ye. A. Morozova and V. Z. Parton (1961). The stable development of a biperiodic system of cracks was studied by V. Z. Parton (1965). G. P. Cherepanov (1966) studied the development of cracks in compressed bodies.

A model of a crack in which the adhesive forces on sectors which are commensurate with the length of the crack are also taken into account was studied, using the condition for the smooth coupling of the edges of the cracks and the finiteness of the stresses on them, by M. Ya. Leonov and V. V. Panasyuk (1959).[2] The solution for a large number of plane problems dealing

---

1. For greater detail about the mechanics of fracture, see pp 427-574 (editor).

2. The monograph of V. V. Panasyuk, "Limiting Equilibrium of Brittle Bodies with Cracks," (1968) is devoted to the static theory of cracks. The monograph also contains a detailed bibliography.

with the limiting equilibrium of a body with cracks in various positions and of various shapes when the body with the cracks was subjected to various loads were obtained. (V. V. Panasyuk and B. L. Lozovoy, 1962, V. V. Panasyuk and L. T. Berezhnitskiy, 1964-1966). This class of problems includes plane problems dealing with the stressed state in the neighborhood of the corner points of the contour of a "hole" (V. V. Panasyuk and Ye. V. Buyna, 1966), in particular, a circle with radial cracks (V. V. Panasyuk, 1965).

The study of G. P. Cherepanov (1963) investigated the initial development of a crack from the corner points in an infinite rectangular cutout pressed at the bottom by a rigid die.

Problems dealing with the stressed state near the edge of a crack extending to the edge of the plate or close to it were studied by V. V. Panasyuk (1960), G. I. Barenblatt and G. P. Cherepanov (1960, 1962). The problem of the limiting values of an external load (bending moment, uniformly distributed pressure) on a strip (beam) with a rectilinear crack perpendicular to the axis of the strip were considered in the studies of B. L. Lozovoy and V. V. Panasyuk (1961-1963). Three-dimensional limiting equilibrium problems of a body with a plane circular crack were studied by M. Ya. Leonov and V. V. Panasyuk (1961). The more complex case of an elliptical crack was studied by V. V. Panasyuk (1962), M. Ya. Leonov and K. N. Rusinko (1963, 1964).

NONLINEAR ELASTICITY THEORY

V. V. Novozhilov, L. A. Tolokonnikov, K. F. Chernykh

Two assumptions are used in the construction of classical linear elasticity theory: that the elongations, shifts and angles of rotation are small and of the same order of magnitude and that it is possible to apply the generalized Hooke law. When one of these assumptions is discarded or replaced by a less stringent assumption, various variants of nonlinear elasticity theory are obtained.

A number of specific problems and difficulties arise in the transition to nonlinear theory:

1) the selection of a coordinate system which defines the positions of the points of the body;

2) the adoption of deformation characteristics of one kind or another and the corresponding generalized stresses;

3) determination, taking into account tensor-invariant and thermodynamic concepts, the type of relation between the stresses and strains, the selection of a convenient set of invariants, expressing this relation concretely for various groups of materials, and carrying out the simplest experiments needed for this purpose;

4) classification of the problems in nonlinear theory and finding approaches to the simplification of nonlinear relations in various special cases;

5) formulation of variational and related principles;

6) formulation of problems in equilibrium stability elastic bodies.

Section 1 of the survey considers studies of a general character which clarify the first five problems that were enumerated. Section 2 analyzes studies dealing with secondary effects accompanying the torsion and bending of prismatic and cylindrical bodies. Section 3 is devoted to studies of

of plane problems. Section 4 considers studies in the equilibrium stability of elastic bodies in which the initial relations are the relations of nonlinear elasticity theory.

The development of nonlinear elasticity theory dates back to the 19th century to the work of O. Cauchy, G. Green, G. Kirchhoff, I. Finger and later E. Treftz, A. Signorini, F. D. Murnagan, M. A. Bio and many contemporary foreign scientists, among whom we first mention R. S. Rivlin, R. Hill and A. E. Green. The results obtained by them overlap in many respects with the results obtained by Soviet scientists.

Since the purpose of this article is to survey the achievements of Soviet scinetists, it cannot give a complete picture of the general state of nonlinear elasticity theory. This should be kept in mind when the article is read.

§1. General Problems.

Although the first publications on nonlinear elasticity theory in the USSR date back to the 30's (N. V. Zvolinskiy, 1939, N. V. Zvolinskiy and P. M. Riz, 1938, 1939, D. Yu. Panov, 1939, P. M. Riz, 1938, 1939), serious attention has only been given to nonlinear problems in the last two decades. This was stimulated to a considerable extent by the appearance of publications dealing with general theoretical problems (K. Z. Galimov, 1946, 1948, 1949, I. I. Gol'denblat, 1950, D. I. Kutilin, 1947, V. V. Novozhliov, 1948) and other publications that were published later. The studies that were mentioned dealt with a wide class of problems and determined the direction of research in nonlinear elasticity theory in the country.

Two types of coordinates are used in the mechanics of continuous media; three-dimensional Eulerian coordinates and material Lagrangean coordinates ("frozen in the body") (K. Z. Galimov, 1946-1955, I. I. Gol'denblat, 1950, 1955, V. V. Krylov, 1956, D. I. Kutilin, 1947, V. V. Novozhilov, 1948). Material coordinates (V. V. Novozhilov, 1958) in which the boundary conditions and the deformation hypotheses are formulated more simply (for example, the hypothesis of the principal normal in the theory of plates and shells and the hypothesis of plane sections in the theory of the flexure of rods) are more convenient in nonlinear theory. When we consider not the deformation process itself (which is done in elasticity theory) but only the initial and final position of the body, the introduction of three-dimensional coordinates is unnecessary (L. I. Sedov, 1962). The magnitudes which characterize the deformation and equilibirum of the body can be referred either

to the undeformed or deformed material coordinate basis. The monograph of L. I. Sedov (1962) discusses in detail the selection of the coordinate vector bases and the relations between them.

The principal characteristics of the deformations used are half the differences in the components of the fundamental metric tensor in the deformed and nondeformed states (K. Z. Galimov, 1946, 1949, 1955; I. I. Gol'denblat, 1950, 1955; V. V. Krylov, 1956; D. I. Kutilin, 1947; V. V. Novozhilov, 1948, 1958). Other characteristics are also used to describe large deformations among which we shall mention, for example, the following: logarithmic (or true) deformations, components of the tensor which coincide on the principal deformation axes with the principal relative elongations and components of a tensor whose contravariant components are half the differences of the corresponding components of the metric tensors in the deformed and nondeformed states. In the study of different problems, preference is given to different sets of characteristics. To treat the results properly, it is important that the generalized characteristics of the deformation that were adopted correspond to the generalized stresses (in the expression for the elementary work) (V. V. Novozhilov, 1951). In the monograph of L. I. Sedov (1962) which summarizes the results of earlier studies (L. I. Sedov, 1960, V. D. Bondar', 1960, 1961, M. E. Eglit, 1961) when the deformation of an element of the body is discussed, the theory of tensor functions is widely used. From this standpoint, any analytic function of the deformation tensor can be used as a characteristic of the deformation. A skew symmetric tensor corresponding to the vector of rotation of the principal axes of the deformation is used in the same study for a deformation of general shape.

The relation between the mean rotation of an element of the deformed body and the rotor of the displacement vector was established earlier (V. V. Novozhilov, 1948).

A great deal of attention was given to the problem of selecting an optimal system of invariants, the calculation of the mechanical orientation of the invariants and the relation between them (K. Z. Galimov, 1946-1955; I. I. Gol'denblat, 1950, 1955; V. V. Novozhilov, 1948, 1958). Thus, it was noted (V. V. Novozhilov, 1952) that with an accuracy up to a constant factor, the intensity of the tangential stresses coincides with the mean value of the tangential stress at the point of the body under consideration. Subsequently, the principal values of the deformation tensors and stresses were represented trigonometrically (V. V. Novozhilov, 1951). The fundamental invariants are the linear invariants, the intensity of the deviator and the inclusion angle of the tensor (deviator).

The connection between the strain and stress tensors is characterized by the generalized volumetric expansion modulus, the generalized displacement modulus and the similarity phase of the deviators (which is equal to the difference of the "inclusion angles" of the tensors under consideration. The differential relations between the generalized moduli that were introduced were determined from the condition for the existence of the potentials for the stresses and strains.

Similar relations were obtained with the aid of Mohr circles (A. K. Sinitskiy, 1958). The trigonometric representation of the principal values of the tensor made it possible (V. V. Novozhilov, 1951) to obtain concretely the coefficients proposed by V. Prager for the relation between two coaxial tensors. The further development of the geometric aspects of the problem of the relation between symmetric tensors of rank two is given in the studies of V. V. Novozhilov (1963), L. I. Sedov (1962), K. F. Chernykh (1967).

An extensive study of the problem of the relation between invariants using the results from the theory of algebraic invariants and group theory was carried out by I. I. Gol'denblat (1950, 1955). The possibility of introducing invariants which made it possible to consider separately the change in the volume of the element and its shape was clarified (L. A. Tolokonnikov, 1956). Relations generalizing the similarity law for the stress and strain deviators were proposed in the same article. L. A. Tolokonnikov (1957) developed on this basis a variant of quadratic theory (with four constants), which was based on the following assumptions: the pressure from all directions depends only on the relative change in the volume, the intensity of the tangential stresses only on the intensity of the shearing strain, "the inclusion angles" of the tensors of the true stresses and the logarithmic stresses are equal to one another.

It was shown (D. D. Ivlev, 1961) that for an isotropic body which resists expansion and compression in a different manner, the set of the simplest experiments does not fully determine the potential of the deformation.

It was established (V. D. Bondar', 1963) that any equilibrium state of the body with stresses and strains which are different from zero can be taken as the initial state provided the body forces are determined in a special way. Thermodynamic concepts have been used relatively frequently in the construction of nonlinear elasticity theory (I. I. Gol'denblat, 1950, 1955; D. I. Kutilin, 1947; V. V. Novozhilov, 1963). The monograph of L. I. Sedov (1962) discusses in detail the problem of the application of the thermodynamics of reversible processes for obtaining a closed system of equations in nonlinear

elasticity theory. Here, all four thermodynamic potentials are used. For their arguments (along with the usual components of the strain and stress tensors, the temperature and entropy), parameters which determine the physical-chemical properties of the materials of the body are also introduced. The latter can also be tensor quantities. The case of the presence of internal reactions has been studied (for example, the incompressibility condition for the material). The case of an isotropic body has also been discussed in detail.

In the monograph of I. I. Gol'denblat (1955), which summarizes his earlier studies (1949, 1950), the case corresponding to the adoption for the arguments of the thermodynamic potentials of the invariants of the strain and stress tensors, the elastic moduli (deformation coefficients) is analyzed in detail. L. I. Sedov (1965) introduced into the discussion stress moments.

The relations that were derived were obtained in concrete form when applied to rubber in the studies of G. M. Bartenev and T. N. Khazanovich (1960), V. L. Biderman (1953, 1957, 1958, 1962), V. L. Biderman and B. L. Bukhin (1960, 1961). A general approach was proposed for the calculation of rubber parts for large strains and displacements (V. L. Biderman, 1958). The forms of the potential of an incompressible material were studied. The possibility of satisfying approximately the incompressibility condition was clarified (V. L. Biderman and N. A. Sukhova, 1963). The four constants in the polynomial for the elastic potential were determined from the experiments. The solutions of certain problems for rubber shock absorbers and seals were obtained (V. L. Biderman, 1962). G. M. Bartenev and T. N. Khazanovich (1960) proposed a form of the stress potential with three constants on the basis of an analysis of the behavior of rubber during a one-dimensional deformation.

In the study of V. V. Lokhin (1963) it was pointed out that it was convenient to classify anisotropic media by their point symmetry groups. It was shown that any tensor which is invariant with respect to a given group of points can be represented as a linear combination of tensors obtained with the aid of tensor operations from a minimum set of tensors. L. I. Sedov and V. V. Lokhin (1963) found such systems of tensors for seven types of structures and all 32 classes of crystals. The general form of the formulas for tensors of arbitrary rank was determined in the form of nonlinear tensor functions of scalar and tensor functions of arbitrary rank, (see also V. V. Lokhin and L. I. Sedov, 1963). It was shown that to construct the tensor functions, a necessary and sufficient condition was the knowledge of the complete system of functionally independent consistent tensor invariants

and tensor arguments under consideration. The structure of the tensor functions describing the state of the structures and certain classes of crystals was clarified (V. V. Lokhin, 1963).

The general theorems of nonlinear elasticity theory are discussed in the studies of L. N. Vorob'ev (1956), N. A. Kil'chevskiy (1963, 1964), D. I. Kutilin (1947), V. V. Novozhilov (1958). The extension of the early variational principles (of the type proposed in linear theory by E. Reissner) was formulated by K. Z. Galimov (1952) and I. G. Teregulov (1962). The proposed variational principles use as the independent functional elements which are varied, the displacements, stresses and strains which are unrelated inside and on the boundary of the body. The variational principles give an alternative approach to the solution of nonlinear problems through the use of direct methods of mathematical physics. When relations are imposed on the elements that are varied, the principles discussed become the classical original displacements and possible changes in the stressed state (The Castigliano principle).

The studies of N. V. Zvolinskiy, D. M. Panov and P. M. Riz (1938-1943) determined the general trend of the applied work in nonlinear elasticity in the country (§2, 3). The latter is characterized by the use of the so-called quadratic theory (a variant of nonlinear theory), which is obtained by retaining in all relations the products and squares of the unknown quantities together with the linear terms.

V. V. Novozhilov (1948, 1958) made a number of critical remarks about the quadratic theory. Briefly, they reduce to the following. The possibility of a complete or partial linearization of the geometric and static (dynamic) relations in nonlinear elasticity theory is based on purely geometric factors: the magnitude of the elongations, shifts, and rotation angles, both compared to one, and to one another. Therefore, the undifferentiated approach used in quadratic theory (as mentioned above) to simplify the static-geometric relations has a formal character. Further, to simplify the relations relating the stresses and strains, the smallness of the strain components compared to one is not sufficient. They must be compared to the physical constants of the material (the proportionality limits), quantities, which, as a rule, are very small compared to one. In addition, quadratic theory is characterized by the retention in the stress potential of cubic terms along with the quadratic terms (the five constant Feucht-Murnagan theory). For the majority of real materials, the deviation from Hooke's law is due to the even powers of the strain components.

An alternative approach to the simplification of the nonlinear relations which is free of the disadvantages that were pointed out has been discussed in detail by V. V. Novozhilov (1948, 1958). In particular, one consequence of this approach is the presently used breakdown of the problems into four groups: 1) problems which are linear physically and geometrically, 2) problems which are nonlinear physically but linear geometrically, 3) problems which are linear physically but nonlinear geometrically, 4) problems which are nonlinear physically and geometrically. The monograph of V. V. Novozhilov (1948) analyzes from the standpoint of the general relations in nonlinear elasticity theory the geometric assumptions which are widely used in the study of the deformation of rods, plates and shells.

It is well known that in the important practical case of a simple load (all stresses in the body vary proportionally with the same parameter) the relations in plasticity theory degenerate into the formulas of nonlinear elasticity theory. L. I. Sedov (1959) has shown that for large deformations, the simple load on the body as a whole can only occur for deformations of a very special form. The study of V. D. Bondar (1960) is devoted to the clarification of the form of the deformations which correspond to a simple load.

V. M. Babich (1954) considered, using kinematic and dynamic consistency conditions, a system of equations of motion of an elastic medium for which the potential for the change of shape is an arbitrary function of the intensity of the strain. The propagation velocities of the waves that depend on the direction of the homogeneous field of initial stresses which create the anisotropy were found.

It was shown in the article of I. A. Viktorov (1963) that in a nonlinear elastic medium the principal longitudinal wave leads to the occurrence of secondary longitudinal and transverse waves and the same applies to the principal transverse wave.

§2. Secondary Effects in Problems of Bending and Torsion of Prismatic and Cylindrical Bodies

The effects predicted by quadratic theory used together with the results of linear theory are called secondary effects. The possibility and usefulness of taking into account the secondary effects was pointed out in 1937 by F. D. Murnagan (Amer. J. Math, Vol. 59, No. 2, 235-260 (1937)). An original approach to a class of problems that occur in the transition to quadratic theory was presented in the studies of N. V. Zvolinskiy and P. M. Riz (1939) and P. M. Riz (1947). As an application of the theory that was developed, the effects related to the axial

deformation of prismatic bodies under the action of torques was considered. The extent to which expansion increases and compression reduces the torsional stiffness of rods was determined. The critical values of the compression forces at which the rod has no stiffness in torsion were determined.

P. M. Riz (1938, 1939) solved the problem of the torsion of a circular cylinder retaining second order twist terms. Axial compression and elongation of radial wires was detected. Analogous effects occured during the torsion of an elliptical cylinder (D. Yu. Panov, 1939).

In order to estimate the mutual effect of the pure bending strains in each principal plane, the oblique bending of a rod was investigated (P. M. Riz and A. I. Pozhalostin, 1942). The studies of A. Ya. Gordidze and A. K. Rukhadze (1941, 1943), N. V. Zvolinskiy (1939), R. S. Minasyan (1962, 1963), P. M. Riz (1939), A. K. Rukhadze (1941, 1947), A. K. Rukhadze and A. Ya. Gordidze (1944) clarified the mutual effect of various actions on the rod (homogeneous or composite): axial expansion by surface and body forces, bending by couples, bending by a force and torsion. In particular, it was shown that the mutual effect of loads is considerable for long bodies with a thin profile, such as airplane propellers.

The quadratic theory was further developed by L. A. Tolokonnikov (1956, 1959). Here, the assumption about the similarity of the strain and stress deviators is important as well as the decomposition of the general elasticity moduli in accordance with two parameters (the relative change in the volume and the degree in the change of the shape). The results that were obtained are illustrated on the problem of the torsion of a circular shaft. The study of N. V. Vasilenko (1965) analyzes the quadratic relations in thermoelasticity.

§3. Plane Problems

Just as in the general case, it is possible to isolate three trends in the study of plane problems in nonlinear elasticity theory.

The first trend studies problems which are nonlinear both physically and geometrically, which is characteristic of the further development of the theory formulated in the work of G. E. Adkins, A. E. Green, R. T. Shield and G. K. Nicholas. The method of a small parameter which is used as the first approximation for the linear solution of the problem is used on a wide scale here.

This approach made it possible to apply effectively (G. N. Savin, 1964) the method of functions of a complex variable and integral formulas of the Cauchy type that were developed earlier and applied to linear problems. The singularities and conditions for single valued complex potentials were studied and various variants of static and geometric boundary conditions in the initial and deformed states were formulated (G. N. Savin and Yu. I. Koyfman, 1961). Next, a number of problems dealing with concentrated stresses around a circular and elliptical opening (free and supported) during a homogeneous stressed state at infinity were considered (Yu. I. Koyfman, 1961-1964). Similar problems for plates with a rigid core were also considered here.

An original approach to the plane incompressible state was proposed by L. A. Tolokonnikov (1958), V. G. Gromov (1959) obtained an exact solution of the axially symmetric problem which made it possible to estimate the accuracy of approximate solution methods. The application of the method of functions of a complex variable was developed further (V. G. Gromov and L. A. Tolokonnikov, 1963). The constraint related to the incompressibility condition was removed in the study of I. G. Teregulov (1962).

The study of V. V. Krylov (1946) belongs to the second trend, (problems which are nonlinear geometrically and linear physically). A thorough analysis of the plane state was made in this publication which was one of the first to appear in the country which dealt with the nonlinear plane problem. The possibility of applying functions of a complex variable was demonstrated.

The third direction (problems, which are nonlinear physically and linear geometrically) studies small deviations from the law governing the change of the shape (according to Kauderer). G. N. Savin (1965) obtained the solution equation in arbitrary isometric coordinates determined from a mapping function of general form. A number of concrete problems dealing with the concentrated stresses around "holes" with different stress fields at infinity have been considered. The effectiveness of an elastic support of the contours has been studied (I. A. Tsurpal, 1962-1965). The solution of a number of problems in the third direction is based on the relations of quadratic elasticity theory (I. N. Slezinger and S. Ya. Barskaya, 1960, 1965). An analysis of the solutions that were obtained shows that taking into account the physical nonlinearity of the material leads to a reduced stress concentration around the holes.

§4. Equilibrium Stability of an Elastic Body

We will only dwell on studies dealing with the equilibrium stability of elastic bodies in which the relations from non-linear elasticity theory are initially used without the assumptions made in the theory of thin-walled structures.

We begin with the study of L. S. Leybenzon (1961) in which the stresses, displacements and strains were clearly broken up for the first time into principal and additional stresses, displacements and deformations formed during the loss of stability. The relations that were obtained for the additional state made it possible to determine the critical values of the differences in the pressures acting on the external and internal surface of a hollow sphere and a long pipe. In subsequent studies L. S. Leybenzon gives a thorough analysis of approximate solution methods for elastic equilibrium stability problems.

A survey of the general formulation of problems in the stability of the equilibrium of an elastic body which follows Hooke's law is available in the monograph of V. V. Novozhilov (1948). This monograph clarified (without any preliminary simplification) the conditions under which a new form of equilibrium can occur and formulated the differential equations and boundary conditions for the elastic equilibrium problem. It also analyzed the simplifications which follow from the assumption that the initial state is described by the relations of classical elasticity theory and proposed an energy criterion for the stability.

The study of V. V. Bolotin (1956) is devoted to general stability problems. The fundamental state, described by the relations of linear elasticity theory is represented in terms of Green's tensor and the problem is reduced to a study of a system of linear integral equations (the latter become under the appropriate assumptions the stability equations for thin-walled structural elements). The effect of a change in the surface and body forces on the stability and also in deformations preceding loss of stability has also been discussed. The general equations of nonlinear elasticity are used by V. V. Bolotin (1958) in the study of the stability problem "in the small" and "in the large." It is assumed that the elongations and displacements are small and the eigenvalues of the general stability boundary value problems are analyzed "in the small" and the stability relations are formulated "in the large."

A. Yu. Ishlinskiy (1943) applied the equations for the equilibrium stability of an elastic body to the stability problem of a compressed strip. He represents the critical stress in a series in powers of a parameter which vanishes together with the thickness of the plate. The first term in the series gives the value of the critical load according to Euler. During the study the stability of a compressible strip with

different boundary conditions was investigated (L. V. Yershov and D. D. Ivlev, 1961).

In the spirit of the studies of A. Yu. Ishlinskiy (1943, 1954) the problem of the stability of a square plate during uniaxial and triaxial compression was studied taking into account the nonlinearity of the law for the change of the shape (I. D. Legenya, 1961, 1962). Subsequent investigations led to the result that it was necessary to take into account the angles of rotation during the formulation of the equilibrium conditions for an element of the body in the perturbed state (I. D. Legenya, 1963). It became apparent that when this was done using V. V. Novozhilov's formulation (1948), the expressions for the critical pressure on the square plate had terms which differed from the classical terms and did not vanish when the thickness of the plate was reduced.

Taking into account the rotation of the incompressible elements of the body (K. N. Semchinov, 1961), the loss of stability of a strip of finite dimensions was studied and the conditions for the bending of the strip during compression were obtained and the critical expanding forces at which a neck is formed on the strip were determined.

The problem of the compression of a circular plate was discussed by L. A. Tolokonnikov (1959) taking into account the strain and displacements in the basic state. It was shown that the critical pressure as a function of the relative length is not monotonic and single-valued. A limiting thickness to radius ratio exists for which the plate no longer looses stability. Using the same method, the critical loads were found for an annular plate, a circular cylindrical shell and a cylindrical panel under the action of transverse pressure (G. B. Kireyeva, 1961, 1966).

The critical value of the compression force for a rod was determined by A. I. Lur'e (1966) from the general relations which he derived.

PLASTICITY THEORY

A. A. Vakulenko, L. M. Kachanov

§1. Introduction

1.1. General Remarks

Sometimes by plasticity is simply meant the ability of the body to undergo a deformation which does not disappear completely

when the causes causing it are removed. In this sense plasticity is a general property of solid bodies. But more often this term has a more narrow meaning which identifies "plasticity" with athermal ("cold") plasticity, i.e., the capability for residual deformations not connected with the thermal mobility of the substance. Externally this manifests itself in a certain kind of independence of the pattern of the process of time.

For simplicity we will consider the case when the model (or an element of volume of the medium) is subjected to a deformation at a temperature which does not change when the external electric and magnetic fields do not change. Let t be time, σ_{ij}, ϵ_{ij} the components of the stress tensor and strain tensor of the element. In accordance with one of the usual initial assumptions, in this case always when $t \leq t_0$, the element is in a state of thermodynamic equilibrium, the $\epsilon_{ij} = \epsilon_{ij}(t)$ $(t \geq t_0)$ are given and the functions $\sigma_{ij} = \sigma_{ij}(t)$ $(t \geq t_0)$ are completely determined.

We say that the behavior of the element is independent of time if for any two processes $\epsilon_{ij}^{(1)}(t)$, $\epsilon_{ij}^{(2)}(t)$ with the same state of the element at $t = t_0$, such that for some $c > 0$ the equalities $\epsilon_{ij}^{(1)}(t) = \epsilon_{ij}^{(2)}(ct + b)$ $(b = (1 - c)t_0,\ t \geq t_0)$ hold for every $t \geq t_0$, we also have $\sigma_{ij}^{(1)}(t) = \sigma_{ij}^{(2)}(ct + b)$. This condition can be generalized to nonisothermal deformation processes and processes with varying electromagnetic fields. For any continuous medium which can undergo residual deformation and at the same time satisfies this condition for the independence of the behavior of time, the name "plastic" in the sense that was mentioned is justified. The characteristic property of the medium from the thermodynamic standpoint is that not every quasistatic process in it is a reversible process.

It must be emphasized that the residual deformation of a real solid cannot be completely athermal. To eliminate creep to a sufficient extent as well as other effects related to the thermal motion of atomic particles, we must bound below the admissible rates of the process, more so the higher the temperature, all other conditions being equal. But for nonmetallic materials this limits the capacity for residual deformations of the materials themselves; during the deformation of a nonmetallic body at rates which ensure the athermal character of the process, the appearance of a residual deformation is usually almost immediately accompanied by fracture. The fracture can only be avoided by applying a sufficiently large hydrostatic pressure (in most cases measured in tens

or even hundreds of thousands of atmospheres). Only metals have considerable athermal plasticity for the usual values of the spherical stress component. Naturally, for this reason the experimental foundations of plasticity theory consist almost exclusively of the data obtained from experiments with metals.

1.2. Short Historical Survey

The beginning of plasticity theory goes back to the 70's of the last century and it is connected with the names A. St.-Venant and M. Levi. St.-Venant was the first man who was able to formulate the equations satisfying the laws of the plastic flow of metals in the language of the mechanics of a continuous medium. This success owed a great deal to the experimental studies of A. Tresk, who made toward the end of the 60's a series of experiments dealing with the pressing and indentation of metals through "holes." The classical study of St.-Venant dealing with equations for the "internal movements which arise in plastic solids beyond the elasticity limit" begins by mentioning these experiments. The study was restricted to the case of plane deformation, but the equations that were derived in it were immediately generalized by M. Levi to the three-dimensional case (the studies of St.-Venant and Levi appeared almost simultaneously in Journal de mathematiques pures et appliquees in 1871. A translation of this article is available in the collection "Theory of Plasticity," Moscow, 1948).

Not much happened during the end of the last century and plasticity problems again started attracting the attention of major scientists at the beginning of our century. In 1909 the studies of A. Haar and T. Karman appeared, which made an attempt to obtain the equations of plasticity theory with the aid of variational principles, and subsequently in 1913 the important study of R. Mises appeared (see the collection of translations "Theory of Plasticity" that was already mentioned). In this study Mises clearly formulated the plasticity condition according to which the transition to the plastic state is determined by the value of the quadratic invariant of the stress deviator (this condition was stated less clearly earlier not in connection with the development of plasticity theory). The main reason why Mises favored this condition was its closeness to the yield condition formulated by Tresk and used by St.Venant (the condition of a maximum tangential stress). This closeness is related to the fact that as a result of the symmetry of the stress tensor

$$\sqrt{3}\, s_* \leqslant \tau_{max} \leqslant 2 s_*,$$

always holds where

$$s_* = \sqrt{{}^2/_3(\tau_1^2 + \tau_2^2 + \tau_3^2)} = \sqrt{{}^1/_2 s_{\alpha\beta}s_{\alpha\beta}}$$

is the intensity of the tangential stresses (τ_1, τ_2, τ_3 are the principal tangential stresses, and s_{ij} are the components of the stress deviator.

A number of important studies appeared in the 20's. Thus H. Hencky and L. Prandtl drew attention to two-dimensional problems in the theory of ideal plasticity, primarily to plane deformation problems. In one study from this period Hencky established the properties of "slippage lines" (the trajectories τ_{max}) in the plane deformation problem of an ideal plastic body (Z. angew, Math. und Mech., (1923), Vol. 3, No. 4, pp. 241-251). In a study published soon afterwards, Prandtl pointed out the ways in which these properties could be applied to the solution of some concrete problems (pressing in of a die, compression of a layer; see the collection "Theory of Plasticity," which contains the translation of Hencky's article). Together with the study of H. Heiringer (1930) which derived the equations for the velocities on the slippage lines, these studies served as an impetus for the extensive development of studies dealing with the plane problem in the theory of ideal plasticity toward the end of the 30's and later (see Section 3 of this survey).

In still another study from the 20's, H. Hencky gave the now well known energy interpretation of the Mises' condition (which is used in many texts on the strength of materials) and using a variational principle analogous to the principle formulated earlier by A. Haar and T. Karman, he obtained the equations for an ideal plastic body as finite relations between the stress and strain tensors. A. Nadai generalized the Hencky equations to the case of an isotropic body with reinforcement. As in Hencky's study the range of applicability of the finite set of equations relating the stress and strain tensors which describe the plasticity are not clearly defined. Clarity with regard to this problem was achieved later after the appearance in the 40's of a number of studies by A. A. Il'yushin (see Section 2.5.).

A considerable step forward in the development of the St.-Venant-Levi theory, in which the medium under consideration is in fact a "rigid-plastic" medium (which can undergo only residual deformations) was made in the 20's. L. Prandtl

was apparently the first man who drew attention to this fact. In one of these studies from the early 20's he gives a generalization of the St.-Venant equations according to which the strain increment $d\varepsilon_{ij}$ at a given point of the medium always consists of an elastic and residual part and the stress tensor is coaxial with the tensor characterizing the residual part, not the entire strain increment. In 1930, E. Reiss generalized in a similar manner the variant of the St.-Venant-Levi theory developed by R. Mises (which differed from the initial variant only by the yield condition, Z. angew, Math. und Mech., Vol. 10, No. 3, 226-274 (1930); see the collection "Theory of Plasticity" that was cited).

The beginning of systemmatic experimental studies connected with problems in plasticity theory also goes back to the 20's. M. Roche and A. Eichinger published the results of their experiments in 1926 and the fundamental study of V. Lode[1] appeared two years later. In both cases, models in the shape of thin-walled tubes were tested and one of the main goals of the experiment was to compare the Tresk and Mises yield conditions for a wider set of stressed states than simple elongation and pure shear. In addition to this, Lode introduced into the discussion a parameter which characterized the "form" of the bivalent symmetric tensor (the ratio of the diameters of the Mohr circles) and he studied in his experiments the relation between μ_σ and μ_ε,, the Lode parameters" of the stress tensor and the strain velocity tensor respectively. In the plane referred to the coordinates μ_σ, μ_ε, the diagram of this relation according to the data from the experiments of Lode, has a characteristic form which was always obtained even in later experiments of this type, which makes it possible to draw important conclusions with regard to the structure of the defining relations.

It must also be noted that the experimental study of the plasticity and strength of metallic monocrystals began in those years. It is known that during the cooling of a liquid metal usually a body with a polycrystalline structure is obtained. The growth of a metallic monocrystal is a difficult matter and, in spite of the long history of metallurgy, the first methods for obtaining monocrystals of typical metals were obtained only in 1918-1920. However, the laws for the plastic deformation on the "crystallographic level" were used almost immediately on a wide scale. S. Elam, M. Polyani, E. Schmidt and other physicists-metallurgists carried out in

1. See the collection "Theory of Plasticity" that was cited. The article of V. Lode includes, in particular, a short survey of previous experimental studies, the experiments of T. Guest, V. Mason, G. Cook and A. Robertson, et al. which were made before World War I but which did not have a great effect on the development of the theory of plasticity.

the 20's hundreds of experiments which studied the elongation and displacement of monocrystalline models beyond the elasticity limits with different orientation of the lattice of the model relative to the principal stress axes. As a result it was established that the plastic deformation of a monocrystal occurrs mainly as a result of the translation ("slippage") of its parts separated by systems of crystallographic planes, and that crystallographic planes and directions in which the points of the lattice are most dense have the smallest resistance to slippage and a number of other simple facts, the most important of which describe the so-called "Schmidt laws" (a survey of these facts is available in the monograph of E. Schmidt and V. Boas, "Plasticity of Crystals," 1935, Russian translation, Moscow-Leningrad, 1938).

The Schmidt laws allow a pure macroscopic formulation. Therefore, when they are used clarity can be introduced in some problems pertaining to the laws of the plastic deformation "of a quasiisotropic" (polycrystalline) body. However, the construction of a sufficiently complete and rigorous theory of the deformation of a polycrystalline model in this way is an extremely difficult problem. For this reason the successes of physical metallurgy did not have a great effect on the rheology of plastic media. The development of the latter followed predominantly the same direction as in 1930, i.e., it was based directly on the experimental data obtained from the usual models.

In the early 30's important experiments were set up by G. Taylor and H. Quinne, R. Schmidt, F. Odquist, and K. Howenemser. The experiments of Taylor and Quinne studied the mutual orientation of the principal axes of the stress tensors and the deformation rates and hardening. The experiments of Schmidt were among the first experiments devoted specially to hardening in the complex stressed state (Ing-Arch. Vol. 3, 215-235 (1932), see the collection "Theory of Plasticity"). Having subjected to an analysis a number of variants of the hardening condition, Schmidt discovered that the most satisfactory variant was the variant according to which the intensity of the tangential stresses is a function of the density of the work of the stresses : $s_* = h(w)$, $dw = \sigma_{\alpha\beta} d\varepsilon_{\alpha\beta}$. (G. Taylor and H. Quinne reached the same conclusion on the basis of their experiments.) It turned out that the pattern of the process on the plane in the coordinates s_*, w changes little in the transition from experiments with "a proportional load to loads with sharp rotations of the principal axes. F. Odquist noticed almost immediately that the condition according to which

$$s_* = g(\lambda), \quad d\lambda = \sqrt{d\varepsilon_{\alpha\beta}\, d\varepsilon_{\alpha\beta}}.$$

was just as unsatisfactory.

In both cases the elastic component of the deformation is not taken into account. When the elastic deformation is ignored the increment $d\varepsilon_{ij}$ must be replaced by its residual part $d\varepsilon^p_{ij} = d\varepsilon_{ij} - d\varepsilon^e_{ij}$. Then for any admissible state

$$s_* \leqslant h(w), \quad dw = \sigma_{\alpha\beta}\, d\varepsilon^p_{\alpha\beta}$$

or, according to Odquist,

$$s_* \leqslant g(\lambda), \quad d\lambda = \sqrt{d\varepsilon^p_{\alpha\beta}\, d\varepsilon^p_{\alpha\beta}}, \tag{1.1}$$

where h and g are monotonic functions, and in both cases $d\varepsilon^p_{ij} \neq 0$ only when the equality holds.

The closeness of the conditions (1.1) which is always verified in experiments almost dictates that the simplest generalization of the Prandtl -Reiss equations be constructed for the case of a medium with hardening. The point is that conditions (1.1) agree completely with one another (i.e., $h(w) = g(\lambda)$ for any process) only in the case when in any state with $d\varepsilon^p_{ij} \neq 0$ the stress tensor is coaxial with and similar to the tensor $d\varepsilon^p_{ij}$. Together with the condition for the plastic incompressibility of the material and the Mises yield condition the coaxality and similarity of these tensors includes also the Reiss equations.

A generalization of the Reiss equations that was mentioned (which was obtained by replacing the Mises conditions by any of the conditions (1.1) was constructed in a somewhat different way by G. Handelman and V. Prager (Prikl. Mat. i. Mekh., Vol. 2, No. 11, 291-292 (1947)). Let $f = \sqrt{2}\,(s_* - g(\lambda))$, where as before $d\lambda = \sqrt{d\varepsilon^p_{\alpha\beta} d\varepsilon^p_{\alpha\beta}}$.

The coaxality and similarity of the stress tensors and the rate of the residual deformation together with the plastic incompressibility condition are equivalent to the relation

$$d\varepsilon^p_{ij} = d\lambda \frac{\partial f}{\partial \sigma_{ij}}, \tag{1.2}$$

and, in accordance with (1.1) for any state $f \leq 0$, where $d\lambda \neq 0$ only when $f = 0$, $df = 0$. When the "load function" f has the concrete form mentioned, these relations coincide with the Handelman-Prager relations. For the elastic component of the deformation, as usual, it is assumed that Hooke's law is valid.

The Handelman-Prager relations define a complete concrete and simple model of an elastoplastic medium with hardening. In 1951 D. Ch. Drucker formulated a postulate as a result of which the residual deformation velocity tensor must be related to the load function by a "gradient" relation of type (1.2) for a large class of cases. When applied to isothermal processes in a medium with hardening and the usual unloading law (which does not change as a result of the plastic deformation), the Drucker postulate is equivalent to the following local maximum principle:

$$\sigma_{\alpha\beta}\, d\varepsilon^p_{\alpha\beta} \geqslant \tilde{\sigma}_{\alpha\beta}\, d\varepsilon^p_{\alpha\beta} \tag{1.3}$$

for any real σ_{ij}, $d\varepsilon^p_{ij}$ (related by the defining equations) and any admissible stressed state $\tilde{\sigma}_{ij}$ (bounded only by the condition $f \leq 0$). For an ideal plastic medium (1.3) always holds with the equality sign. It follows that the region in the space of the stresses occupied by the trajectories of the reversible changes is never concave, and the tensor $d\varepsilon^p_{ij}$ for each smooth sector of the boundary of this region ("the loading surface") is related to the normal to it by a relation of type (1.2).

The defining relations in which the load functions play the role of a "plastic potential" are usually called the associated law. In the case of an ideal plastic medium with a smooth loading surface which is most frequently called the yield surface (when applied to such media) the adoption of the Drucker postulate exhausts the problem of defining the relations, at least for processes for which the temperature field does not change. In the case of a medium with hardening additional assumptions must be made. When the loading

surface has a singular point, the problem of the relation of the tensor $d\epsilon^p_{ij}$ and other variables arises during changes of state corresponding to displacements from these points.

The problem of the yield law in the case of a plastic potential with singularities was touched on already by E. Reiss in the early 30's and later in the studies of V. Prager. The studies of V. T. Koiter, 1953-1956 gave an elegant solution of this problem for a medium with a piecewise-smooth loading surface of general form (Quart. App. Mech., Vol. 11, No. 3, 350-354 (1953) and other articles, the fundamental results and bibliography of which are available in the survey study of V. T. Koiter, "General Theorems in the Theory of Elastoplastic Media," 1960, Russian Translation, Moscow, 1961).

When the piecewise-smooth loading surface consists of $n \geq 1$ smooth sectors to which the loading functions f_1, f_2, . . ., f_n correspond, according to Koiter for any process

$$d\varepsilon^p_{ij} = \sum_{m=1}^{n} d\lambda_m \frac{\partial f_m}{\partial \sigma_{ij}}, \quad f_m \leqslant 0, \quad d\lambda_m \geqslant 0, \tag{1.4}$$

and for each m = 1, 2, . . ., n, $d\lambda_m > 0$ if and only if,

$$f_m = 0, \quad \frac{\partial f_m}{\partial \sigma_{\alpha\beta}} d\sigma_{\alpha\beta} > 0 \tag{1.4'}$$

(for an ideal plastic medium the last condition is formulated somewhat differently). For a point on the loading surface belonging only to one smooth sector, according to (1.4') only one term in the sum (1.4) is different from zero, and the segment which has the direction of the normal to the surface corresponds to the tensor $d\epsilon^p_{ij}$ as before in the space of the stresses. For the points of the surface at which the normal is not defined, the maximum principle (1.3) admits a great deal of arbitrariness. Hence, in these cases, on the basis of (1.4) and (1.4') several partial loading functions can be "active" simultaneously.

In particular the studies of V. T. Koiter made it possible to understand the connection between the theories of the usual type and theories claiming a microstructural approach. One of the most important facts was established by Koiter himself who has shown that for an appropriate selection of the functions f_m and a transition to the limit as $n \to \infty$

relations (1.4) reduce to the relations of S. B. Batdorf's and B. Budyanskiy's "slippage theory."

This theory was published in 1949 and it was the first theory which attracted attention by its attempt to construct the equations of plasticity theory on the basis of the laws of the plastic deformation of monocrystals (NASA Techn. Note, No. 1871, 1949, Russian Translation in the collection of translations "Mekhanika," (Mechanics), No. 1, 1962). In the 50's several tens of articles dealing with an analysis and certain improvements of the Batdorf-Budyanskiy theory were published. However, it became clear toward the end of the 50's that its fundamental assumptions oversimplified the "slippage" pattern in a polycrystal. Experimental studies which demonstrated unambiguously the unsound characteristic predictions of this theory played an important role.

A new important contribution was made in this period to the theory which was developed within the framework of the classical approach. In accordance with (1.1) the loading surface in any state is a Mises cylinder with a fixed axis and only the radius of the cylinder changes during the plastic deformation. Above all, this eliminates taking into account the Bauschinger effect. The first concrete models of an elastoplastic medium with deformation strengthening anisotropy and the Bauschinger effect were constructed in the studies of V. Prager and other scientists in the 50's. Later studies which made these models more precise appeared. The main source for the improvements were the results of experiments with multiple loads with changing signs that were carried out in the 50's and 60's by many experimenters and which made it possible to advance considerably the understanding of the causes and forms of the Bauschinger effect in real metals.

Other interesting studies of models under complex loads were also carried out in these years. Experiments with "small additional loads" and studies of the "delay" effect which will be discussed in greater detail in Section 2 were of fundamental importance. Gradually it became clear that no theory in which the boundaries of the elastic behavior for each state of the body were described by one surface in the space of stresses or strains gave satisfactory agreement with the experiment. As a result of this, recently the interest in theories which can be called with some justification microstructural was again revived.

In conclusion we note that the first studies in plasticity theory in our country appeared in 1936-1938. In the last few years the publications of the USSR Academy of Sciences alone published over 200 studies. The next section of this survey is devoted to the studies of Soviet scientists in the field of rheology of plastic media and Section 3 to studies in boundary value problems. The survey makes no claims to completeness. We avoided the discussion of partial problems or special problems. Thus, the theory of plastic shells and plates, the flow of thin plastic layers, application of the theory to technological problems, the problem of stability beyond the elasticity limit, dynamic problems and certain other problems were not touched at all.

§2. Relations Between the Local Characteristics of the State and the Deformation of the Medium

2.1. Ideal Plastic Media

According to the definition of an ideal plastic body, in processes in which the temperature does not change, a fixed region in the space of stresses corresponds to its admissible states. Therefore, the function f in the equation $f = 0$ of the boundary of this region must be only a function of the stresses. In the case of a piecewise-smooth yield surface this holds for every function $f_1, f_2, \ldots, f_n$, corresponding to smooth sectors. As a result, relations (1.4) together with the usual equations for the elastic component of the deformation (which describe Hooke's law) form a complete system of defining relations. (1.4') is replaced by the condition by virtue of which $d\lambda_m > 0$ holds only when $f_m = 0$, $df_m = 0$, which also follows directly from the definition of an ideal plastic medium.

For an isotropic medium the functions f_m must be invariant with respect to a complete orthogonal group and may therefore depend on the stress tensor only by way of its "absolute" invariants. The condition for plastic incompressibility is equivalent to the condition that the f_m do not depend on the invariant $\sigma_{\alpha\beta}\delta_{\alpha\beta}$ and, hence, can be represented in the form of functions of scalar invariants of the stress deviator, among which we can always consider the intensity

$$s_* = \sqrt{{}^1/_2 s_{\alpha\beta} s_{\alpha\beta}}$$

and the "inclusion angle"

$$\alpha_s = \frac{1}{3}\arccos\left(-\frac{\sqrt{3}\, s_{\alpha\beta} s_{\beta\gamma} s_{\gamma\alpha}}{2 s_*^3}\right).$$

as being independent. The concrete form of the functions $f_m = f_m(s_*, \alpha_s)$ must satisfy the non-concavity condition for the yield surface. In addition, it is usually assumed that the yield points during expansion and compression are the same. But even when this condition is introduced, a great deal of arbitrariness still remains. In particular, both classical yield conditions, the Tresk and the Mises condition satisfy all the conditions that were mentioned.

The idea of using the maximum dissipation power principle for a comparison of the yield conditions is due to D. D. Ivlev (D. D. Ivlev, 1958, 1966). The preference of the Tresk condition is proved with the aid of such a comparison. However, in addition to the maximum principle it is necessary to use an assumption which stipulates the manner in which the yield point is measured (which must always be determined from pure shear experiments).

Reasons in favor of good agreement between the Tresk condition and the physics of the plastic deformation were also given by other authors. On the other hand, it is known that the Mises condition agrees more satisfactorily in most cases with the experimental data. In this regard, the experimental data for the Lode parameter relation are especially characteristic since this relation depends on arbitrary yield functions and the difference in the Tresk and Mises conditions becomes more and more appreciable.

It is known that the "inclusion angle" of the given symmetric tensor determines the direction of its component in the "octahedral" plane of the sector (which subtends the same angle with the principal axes). Taking this into account, it can be easily seen that within each face of the Tresk prism "the inclusion angle" of the tensor $d\varepsilon_{ij}^p$ preserves a fixed value which changes by 1/3 π during the transition to the neighboring face (Fig. 1). For the Mises condition in states with $d\varepsilon_{ij}^p \neq 0$, $\alpha_s = \alpha_{d\varepsilon^p}$ always holds. The Lode parameter is uniquely determined by the "inclusion angle," and in the transition to the Lode parameters, we obtain the diagram plotted in Fig. 2. The dashed line TOT corresponds to the Tresk condition,

which coincides for $\mu_{d\varepsilon^p} = 0$ with the interval $[-1, 1]$ on the μ_s axis and the line $\mu_s = \mu_{d\varepsilon^p}$ corresponds to the Mises condition. After the experiments of V. Lode that were mentioned in Section 1, the relation between the Lode parameters was studied by G. Taylor and H. Quinne in experiments with different metals, and in the 40's by E. Davis. In the USSR several experiments of this type were made by Yu. I. Yagnom and his collaborators (Yu. I. Yagn and I. I. Vinogradov, 1954, N. M. Mitrokhin and Yu. I. Yang, 1960, et al.). According to the data from all these experiments, curves which have the shape given by the dotted curve in Fig. 2 which are smooth and nearly the line $\mu_s = \mu_{d\varepsilon^p}$ are obtained in all cases (including metals with a high yield point).

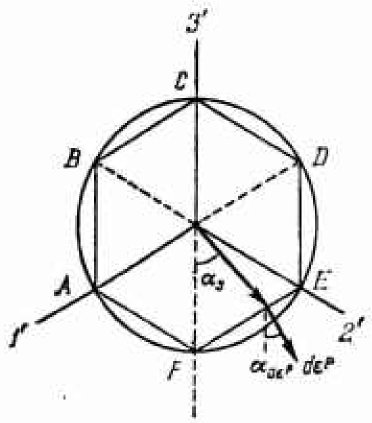

Fig. 1

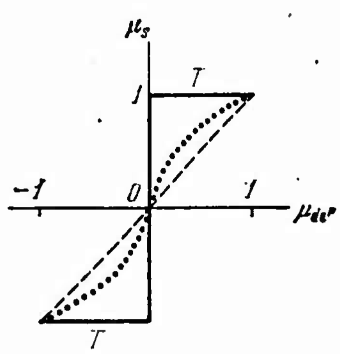

Fig. 2

The studies of N. K. Snitko (1948) and V. V. Novozhilov (1952) should also be mentioned in connection with the problem of the forms of the yield condition. In accordance with the first study, the ratio of the elasticity limits during elongation and pure shear for a polycrystalline sample depends on the type of lattice of its monocrystalline elements. It was shown in the study of V. V. Novozhilov that the intensity of the tangential stresses can be considered as the mean square of the tangential stresses on the sectors oriented in all possible ways at the given point of the body.

2.2. Hardened Media with Smooth or Piecewise- - Smooth Loading Surface

For a hardened medium the load surface varies for $d\epsilon^p_{ij} \neq 0$. Even here the smoothness of the surface may vary in different states of the medium, for example, on a surface which is initially smooth at all points of the loading surface pointed points may appear as a result of a plastic deformation and the number of smooth sectors on the piecewise-smooth smooth surface may differ in different states of the medium, etc.

On the other hand, the experiments that were made already in the 30's that were mentioned in Section 1, have shown that in some cases we can restrict ourselves to the simplest assumption. This assumption is included in (1.1) and it consists of the fact that the change on the loading surface when $d\epsilon^p_{ij} \neq 0$ can always be reduced to a similarity transformation with respect to the center or its axis of symmetry ("isotropic hardening").

The Bauschinger effect is not taken into account under this assumption. In the early 50's it was understood that to describe this effect it was necessary that one element for the change on the loading surface when $d\epsilon^p_{ij} \neq 0$ be a translation in the direction of the displacement of the point in the space of stresses. This fact was noted in different ways in the studies of G. Edelman, D. Ch. Drucker and V. Prager. The 1954-1955 studies of V. Prager developed concrete models of the medium with a translation of the loading surface.

One such model was discussed in 1954 by A. Yu. Ishlinskiy. The fundamental relations for this medium follow from (1.2) with the following concrete form of the stress function:

$$f = (s_{\alpha\beta} - \mu_{\alpha\beta})(s_{\alpha\beta} - \mu_{\alpha\beta}) - 2k^2, \tag{2.1}$$

where k is a constant and the deviator with components μ_{ij} is a linear isotropic function of the deviator of the residual deformation. In the initial state $\mu_{ij} = 0$ and (2.1) coincides with the Mises condition. Beyond the elasticity limit, the Mises cylinder is gradually displaced as a rigid whole.

Further progress was made by V. V. Novozhilov and Yu. I. Kadashevich (1958) who started out with the fact that in real metals the Bauchinger effect and the hardening deformation anisotropy are related to the "microstresses" (inhomogeneities in the field of internal forces in volumes whose dimensions are on the order of a grain or smaller). The

effect of the latter on the macroscopic properties of the material were analyzed with the aid of a mechanical model with a dry frictional element on a plane and a system of springs which simulated the macroscopic and residual microscopic stresses. The plasticity law obtained in this study follows from (1.2) when the stress function has the form (2.1), but, unlike in the study of A. Yu. Ishlinskiy, the μ_{ij} are related to the components of the deviator of the plastic deformation by nonlinear equations and k is a monotonic function of the scalar λ, and

$d\lambda = \sqrt{d\varepsilon^p_{\alpha\beta}\, d\varepsilon^p_{\alpha\beta}}$. In particular the authors single out the case case when k = const (a medium with an ideal Bauschinger effect); however, in the general case the loading surface undergoes simultaneously a translation and isotropic expansion when $d\epsilon^p_{ij} \neq 0$. It was shown later that to obtain a concrete relation for the tensor μ_{ij} and other variables the results of experiments with multiple loads with changing signs on the samples were important (R. A. Arutyunyan and A. A. Vakulenko, 1965). It also became evident that the interpretation of the tensor μ_{ij} as a "microstress tensor" which was proposed by V. V. Novozhilov and Yu. I. Kadashevich was well founded also from the standpoint of dislocation theory (A. A. Vakulenko and L. M. Kachanov, 1969).

Experience has shown that the hardening of real metals has always an anisotropic character. Under appropriate loads the Bauschinger effect and the hardening deformation anisotropy are effects which basically have the same order of magnitude as the hardening itself. Therefore, for any model of a medium with anisotropic hardening agreement with the experiments can only be fully satisfied for processes whose trajectory in the deviator hyperplane in the space of stresses lies in a sufficiently narrow cone with apex at the point $s_{ij} = 0$. For media for which the loading surface is translated, this cone is replaced by a cylinder which intersects the surface in the neighborhoods of each end of some diameter, since loads for which the sign of the stresses changes are now permitted. But in both cases the class of processes in which we can expect satisfactory agreement between the theory and the experiment is further narrowed by certain additional conditions imposed on the curvature of the trajectories. These constraints are more stringent for media with isotropic hardening whose behavior during sharp rotations of the principal axes of the stress increment tensor do not even agree qualitatively with the experiment. This was clearly detected for the first time in experiments with so-called small additional loads.

2.3. Theories of the "Slippage" Type

The first experiments with small additional loads in the USSR were made by A. M. Zhukov and Yu. N. Rabotnov (1954). The samples which had the shape of thin-walled pipes were first subjected to expansion during which they were subjected to a residual strain, after which torque couples were applied during a fixed expanding force which caused the tangential stresses $\Delta\tau$ (the trajectory OMM_1 in Fig. 3). If the loading surface remains smooth at its "active" points at the instant when the additional load is applied, the displacement MM_1 lies in the tangent plane (Fig. 3), and by virtue of the "neutrality" of such displacements with an accuracy up to small higher order magnitudes $\Delta\gamma = \Delta\gamma^e$, i.e., $\Delta\tau = G\,\Delta\gamma$, where G is modulus of elasticity in shear. In the experiment the ratio $\Delta\tau/\Delta\gamma$ was always much smaller than the modulus G. Experiments have been known in which a "fan" of additional loads from the given state occurred (the experiments of P. M. Nahdi and G. Rowe, see the collection of translations "Mekhanika," No. 3, 1955, and others). Considerable nonelastic changes in the deformation of the sample were usually observed already during additional loads with displacements in the space of stresses at angles $> 1/2\ \pi$ relative to the vector connecting the coordinate origin with the point under consideration on the loading surface.

Within the framework of the usual definition of this surface, the conclusion must be drawn that for an initially smooth surface a pointed point can occur on it at the additional load instant. Another argument in favor of this possibility are the conclusions which follow from the Battorf-Budyanskiy "slippage theory" and from essentially similar theories of other authors.

Thus, V. D. Klyushnikov (1958) proposed a plane model for a plastic medium in which, as in the Batdorf-Budyanskiy theory, the plastic deformation is the result of the differently oriented displacements on the areas at the given point of the body. However, because of its greater simplicity, the V. D. Klyushnikov model is more amenable to an analysis of the relation between the stresses and strains during different "loading ways." An even simpler two-dimensional model was proposed by Yu. N. Rabotnov (1959). Both these models lead, during a plastic deformation, to a change in the pattern of the loading surface which is similar in many respects, which, in turn, is similar to that which follows from the Batdorf-Budyanskiy theory and differs qualitatively from that which corresponds to the isotropic or translational hardening. In contrast to the models of a medium with hardening considered in Section 2.2, a pointed point is developed on the loading surface for a wide class of loading regimes,

whose apex coincides with the loading point while the "rear" part of the surface does not change and remains fixed.

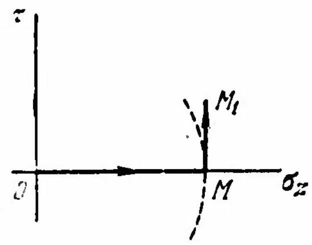

Fig. 3

As we already mentioned, the relations from the Batdorf-Budyanskiy theory can be obtained from the relations (1.4) for the associated law (see the Russian translation of the work of V. T. Koiter in the collected translations "Mekhanika," No. 2, 1960). For a somewhat different selection of the functions f_m and also during the transition to the limit as $n \to \infty$, the relations from the theory of "local deformations" developed by A. K. Malmeister (1957) are obtained from (1.4). In both theories the stresses on the slippage surfaces (local displacement) coincide with the stresses resulting from external forces on the surfaces with a given orientation. However, it is known that in a real polycrystal the stresses in the grains and parts of the grains differ from the mean stresses in large volumes. With the appearance of the macroscopic residual deformation, the microinhomogeneity of the field of stresses in the sample is strengthened in a certain sense, which is the reason for the deformation hardening anisotropy and the Bauschinger effect. Therefore, it is natural that the predictions based on the Batdorf-Budyanskiy theory do not agree well with the experiment. This also applies to the derivation of the "pointedness" on the loading surface.

A series of experimental studies of the changes on the loading surface during plastic deformation are available at the present time. In the USSR such studies were carried out by A. M. Zhukov (1957), Yu. I. Yagn and O. A. Shishmarev (1958), G. B. Talypov and V. N. Kamenets (1958, 1961), G. B. Talypov (1961), O. A. Shishmarev (1962, 1966). In all these experiments, the behavior of the samples which had the shape of thin-walled pipes was studied, but in details the experimental design of different authors differed, so that their conclusions do not agree in all respects. One general conclusion which can be made on the basis of the results of these experiments is that an important element in the change in the geometry of the surface (which is often the only element

or is only combined with isotropic expansion) as a result of a given plastic deformation is the translation, and that the surface remains smooth. Nothing resembling a singularity at the loading point was observed in all the studies that were enumerated.

In the experiments of certain foreign investigators, the translational displacement of the loading surface was accompanied by a relatively moderate "tendency to form corner points" (the experiments of P. K. Birch and V. N. Findley, P. M. Nahdi, et al., and a number of other experiments).

2.4. Other Models of a Plastic Medium with Hardening

In conjunction with these experimental facts, attempts have been made to construct a theory which was satisfactory when additional orthogonal loads were applied to the loads with a smooth loading surface of the medium in any state. Thus, the studies of G. A. Hemmerling (1964) should be mentioned which proposed a certain generalization of the Drucker postulate. A variant of the unassociated plasticity law was developed on the basis of the generalization.

A different generalization of the Drucker postulate was proposed earlier by A. A. Il'yushin (1961). In this study it is postulated that for any isothermal processes closed with respect to the deformation

$$\int \sigma_{\alpha\beta}\, d\varepsilon_{\alpha\beta} \geqslant 0,$$

where the equality only holds when the process is reversible.

As we already mentioned, the different results of the experiments that were discussed in Section 2.3 are related to a considerable extent, to differences in the formulation of the study, more precisely the method by which the points on the loading surface are determined. This can already be seen on the example of the usual tests of metals in engineering during uniaxial expansion or compression of the samples. It is well known that a sharp dividing line between elastic and elastoplastic states has not been detected and that the elasticity limit must be determined in such experiments, by convention, as the stress which corresponds to some given small value of the residual deformation. Naturally, the situation in tests during the complex stressed state is no better. The dimensions and shape of the loading surface depend on the residual deformation "tolerance" with which the points are determined on this surface.

Thus, in fact, the elasticity boundary is not as clear cut as defined by the concept of a loading surface in its usual form. The facts that were detected in experiments with a small additional load are connected with the "spread" of the real elasticity boundary. The first step which takes into account this spread is to abandon the condition for the neutrality of the loads to which the displacements on the loading surface correspond. However, when this is done, the continuity of the relation relating the rates $\dot{\sigma}_{ij}$ and $\dot{\varepsilon}^p_{ij}$ for the given point on the loading surface disappears and certain difficulties arise in the formulation of theoretical boundary value problems. Therefore, it is natural to take the next step and consider as the changes with $d\varepsilon^p_{ij} \neq 0$ also those changes in the state of the medium which correspond to the displacement of the loading point inside the region bounded by the loading surface with a corresponding improvement in the direction of the latter. More precisely this surface must now be considered not as the boundary in the space σ_{ij} in the elastic region of the material, but as the locus of the points corresponding to a given small "tolerance" for the magnitude of the residual deformation during a load "on the rays" from a given state (we emphasize that the surface is determined experimentally in this way). In essence, such an approach was outlined in a study of V. D. Klyushnikov (1964), although the reasoning was somewhat different.

In fact, on the basis of the Cauchy-Bunyakovskiy inequality, we can write

$$\frac{\partial f}{\partial \sigma_{\alpha\beta}} d\sigma_{\alpha\beta} = \left(\sqrt{\frac{\partial f}{\partial \sigma_{\alpha\beta}} \frac{\partial f}{\partial \sigma_{\alpha\beta}}} \sqrt{d\sigma_{\mu\nu} d\sigma_{\mu\nu}} \right) \cos \varphi .$$

With the usual assumption about the "neutrality" of the load with a displacement f = 0 on the surface, we have for the differential form dλ in the relations for the associated law

$$d\lambda = \frac{\partial f}{\partial \sigma_{\alpha\beta}} d\sigma_{\alpha\beta}, \tag{2.2}$$

and using this and the preceding expressions, relation (1.2) can be written as

$$d\varepsilon^p_{ij} = \eta \frac{\partial f}{\partial \sigma_{ij}} \sqrt{d\sigma_{\alpha\beta}\, d\sigma_{\alpha\beta}}\, \psi(\varphi). \tag{2.3}$$

where $\eta > 0$ is a function of the stresses and the history of the stresses and

$$\psi(\varphi) = \cos\varphi \ \underset{a}{\text{при}}\ 0 \leqslant |\varphi| \leqslant {}^1\!/_2\pi, \quad \psi(\varphi) = 0 \ \underset{a}{\text{при}}\ {}^1\!/_2\pi \leqslant |\varphi| \leqslant \pi. \tag{2.3'}$$

Key: a. for

We emphasize that (2.3) and (2.3') includes the usual concrete associated law in a somewhat different form for a hardening medium with a smooth loading surface. According to (2.3) the function $\psi(\varphi)$ is continuous but not differentiable at $\varphi = \pm 1/2\,\pi$. Taking into consideration that this is one of the main reasons for the complexity of the boundary value problems in the theory of elastoplastic media with hardening, V. D. Klyushnikov proposed instead of (2.3') that $\psi(\varphi)$ be defined as an analytic function which approximates the function defined by relations (2.3'). It is difficult to say to what extent this will simplify the boundary value problems, but it is clear that the description of the behavior of the models under small additional loads can be improved in this manner by obtaining the concrete form of the function $\psi(\varphi)$ directly with the aid of experimental data. It is essential that the loading surface (in the sense described above) remain smooth in the neighborhood of the point where the additional load is applied).

2.5. Deformation when the Position of the Principal Axes does not Change. Deformation Theory.

Suppose that the homogeneous deformation of the medium is such that during the entire process the position of the principal axes of the deformation tensor (relative to the fixed axes of the material) does not change. If the medium is isotropic in the initial state, the position of the principal axes of the stress tensor will also not change and we can assume without loss of generality that the principal axes of the two tensors coincide. Then, at each instant during the process, at least one of the following tensor equations holds (V. V. Novozhilov 1951, 1954):

$$
\left.
\begin{aligned}
s_{ij} &= \frac{s_*}{\text{э}_*}\left[\frac{\sin(\alpha_s - 2\alpha_\text{э})}{\sin 3\alpha_\text{э}}\,\text{э}_{ij} + \frac{\sqrt{3}\sin(\alpha_s - \alpha_\text{э})}{\text{э}_*\sin 3\alpha_\text{э}}\left(\text{э}_{i\alpha}\text{э}_{\alpha j} - \frac{2}{3}\,\text{э}_*^2\delta_{ij}\right)\right],\\
\text{э}_{ij} &= \frac{\text{э}_*}{s_*}\left[\frac{\sin(\alpha_\text{э} - 2\alpha_s)}{\sin 3\alpha_s}\,s_{ij} + \frac{\sqrt{3}\sin(\alpha_\text{э} - \alpha_s)}{s_*\sin 3\alpha_s}\left(s_{i\alpha}s_{\alpha j} - \frac{2}{3}\,s_*^2\delta_{ij}\right)\right],
\end{aligned}
\right\}
\qquad (2.4)
$$

where s_{ij} and э_{ij} are the components of the stress deviator and the strain deviator

$$
s_* = \sqrt{\frac{1}{2}s_{\alpha\beta}s_{\alpha\beta}}, \quad \alpha_s = \frac{1}{3}\arccos\left(-\frac{\sqrt{3}\,s_{\alpha\beta}s_{\beta\gamma}s_{\gamma\alpha}}{2s_*^3}\right),
$$

and э_* and $\alpha_\text{э}$ are defined analogously (when $\sin 3\alpha_s \neq 0$ and $\sin 3\alpha_\text{э} \neq 0$ equations (2.4) are equivalent). These equations describe only the coaxiality of the stress and strain tensor and are, therefore, valid in the case under consideration regardless of other properties of the medium (initial super isotropy). The specific characteristics of the medium are reflected in the equations which relate s_* and α_s to the scalars of the deformation tensor, which must be added to equations (2.4), which, for a plastic medium, are generally not holonomic in this case. This relation will only be holonomic with an additional constraint on the change of the deformation tensor.

In particular, we will assume that along with the position of the trihedron of the principal axes the "inclusion angle" $\alpha_\text{э}$ of the deformation deviator does not change and that at any instant during the process $d\text{э}_*/dt > 0$. Then, it can be shown that when the medium is isotropic in the initial state and its behavior does not depend on time $\sigma = \sigma(\varepsilon, \text{э}_*, \alpha_\text{э})$, $s_* = s_*(\varepsilon, \text{э}_*, \alpha_\text{э})$, $\alpha_s = \alpha_s(\varepsilon, \text{э}_*, \alpha_\text{э})$, where $\sigma = 1/3\,\sigma_{\alpha\beta}\delta_{\alpha\beta}$, and ϵ is the analogous invariant of the deformation tensor. Usually a special form of these functions is considered for which relations (2.4) reduce to a "linear tensor" equation and, with an appropriate stipulation for the unloading case for a plastic medium

$$\sigma = 3K\varepsilon, \quad s_{ij} = \frac{s_*}{э_*} э_{ij},$$

$$s_* = \begin{cases} \Phi(э_*) \ \text{при} \ s_* = s_M, \ ds_* > 0, \\ 2G(э_* - э_*'') \ \text{при} \ s_* < s_M \ \text{или} \ s_* = s_M, \ ds_* \leqslant 0, \end{cases} \tag{2.5}$$

Key: a. for
b. or

where K and G are constants, Φ is a monotonic function of $э_*$ and s_M is the maximum value of s_* that was obtained (including the current state).

The condition that the position of the trihedron of the principal axes and the value of the "inclusion angle" $\alpha_э$ of the deviator of the deformation do not change is equivalent to the condition that $э_{ij} = э_*(t)\tilde{э}_{ij}$, where $\tilde{э}_{ij}$ does not depend on the parameter t of the process. The deformation process during which this condition is satisfied is called the simple or proportional deformation process. The simple loading process is defined analogously (A. A. Il'yushin, 1948). According to (2.5) during a simple load, the deformation will also be simple.

L. I. Sedov (1959) has shown that for arbitrary (finite) deformation processes the deformation can only be simple for some exceptional values of $\alpha_э$. This is due to the fact that for a finite homogeneous deformation, the angles between the material lines (those that are "frozen" into the material) vary in such a way that the orientation of the principal axes of the symmetric tensor cannot be retained with respect to these lines (the exception being uniaxial expansion and other cases corresponding to $\sin 3\alpha_э = 0$).

Equations (2.5) describe the fundamental and simplest variant of the so-called deformation theory of plasticity. Historically the latter dates back to the well-known studies of H. Hencky and A. Nadai that were mentioned in Section 1. However, these studies were based on concepts which did not make allowance for a definite judgment about the range of applicability of the theory to real metals. The development of the concepts, foundations and sphere of applicability of the theory are connected with the studies of A. A. Il'yushin published in the 40's which were summarized in his monograph (A. A. Il'yushin, 1948).

Under a simple load the trajectory of the process in the deviator hyperplane of the space of stresses represents a segment of a line with origin at the point $s_{ij} = 0$. If an arbitrary point in this hyperplane moves along the line passing through the point $s_{ij} = 0$ and intersects the latter, the load will not be simple. V. V. Moskvitin (1952, 1965) generalized the equations of deformation theory and the theorems of A. A. Il'yushin for a simple load to the case of such a "sign changing simple" load. Effects of the Bauschinger type in these studies are taken into account with the aid of the so-call "Mazing principle" and the generalization of this principle proposed by V. V. Moskvitin. A detailed presentation of all these results can be found in the monograph (V. V. Moskvitin, 1965).

2.6. The "Isotropy Postulate" and Studies in Problems in the General Theory of Tensor Functions and Functionals which Arise in Connection with Rheology Problems of Plastic Media

The Set III of all symmetric bivalent tensors which can be defined for a fixed point of a continuous medium is closed with respect to linear combinations of its elements and represents some six-dimensional linear system. From the standpoint of its linear properties, this system is completely analogous to a six-dimensional Euclidian space. Thus, a vector in Euclidian space has only one "scalar invariant" (which is independent of the number of coordinate systems) while an element of the system III has three such independent invariants. This fact was the main argument of one school in the discussion about the "isotropy postulate" (D. D. Ivlev, 1960, V. V. Novozhilov, 1961). Later, V. V. Novozhilov characterized more accurately the specific characteristics of the linear system III and outlined a way for the construction of an orthonormal basis for this system (1963). K. F. Chernykh (1967) worked out in detail these concepts and constructed a concrete example of such a basis.

In the classical mechanics of continuous media, the stress tensor and the strain tensor are symmetric bivalent tensors and, hence, elements of the set III. By specifying concretely the physical dimensions of the basis elements, it is possible to study two representatives of this set in the corresponding manner, "the space of stresses" and the "space of strains." The deviators in each of these spaces form a linear subset (subspace), which we will denote, respectively, by D_s and $D_з$. "The Isotropy Postulate" (A. A. Il'yushin, 1954) is the statement according to which for an initially isotropic medium the trajectory of the process in D_s depends only on those properties of the trajectory in $D_з$ which are invariant with respect to

orthogonal transformations of $D_{\text{э}}$. By orthogonal transformations are meant linear transformations of the space $D_{\text{э}}$ for which the quadratic scalars of the deviators are preserved (the deviator with components э_{ij} is transformed into the deviator $\bar{\text{э}}_{ij}$ for which $\bar{\text{э}}_{\alpha\beta}\bar{\text{э}}_{\alpha\beta} = \text{э}_{\alpha\beta}\text{э}_{\alpha\beta}$). Since cubic scalar invariants of the deviators are not preserved under an arbitrary orthogonal transformation, the sphere of applicability of the isotropy postulate as defined is limited and includes only media for which the "material law" is described by equations not containing the products of bivalent tensors (tensors with components of the form $a_{i\alpha}b_{\alpha j}$, $a_{i\alpha}b_{\alpha\beta}c_{\beta j}$, etc.) and scalar invariants of the "inclusion angle" type.

In one chapter of the monograph (A. A. Il'yushin, 1963) an attempt is made to generalize the isotropy postulate on the basis of an analytical representation of the trajectory of the process in $D_{\text{э}}$. It should be mentioned that a number of experimental studies have been made in connection with the isotropy postulate (V. S. Lenskiy, 1958, 1961).

2.7. Some Results

In conclusion we emphasize first of all that everything that was done until now in the field of developing the "defining equations" represents a treatment of the problem in its classical formulation (Section 1). The concept of an ideal plastic medium is naturally defined in this framework and when the theory is developed only the most fundamental elements of the macroscopic pattern of the plastic deformation of metals are taken into account. The models of an ideal plastic medium play, in the theory of plasticity, basically the same role as an ideal liquid and an ideal gas in the mechanics of fluids and gases.

Models of a plastic medium with hardening must reflect finer details of the plastic properties of metals. The great variety and complexity of these details make the problem of constructing a fully satisfactory theory of such media very difficult. The models of a plastic medium with hardening known until now are in satisfactory agreement with experimental data only for a class of processes which in addition to the constraints defined by the conditions for the independence of the process of time and the constant temperature field are also limited considerably with respect to the admissible deformation or loading paths (the trajectories of the process in the space D_s or $D_{\text{э}}$. Particular difficulties arise in the description of the behavior of real metals during abrupt changes of the position of the principal stress axes

which correspond to trajectories of the type encountered in experiments with "an orthogonal" additional load. In these cases, the "spread" of the actual elasticity boundary of the material manifests itself most clearly. To take it into account, it is necessary to abandon certain customary assumptions made in the mechanics of plastic media. It should be noted that this "spread" also plays a role in the results of experiments which study the "delay" pattern (V. S. Lenskiy, 1958, 1961).

Some important effects, as a matter of fact, are not included at all in the rheology of plastic media in its contemporary state. One of these effects is, for example, the "aging" and other forms of the effect of a change in the composition of "solid solutions" on their macroscopic mechanical properties, even when this effect is considerable. Thus, a number of studies of Soviet physicists-metallurgists have shown that the plastic deformation of some metastable alloys is accompanied by changes in composition as a result of which the volume of the sample is changed irreversibly. Another factor which when taken into account may show that the assumption $\varepsilon^{p}_{\alpha\beta}\delta_{\alpha\beta} \equiv 0$ is not sufficiently accurate, is the so-called "plastic elongation" (the development of a grid of pores and cracks along the edges and inside the grain of the polycrystal during the plastic deformation). V. V. Novozhilov (1964) pointed out the important fact that this "elongation" which is usually small until the visible fracture of the sample, may become considerable under multiple cyclic loads.

Recently, certain concrete forms were used in the rheology of plastic media as a result of the achievements in the physics of a solid and thermodynamics.

It should be noted that the first and second postulates of thermodynamics make it possible to draw a number of important conclusions already with the usual general assumptions about the properties of the medium. Thus, it was discovered that the "energy balance" for which the work p dissipates completely is only characteristic of an ideal plastic medium. For a medium whose properties change as a result of a plastic deformation, a part of this work is always converted into the so-called "latent energy of the deformation." (A. A. Vakulenko, 1961). When this fact is taken into account, it is possible to use in the analysis of existing and in the development of new models of a plastic continuous medium a number of experimental results obtained in modern physics of metals.

The model (elementary volume) of the plastic medium represents a system, one of whose characteristics is the nonlinearity and nonholonomy of the relations between the external and internal parameters. The studies of L. I. Sedov and M. E. Eglit (1962) outline a way of constructing general forms of the "defining equations" for such media using thermodynamics. The assumption that "phenomenological connections" exist (relations between thermodynamic "forces" and "fluxes") which are the foundation of modern thermodynamics of irreversible processes can also be used for this purpose (A. A. Vakulenko, 1958, 1961; V. N. Nikolayevskiy, 1966).

The boundaries of another "bridge" between the rheology of plastic media and physics was attained with the development of dislocation theory. Such parameters of deformation reinforcement anisotropy as, for example, the tensor μ_{ij} in the theory of plastic media with a translated loading surface (Subsection 2.2) can be interpreted on the basis of the concept of continuous dislocation theory. For this reason undoubtedly progress in dislocation theory will have an effect on the development of the rheology of plastic media. This effect may be mutual, as the details of the relation between the concepts in continuous dislocation theory and the "usual plasticity" theory are clarified, the facts available to the latter, may turn out to be useful in the solution of problems in the theory of dislocations and other "defects" in solids.

3. Boundary Value Problems

3.1. General Remarks

The solution of many engineering and geophysical problems presents considerable demands on plasticity theory. Contemporary plasticity theory can only provide partial answers to these problems. First of all, as we have shown in Section 2, even the most general known equations in the theory of plasticity are valid only when a number of constraining conditions are satisfied. As a rule, it is not possible to verify whether these conditions are satisfied in the body for the given external forces. Therefore, the use of a particular set of defining equations in concrete problems is almost always based on intuitive concepts. On the other hand the nonlinearity and nonholonomy of the plastic deformation equations leads to difficult mathematical problems even in relatively simple boundary value problems (from the standpoint of the shape of the body and the external effects). In addition, difficulties of a theoretical nature arise often (besides the purely computational difficulties.

The peculiar situation which exists in the theory of plasticity reflects these contradictions. The practical needs force us to formulate and solve, at least approximately, a variety of boundary value problems. At the same time the nonavailability of reliable and sufficiently general equations for the plastic state and also the complex structure of the equations hamper to some extent the development of the corresponding theoretical branches.

Another aspect of the problem that was mentioned is the following. Although computers are used on an increasing scale, the theory of plasticity cannot be reduced only to computational schemes. Concepts about the laws governing plastic flow and the particular features of the patterns are important. This leads to the quest for the simplest models of the plastic medium which have only a limited range of applicability, but are useful in the formulation and solution of boundary value problems.

The models of an ideal rigid-plastic body and an ideal elasto-plastic body which use the concept of a fixed yield surface and are used as the basis for the solution of many concrete problems have been defined clearly. The plasticity condition has been verified well in experiments in a sufficiently wide range in which the stresses vary. It is also necessary to take into account the indirect validations of these models obtained from comparing the solutions of many problems with the experimental data.

The model of an ideal rigid-plastic body ignores completely elastic deformations. The body is not deformed until the necessary stress level is attained, after which plastic flow occurs. This scheme is useful in determining the load bearing capacity of the body ("limiting loads") and in the analysis of developed plastic flow ("technological" problem).

An ideal elastoplastic scheme is necessary in the study of problems in which the elastic and plastic deformations have the same order of magnitude. The use of this scheme hinges on overcoming great mathematical difficulties.

The models that were discussed are good approximations also in cases when the medium is slightly strengthened.

When it is considerably strengthened the situation is less clearcut. The study of boundary value problems for a strengthened body is based in the majority of cases on the simplest model of isotropic hardening. The limited value of the scheme was already mentioned above and its improvement as a result of a rigid translation of the loading surface does not eliminate all discrepancies with the experiments, while,

at the same time, it complicates considerably the initial relations. For these reasons it is convenient to study the problems for a strengthened medium only when the loading conditions are not complex, and when the character of the external forces allows us to expect that the elements of the body are subjected to a load which is nearly a simple load in a certain sense. The majority of one-dimensional problems that are important in the applications (axisymmetric problems for pipes, discs, plates, etc.) usually satisfy the condition that was mentioned. No matter how paradoxical it may seem, the mathematical difficulties here play a positive role since they force us to limit the analysis only to the most important and at the same time sufficiently simple problems (with regard to loading conditions).

As we already mentioned, no theorems are available which would allow us to estimate the "simplicity" of a particular problem. The estimate of the usefulness of the solutions that were obtained is usually based on intuitive concepts and perhaps also a number of experimental observations.

3.2. The Rigid-Plastic Body

An ideal rigid-plastic body begins to deform only when the limiting load is attained. At the same time certain parts of the body may remain rigid. The speeds of the particles on the boundary of the plastic zone must agree with the speeds with which rigid parts of the body move.

The scheme of a rigid-plastic body has already been used intuitively in early studies in the theory of plasticity (the rigid zones were sometimes called elastic zones). However, the necessity of the agreement between the stress fields and the speeds has not been recognized for a long time. Only toward the end of the 40's the idea of applying the scheme of a rigid-plastic body has been widely accepted.

The scheme of a rigid-plastic body is useful if the plastic flow which encompasses the entire body or a part of it does not undergo crowding. Another pattern occurs if, for example, a pipe subjected to internal pressure is in an undeformed clamp. Here the rigid-plastic scheme cannot be used.

The scheme of a rigid-plastic body is a highly abstract idealization and its interpretation is connected with a number of difficulties. The solution using this scheme, generally, may differ from the solution of the same elastoplastic problem when the Young modulus $E \to \infty$. The isolation of the rigid zones is arbitrary to some extent and the stresses in them are not defined. This is related to the absence of a

unique way of constructing the solution which is characteristic of a rigid-plastic body and with certain other paradoxical conclusions.

The problems of the nonuniqueness of the solution are eliminated when the rigid-plastic body is studied as the limiting case of an elastoplastic medium. However, the application of this idea is beyond the scope of the rigid-plastic scheme, and it is connected with great mathematical difficulties. In fact, we are forced to work within the rigid-plastic scheme framework and tolerate its shortcomings.

Nevertheless, the idea of the gradual application of the scheme of a rigid-plastic body is natural when certain conditions are satisfied and turned out to be useful not only in the solution of static problems, but also because it pointed out great advantages in the analysis of a number of dynamic problems. The difficulties connected with the nonuniqueness of the solution are overcome by evaluating the latter on the basis of extremal theorems for the limiting load.

In the plastic zones the solution satisfies the differential equilibrium equations

$$\frac{\partial\sigma_{ij}}{\partial x_j}+X_i=0, \tag{3.1}$$

the law for the flow

$$\dot{\varepsilon}_{ij}=\lambda s_{ij} \tag{3.2}$$

and the Mises plasticity condition

$$s_{ij}s_{ij}=k^2. \tag{3.3}$$

Here, we wrote down the Mises plasticity theory equations for an isotropic body. More general equations are easily obtained by introducing a yield condition of the form

$$f(\sigma_{ij})=k^2$$

and the associated flow law

$$\dot{\varepsilon}_{ij} = \lambda \frac{\partial f}{\partial \sigma_{ij}} \tag{3.4}$$

for smooth points on the yield surface. The Tresk-St.-Venant plasticity condition

$$\tau_{max} = \text{const},$$

which corresponds on the deviator plane to a hexagon inscribed in the Mises circle (Fig. 1) is of greatest interest.

The Mises condition agrees better with the experimental data; however, the general concepts (the closeness of the maximum tangential stress to the intensity of the tangential stresses) and the smallness of the deviations that are observed indicate in practice the equivalence of the Mises and Tresk-St.-Venant plasticity conditions. Various concepts which are of interest from one point of view or another, are sometimes used to favor one of the two conditions. Thus, some schemes for the static analysis of polycrystalline sets based on a number of assumptions about the mechanics of a plastic deformation lead to the Mises condition. On the other hand, the Tresk condition is defined in a certain sense by extremal properties (D. D. Ivlev, 1966). However, the attempts to present the Tresk-St.-Venant plasticity condition as the condition which corresponds most to the character of plastic flow are not convincing. The basis for this is the pattern of the plastic flow in monocrystals. An extension of these concepts to polycrystal metals cannot be considered justified.

The use of the Tresk-St.-Venant condition makes it possible to simplify in many cases the mathematical formulation of the problem. The possibility of using this yield condition was discovered relatively late after the studies of V. Prager and V. T, Koiter (1953) were published in which the scheme of the plastic potential (associated flow law) was extended to single yield surfaces. The flow on the edge is represented by a linear combination of the flow on the left and right of the edge:

$$\dot{\varepsilon}_{ij} = \lambda_1 \frac{\partial f_1}{\partial \sigma_{ij}} + \lambda_2 \frac{\partial f_2}{\partial \sigma_{ij}}, \tag{3.5}$$

where f_1 = const, f_2 = const are the equations of the yield surface on the two sides of the edge. The undetermined multipliers λ_1, λ_2 are nonnegative, as a result of which the flow develops in a direction which lies inside the angle formed by the normals to the two adjacent edges. An additional multiplier is needed to satisfy the consistency conditions for the deformation related to the "redundant" constraint on the stressed state.

Such an extension of the associated flow law makes it possible to obtain a consistent system of equations and to derive the corresponding general theorems.

The physical interpretation of the flow on the edge which was defined in this manner is connected with certain difficulties which arise already in the case of simple elongation corresponding to a corner point C (Fig. 1) of the Tresk-St.Venant hexagon. Here the flow is given by the equations

$$\dot{\varepsilon}_1 = \lambda_1 + \lambda_2, \quad \dot{\varepsilon}_2 = -\lambda_1, \quad \dot{\varepsilon}_3 = -\lambda_2,$$

in which the first principal direction is oriented along the axis of the rod. Thus, the transverse deformations are arbitrary, and only the incompressibility condition is satisfied. This pattern does not agree with the usual concepts about the flow of an isotropic rod. Nevertheless, such paradoxical results occur only in extreme cases and apply mainly to the velocity field. The general evaluation of the solutions obtained on the basis of the associated law undoubtedly is well founded and the limiting loads are a good approximation.

In a number of cases the Prager-Koiter scheme has considerable computational advantages. It is this fact which explains the rapid and wide application of this scheme to the plane stressed state problem in the theory of plastic shells and plates and to the axisymmetric flow problem. At the same time the difficulties that were mentioned above force us to evaluate the Prager-Koiter scheme as an idealized approximation of the more realistic Mises theory, and from this standpoint it is not very useful to try to interpret physically the individual paradoxical results.

The system of equations (3.1)-(3.3) which involves ten unknown functions σ_{ij}, λ, v_i can be written down in various forms. In particular, by eliminating the components of the stress deviator s_{ij}, we can obtain a system of five equations in the five unknown functions λ, v_i and the mean pressure σ.

Three-dimensional plastic flow problems are extremely difficult and not well understood. T. Thomas[1] has shown that the system of equations, as a rule, is elliptic. Only in individual problems (plane deformation, torsion and certain other cases) the equations have real characteristics. Since nonlinear hyperbolic equations are analyzed more easily and the formulation of the boundary value problems is simplified considerably, attempts were made to extend the hyperbolic boundaries. This is sometimes achieved by using the Tresk-St.-Venant yield condition. The existence of characteristic surfaces where this condition is used was noted by T. Thomas, who analyzed systematically the discontinuities in the plastic medium.

The so-called "complete plasticity" condition leads to a considerable simplification, according to which the stressed states corresponding to the edges of the Tresk-St.Venant prism (i.e., the vertices A, B, . . ., F of the hexagon in Fig. 1) occur. For such ("statically determinate") stressed states (D. D. Ivlev, 1966) the system of equations will be hyperbolic. The physical arguments which are sometimes used in favor of this system are based on the deceptive simplicity of the mathematical analysis rather than the essence of the problem. Many problems simply cannot be solved within the framework of this scheme (for example, problems in the plane stressed state). At the same time the point of view which favors the complete plasticity condition is much too rigid and negative, which is clearly pointed out in the book of R. Hill ("artificial and unreal yield conditions," "such calculations have little or no value at all"). Solutions of this type are undoubtedly sometimes of interest. Nevertheless, an evaluation of the solutions constructed with the aid of the complete plasticity condition must be based on extremal theorems. If a kinematically admissible field corresponds to a solution obtained on the basis of this scheme, the solution leads to the upper bound on the limiting load. When the stressed state can be extended to the entire body without violating the yield condition, we obtain the lower bound. In the cases when the solution obtained does not belong to either of the two classes that was mentioned, the question of the usefulness of the solution remains open.

1. T. Thomas, Plastic Flow and Fracture in Solids, 1961 (Russian translation, Moscow, 1964).

With regard to the usefulness of discontinuous solutions in the theory of plasticity (in particular, for finding approximately the limiting load) the relations on the discontinuity surfaces have been studied in detail. As R. Hill has shown in 1961 when the yield conditions are convex, the deformation rates are zero and the rates are continuous on the discontinuity surfaces of the stresses. On the other hand, on the discontinuity surface of the velocities, the deviator of the stresses is generally continuous and only in the case of a Tresk prism face the intermediate principal stress may be discontinuous (G. I. Bykovtsev and Yu. M. Myasnyankin, 1966).

The plasticity condition imposes certain constraints on the magnitude of the jump in the stressed state on the discontinuity surface of the stresses. With the Tresk yield condition a mottled pattern occurs, since stressed states can occur on different sides of the discontinuity surface which correspond to different flow regimes. For stressed states corresponding to the edges of the Tresk prism, the relations on the discontinuity surface were studied by D. D. Ivlev (1966).

The solutions of the system of equations for the plastic flow are constructed from different special cases of the stressed and deformed states which have a "common" mechanical meaning (plane deformation, plane stressed state, torsion, etc.). Sometimes more specific cases are also studied.

For example, for a spherical deformed state it is assumed that the velocity components of the deformation and stress in a spherical coordinate system r, θ, φ depend only on θ, φ where $\tau_{r\theta} = \tau_{r\varphi} = 0$. For pure shear, the problem can be determined statically and the corresponding system is of the hyperbolic type (D. D. Ivlev, 1966). Problems dealing with a cone and the pressing of a wedge in the plane of a die have also been considered.

Using the characteristics of the stressed and deformed states that were discovered in the well-known solution of L. Prandtl for the compression of a thin plane layer, A. A. Il'yushin (1954) developed a general theory for the flow of a thin plastic layer on nondeformable surfaces. The equations that were derived were applied to the calculation of a number of problems in which metals are treated by pressure.

The available solutions for the layer apply to the final stage of plastic flow when large tangential stresses are developed on the contact surface. The change in the stressed state in intermediate layers as the load increases (from simple compression to a complex stressed state in the final stage) was studied by L. M. Kachanov (1954, 1962).

In addition to the traditional formulation of the problem of finding the limiting load for a body of a given shape, optimal design problems ("limiting designs") are also of interest. By this is meant the selection of the contours of the body with the required load bearing capacity while satisfying simultaneously certain additional optimality conditions. Usually this condition is the minimum weight requirement. This problem, in the absence of appropriate constraints on the possible outlines of the body, is generally undefined. When rod systems (grids, frames) are considered, the problem is easily reduced to a mathematical programming problem. This method is used in the solution of a number of engineering problems in structural mechanics. The limiting design for bodies with minimum weight with a more complex configuration is much more difficult. Problems of this type have not been studied extensively.

A presentation of the contemporary state of the theory of optimal design and the corresponding literature references are available in the book by M. I. Reitman and G. S. Shapiro (1966).

3.3. Limiting Load Theorems

We will restrict ourselves to a study of small deformations and we will ignore changes in the geometry of the body during the deformation process. Incidentally, under certain known conditions, the results that are obtained can be extended to problems of steady state plastic flow.

The concept of a limiting load ("plastic fracture," "load bearing capacity") for an ideal plastic body is of great practical importance. Calculations based on the limiting load make it often possible to determine more correctly and economically the dimensions of structures and equipment. The concept of a limiting load can be approached in two ways. One can start directly with the scheme of the rigid-plastic body, in which case the instant when the limiting load is attained will correspond to the instant when plastic flow begins, the deformations are small, and the body has the initial configuration.

The other approach is based on the concept of an elastoplastic body. Here the limiting load corresponds to the final stage of the elastoplastic deformation of the body which is often accompanied by large (sometimes infinitely large) deformations (for example, during bending and torsion). In fact, this process is not investigated and the final state of the body is determined immediately when the changes in its configuration are small. This can be justified by the relative smallness of the deformation of an elastoplastic body under loads which are close to the limiting load. In both cases, the theorems are identical and the discussion pertains only to

the interpretation of the final results. We will start out with the concept of a rigid-plastic body which does not require discussion and which is internally more consistent. For this scheme the formulation of the concrete boundary value problems is also more natural. Of course, it must not be forgotten that all the assumptions inherent in the idea of a rigid-plastic body and the usefulness of this concept must be subjected to an analysis every time. Important problems about the suitability of structures related to the presence of residual stresses cannot be evaluated on the basis of this scheme. This problem brings us inevitably back to the elastoplastic body.

Suppose that a rigid plastic body is subjected to the action of forces F_n that are given on a part of the surface of the body S_F and the velocities v are prescribed for points on a part of the surface of the body S_v. The body forces X_i are omitted for the sake of simplicity. Since continuous fields are not of great interest, it is assumed that the stresses are discontinuous on some surfaces S_k, and so are the velocities on some surfaces S_1.

Let v' be any kinematically possible field which is discontinuous on some surfaces S_1'. Then

$$\int F_n v \, dS_v \leqslant k \int H' \, dV - \int F_n' v' \, dS_F + k \int \Delta v_t' \, dS_1', \qquad (3.6)$$

where the last integral extends over all discontinuity surfaces S_1', $\Delta v_t'$ is the absolute value of the jump in the tangential velocity component v' and H' is the intensity of shearing stress velocity

This important inequality which characterizes the minimum properties of the real velocity field has been proved rigorously by A. A. Markov (1947) in the case of continuous velocity fields. In the terminology of structural mechanics, the minimum properties of real displacements have been pointed out earlier by A. A. Gvozdev (1938). The contemporary formulation crystallized as a result of the later studies by G. Greenberg, D. Ch. Drucker, V. Prager, R. Hill and other authors.

The second inequality which follows from comparing the real stress field σ_{ij} with any statically possible field σ_{ij}' inside the yield circle or on it has the form

$$\int F_n v \, dS_v \geq \int F'_n v \, dS_v - \int (k \pm \tau') \Delta v_t \, dS_k, \tag{3.7}$$

where F'_n is a surface force on S_v corresponding to the selected field σ'_{ij} and $\tau' \leq k$ is the tangential component of the field σ'_{ij} on the discontinuity surfaces S_1 in the direction of the relative velocity vector.

The theorem on the maximal properties of the real stressed state has a relatively long history. For certain problems in structural mechanics, this theorem was already stated in the work of G. Kazinchi (1934). A clear formulation of the theorem in the terminology of structural mechanics is available in the study of A. A. Gvozdev that was mentioned (1938). A rigorous proof of the theorem was given by S. M Feinberg (1948). As was noted recently by V. T. Koiter, the adaptability theorem of E. Melan which was proved by him in 1938, includes in essence the theorem that is discussed here as a special case.

The formulations that were given above refer to a Mises medium. However, the corresponding theorems are easily established for an arbitrary convex yield surface and associated flow law. The significance of this law was emphasized by V. T. Koiter who showed that for a medium obeying the Tresk-St.-Venant condition and the Mises relations (3.2) there are no extremum theorems.

The application of the theorems that were stated is especially simple in the case of a proportional load (the external forces increase proportionally with some parameter m), and the velocities are assumed to be zero on a part of the surface S_v (supports). The kinematically possible coefficient m_k corresponds to the kinematically possible field v'. To the statically possible stressed state σ'_{ij} corresponds the static coefficient m_s. The limiting load coefficient m_* which corresponds to a true value is bounded below and above

$$m_s \leqslant m_* \leqslant m_k. \tag{3.8}$$

Thus, we must find a kinematic mechanism for the plastic fracture, for which the boundary m_k is as small as possible. On the other hand, we must find a statically possible stressed field σ'_{ij} which lies inside the yield circle or on it, for which the boundary m_s is as large as possible.

In the solution of more or less simple problems, m_k and m_s are usually brought closer to one another by guessing the appropriate fields. Sometimes it is even possible to find an upper and lower boundary which coincide. In complex problems it is much more difficult to obtain good results.

Mathematical programming methods which have been used on a wide scale in the last few years are suitable for the construction of the limiting surface. In a number of cases, the yield condition has a linear form (problems in the structural mechanics of rod systems and certain axisymmetric problems). In this case the well developed linear programming apparatus can be applied. Many studies were carried out along these lines both in the USSR and abroad.

3.4. Plane Deformation

The plane deformation is described by the system of equations (St.-Venant, 1870)

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \quad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0, \tag{3.9}$$

$$(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2 = 4k^2, \tag{3.10}$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0, \tag{3.11}$$

$$\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)(\sigma_y - \sigma_x) - 2\tau_{xy}\left(\frac{\partial v_x}{\partial x} - \frac{\partial v_y}{\partial y}\right) = 0. \tag{3.12}$$

This system has two different real families of characteristics which coincide with the slippage lines. The Hencky conditions (1923)

$$\frac{\sigma}{2k} - \theta = \text{const} \equiv \xi, \quad \frac{\sigma}{2k} + \theta = \text{const} \equiv \eta,$$

are satisfied along the latter, where σ is the mean pressure, θ is the inclination of the slippage line and the Heiringer relations are

$$du + v\,d\theta = 0, \quad dv - u\,d\theta = 0,$$

where u, v are the velocity components along the slippage lines. The system of equations (3.9), (3.10) for the stresses is transformed to the form

$$\frac{\partial \xi}{\partial x} + \operatorname{tg}\theta \frac{\partial \xi}{\partial y} = 0, \quad \frac{\partial \eta}{\partial x} \operatorname{tg}\theta - \frac{\partial \eta}{\partial y} = 0.$$

This is a reduced system, i.e., it is transformed to a linear system by a change of variables. After the transformation, an important class of solutions with rectilinear characteristics is obtained (simple stressed states), which is widely used in applications. The well-known study of S. A. Christianovich (1936) investigated the solvability $x = x(\xi, \eta)$, $y = y(\xi, \eta)$ of the solutions and also discussed in detail solutions with straight line characteristics.

The possibility of obtaining a separate solution of equations (3.9), (3.10) for the stresses when certain changes in the boundary conditions are permitted,led in the initial development of this theory to an extension of the solutions of the so-called statically determinable problems in which the velocity field was usually not discussed. Various problems in this field were studied by H. Hencky, L. Prandtl, V. V. Sokolovskiy (1950) and other authors.

Although the simplest discontinuities in the stresses (for example, during bending) have been known for a long time, the significance of discontinuous solutions has been recognized much later, after the 1948 study of V. Prager. The slippage line is the boundary of the plastic region. This fact, which has been used intuitively for a long time was established by R. Hill (and the scheme of rigid plastic body).

Concrete problems are usually solved by a semireversed method. First, the boundary value problem for the stresses is considered and an attempt is made to guess the conditions which are needed. After this, the velocity field is determined and its compatibility with the stress field that was constructed earlier is clarified.

The application of this scheme to contact problems is generally difficult. This approach can be applied if the contact lines and the given conditions on them are simple. If the contact line is a curve, it is practically impossible to guess the contact stresses. In this case, the method

of constructing the compatible stress and velocity fields developed by B. A. Druyanov (1961) and V. V. Sokolovskiy (1961) must be applied with the condition that the structure of the slippage field can be defined. The solution of the problem begins with the determination of the velocity field.

The Masseau finite difference method is usually used in the solution of the boundary value problems. Graphical methods are used less often (V. Prager, 1955, S. S. Golushkevich, 1948). L. S. Agamirzyan (1961) has shown that the analytic solution of boundary value problems using the Rieman method can be effective when metacylindrical functions and their tables are used. Nevertheless, the Masseau method has the advantage of simplicity.

The studies of V. V. Sokolovskiy devoted a great deal of attention to the construction of stress fields around holes near the boundary ("plastic boundary layer").

A large number of various problems which can be broken up into three groups (of course, somewhat arbitrarily) has been studied by the method of characteristics.

These are primarily problems of finding the limiting load. Here changes in the configuration of the body are ignored and the formation of plastic flow is studied. These include, for example, problems dealing with the limiting state of strips weakened by cutouts, the pressing of dies on a plastic body, the compression of a layer, etc.

The second large group consists of problems in steady state plastic flow connected with the description of continuous processes in the treatment of metals (rolling, drawing, pressing, cutting, etc.).

Problems of nonstationary plastic flow with geometric similarity of the flow pattern represent a more narrow class of problems which were studied in detail by R. Hill (similarity problems). An example is the problem of the indentation of a rigid wedge in a halfplate.

In those cases when the solutions are accompanied by the construction of the appropriate velocity field, the load that is found is the upper boundary. It is not possible to present here all or even a large part, of the many problems that were solved using this method by Soviet scientists. We only mention the books by D. D. Ivlev (1966), A. A. Il'yushin (1948), L. M. Kachanov (1969), V. V. Sokolovskiy (1969), A. D. Tomlenov (1953), which give the solution of a great variety of problems.

3.5. Plane Stressed State

The plane stressed state problem which arises in thin plates subjected to the action of loads in the middle plane is somewhat more complex. The case of a plane stressed state is also important because such a state occurs during the bending of thin plates and shells.

Unlike in the elastic problem where the mathematical formulations for the case of a deformation and the plane stressed state are identical, these two problems are different in plasticity theory. The stress components σ_x, σ_y, τ_{xy} must satisfy the differential equilibrium equations (3.9) and the Mises condition

$$\sigma_x^2 - \sigma_x\sigma_y + \sigma_y^2 + 3\tau_{xy}^2 = 3k^2 \tag{3.13}$$

or the Tresk-St.-Venant condition

$$\max\{|\sigma_1 - \sigma_2|, |\sigma_1|, |\sigma_2|\} = 2k. \tag{3.14}$$

The first equation describes an ellipse in the plane of principal stresses σ_1, σ_2 (Fig. 4), and the second the hexagon inscribed in it

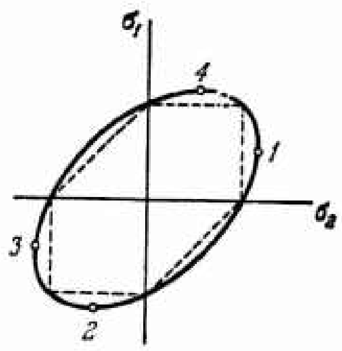

Fig. 4

The system of equations for the stresses was studied by V. V. Sokolovskiy (1945). With the Mises condition, the system may be hyperbolic (for the interior points of the arcs 1-2 and 3-4, here $|\sigma| < \tau_{max}$), parabolic (points 1, 2, 3, 4) and elliptic (for the interior points of the arcs 2-3 and 4-1, here $|\sigma| > \tau_{max}$).

In the hyperbolic case, the characteristics are not orthogonal and do not coincide with the slippage lines. At the same time the system of equations for the stresses is reducible and simple integrals exist which correspond to straight line families of characteristics.

The parabolic case is distinguished by its simplicity. Here the principal stresses are constant and only the inclination of the principal plane is unknown.

In the elliptic case the construction of a solution is connected with great difficulties.

The equations for the velocities and the discontinuous solutions were analyzed in 1952 by R. Hill in the hyperbolic and parabolic cases. He determined the characteristics of the discontinuities in the velocities which occur in the plane stressed state, the normal velocity component is also discontinuous which leads to a local narrowing ("neck") or thickening ("cylinder") on the characteristic.

The solution of rigid-plastic problems, provided it can be obtained without studying the elliptic region, is obtained generally using the same technique as in the case of a plane deformation, although the technique is somewhat more complex. Many concrete problems have been studied in the articles of V. V. Sokolovskiy (1950), R. Hill, A. P. Green, G. Ford and G. Lianis.

A considerable simplification of the mathematical formulation of the problem is obtained by using the Tresk-St.-Venant plasticity condition. The corresponding system of equations for the stresses was studied by V. V. Sokolovskiy (1945). For $\sigma_1\sigma_2 < 0$ it is of the hyperbolic type and it coincides with the equations for the plane deformation. On the horizontal and vertical edges of the hexagon, the system of equations is of the parabolic type, and it is easily integrated. Different types of slippage surfaces correspond to different types of equations. The associated flow law can be used to derive the equations for the velocities. Further attempts have been made along these lines, namely the attempt was made to select an approximate plasticity condition for which the system of equations is hyperbolic everywhere. In particular,

such a condition was proposed by R. Mises. According to his proposal the ellipse is approximated by two branches of parabolas. However, it should be noted that this approximation is relatively coarse, and that the Tresk-St.-Venant condition simplifies the formulation of the problem to such an extent that there is no need for further simplification of the mathematical formulation.

The Tresk-St.Venant condition with the associated flow law has been widely used in the analysis of the bending of plates and shells.

In the last two decades, the plane problem with a plasticity condition of general form

$$f(\sigma_1, \sigma_2) = \text{const.} \tag{3.15}$$

has been developed to a considerable extent.

The deformation rates are usually determined by means of the associated flow law. We will point out a number of reasons which stimulated an analysis of this problem. The different yield conditions in the case of a plane deformation and a plane stressed state, the somewhat different limiting conditions in the mechanics of soils, make the analysis of the problem with a general plasticity condition quite natural. The quest for simple approximate solutions that can be obtained for particular formulations of the yield condition has some value. Finally, the important case of the generalized plane deformation when a long cylindrical body undergoes an axial deformation with a constant velocity ($\dot{\varepsilon}_z$ = const) caused by a given axial load can be related to some extent to a generalized plasticity condition.

We return to the system of equations for the plane problem with plasticity condition (3.15), and note that generally this system cannot be obtained from the equations for the three-dimensional problem. In spite of the disadvantage that was mentioned, a mathematical analysis of the system is undoubtedly of great interest. The type of system depends on the form of the yield curve and the position on it. For "hyperbolic points" on the yield curve, the theory of characteristics and discontinuities was developed (J. Mandel, H. Heiringer, R. Hill, V. V. Sokolovskiy, et al.). Various cases of the yield condition (3.15) were studied. Above, we already mentioned the parabolic Mises condition. We also mention the case of a cycloid (V. V. Sokolovskiy, 1950) for which the system is hyperbolic everywhere and the characteristics are straight lines.

3.6. Torsion

The problem of the pure plastic torsion of a prismatic rod which was mainly studied by A. Nadai (1923) is particularly simple. The stress function F(x, y) satisfies the differential equation

$$\left(\frac{\partial F}{\partial x}\right)^2+\left(\frac{\partial F}{\partial y}\right)^2=k^2$$

and the condition that F be constant on each of the bounding contours. The stress surface z = F(x, y) is a surface with a constant slope, a "roof" constructed on the rear contour and it is determined without particular difficulties. The edges and conical point on this stress surface correspond to the lines and points of discontinuity of the tangential stresses τ_{xz}, τ_{yz}. The magnitude of the tangential stress vector is constant and its direction changes in jumps. The limiting torque is also calculated relatively simply.

When the torsion is complicated by additinal axial elongation (or bending), the problem becomes more difficult. The axisymmetric problem of the combined torsion and elongation of a circular cylindrical rod has been studied in detail.

A generalization of the torsion of a straight rod is the problem of the torsion of a section of a circular ring with a constant cross section, which was considered by V. Freiberger, and also by A. Wang and V. Prager (in 1953-1954). The stress components $\tau_{r\varphi}$ and $\tau_{z\varphi}$ which are different from zero (in a cylindrical coordinate system r, φ, z, the z axis is oriented along the axis of rotation of the ring) satisfy the yield condition using the substitution

$$\tau_{r\varphi}=k\sin\psi,\quad \tau_{z\varphi}=k\cos\psi \tag{3.16}$$

The function $\psi(r, z)$ is determined from the differential equation

$$\sin\psi\frac{\partial\psi}{\partial z}-\cos\psi\frac{\partial\psi}{\partial r}-2\frac{\sin\psi}{r}=0. \tag{3.17}$$

Unlike in the torsion of a straight rod, here the characteristics can be curvilinear.

The problem of the torsion of a straight circular rod with a variable diameter (the z axis is oriented along the axis of the rod), which was studied by V. V. Sokolovskiy (1945, 1950) leads to a similar system of equations for the stresses. Here the same stress components $\tau_{r\varphi}$, $\tau_{z\varphi}$ are different from zero.

The solution of equation (3.17) with the same boundary conditions must be found and the velocity field is easily constructed. A general analysis of the stress field is not needed in the calculation of the limiting moment. It is easily shown that in the limiting state there is a cutoff in the smallest cross section, a part of the rod above and below the section remains rigid. The exact value of the limiting moment is equal to the value of the limiting moment for the cylindrical rod with the smallest diameter that was mentioned.

3.7. Axisymmetric Problems

During axisymmetric deformation the stress and velocity components of the deformation are independent of the polar angle φ. If we exclude torsion, the angular velocity component $v_\varphi = 0$. The differential equilibrium equations in cylindrical coordinates r, φ, z have the form

$$\left.\begin{aligned}\frac{\partial\sigma_r}{\partial r}+\frac{\partial\tau_{rz}}{\partial z}+\frac{\sigma_r-\sigma_\varphi}{r}=0,\\ \frac{\partial\tau_{rz}}{\partial r}+\frac{\partial\sigma_z}{\partial z}+\frac{\tau_{rz}}{r}=0.\end{aligned}\right\} \tag{3.18}$$

The Mises yield condition is:

$$(\sigma_r-\sigma_\varphi)^2+(\sigma_\varphi-\sigma_z)^2+(\sigma_z-\sigma_r)^2+6\tau_{rz}^2=6k^2. \tag{3.19}$$

The components of the deformation velocity

$$\dot{\varepsilon}_r=\frac{\partial v_r}{\partial r},\quad \dot{\varepsilon}_\varphi=\frac{v_r}{r},\quad \dot{\varepsilon}_z=\frac{\partial v_z}{\partial z},\quad \dot{\gamma}_{rz}=\frac{\partial v_r}{\partial z}+\frac{\partial v_z}{\partial r}$$

are related to the stress components by the Mises relations. We have a system of six equations in the six unknown functions v_r, v_φ, σ_r, σ_z, σ_φ, τ_{rz}. Generally, this system is of the elliptic type (R. Hill, 1948), and the formulation and solution

of the boundary value problems is connected with great mathematical difficulties. Only special particular solutions have been obtained.

Often the total plasticity hypothesis is used in the solution of various engineering problems, i.e., the condition for the equality of the two principal stresses. In this case as H. Hencky has shown in 1923, the problem becomes statically determinate and the system of equations (3.18), (3.19) for the stress components will be hyperbolic. The characteristics coincide with the slippage lines in the r, z plane. Using techniques which are similar to the techniques used in the case of a plane deformation, various special problems can be considered. Generally the velocity field when the Mises relations are used cannot be constructed because of the redundant equations. Because of this, it is difficult to evaluate such solutions since usually they cannot be referred either to the possible static or kinematic solutions.

In individual special cases, the total plasticity condition can sometimes be justified. Apparently, the solutions when the total plasticity condition is used, give in a number of cases, an acceptable approximation to the limiting load.

The well-known prospects for the analysis of the axisymmetric problem are realized when a transition is made to the Tresk-St.-Venant plasticity condition and the associated flow law. When this is done flows corresponding to the stresses on the edges of the yield prism and on its faces must be analyzed separately. In the first case, the problem is statically determinate and hyperbolic and the characteristics coincide with the slippage lines. The use of the associated law makes it possible to formulate the problem of finding the compatible velocity field. Solutions of this class of problems were discussed by R. T. Shield, D. D. Ivlev (1959) and by other authors. They can be considered to be kinematically possible (provided the velocity field has been defined) and consequently interpreted as the upper boundary. Using the complete plasticity condition, the problem of the total pressing in of a smooth circular die into a halfspace was studied (A. Yu. Ishlinskiy, 1944, R. T. Shield, 1957). The case of an annular die has also been studied (D. D. Ivlev, 1966).

An analysis of the flow corresponding to the stresses on the face of a prism made by A. D. Cox, G. Eison and G. G. Hopkins (1961), H. Lippman (1962) and other authors has shown that it is kinematically determinate. According to the associated flow law, the velocity of the principal deformation in the direction of the mean principal stress is zero on the edge. This condition gives an additional equation for the velocities. As a result we obtain a system of three differential equations for the velocity components v_r, v_z and the angle ψ which defines the principal

direction. This system is of the hyperbolic type, its characteristics are orthogonal and coincide with the trajectories of the principal stresses in the centerline section r, z.

Unlike in the case of total plasticity, here the solution is connected with well known difficulties which are due to the comparative complexity of the system of equations for the velocities and the necessity of constructing a compatible field of stresses. Several particular solutions were obtained which are characterized by simple velocity fields.

Progress in the solution of the axisymmetric problem will probably be connected with the development of a scheme based on the Tresk-St.-Venant plasticity condition. It is necessary to combine properly the flows on the edges and faces of the yield prism, which requires a thorough analysis of the velocity field and of possible discontinuities. Such solutions are a good approximation of the limiting load.

3.8. Anisotropy and Inhomogeneity

Two directions have developed in the theory of an anisotropic plastic medium. In the first, the plasticity condition is introduced in the form of a generalized quadratic Mises condition for an isotropic medium. The second approach is based on a generalized Tresk-St.Venant plasticity condition.

The problem of an anisotropic plastic medium was first formulated by R. Mises in 1928. The plasticity condition was formulated by him as the condition that a certain quadratic form in the stresses whose coefficient k_{ij} characterizes the anisotropy of the medium be constant, namely:

$$\begin{aligned} f = {} & k_{12}(\sigma_x-\sigma_y)^2 + k_{23}(\sigma_y-\sigma_z)^2 + k_{31}(\sigma_z-\sigma_x)^2 + \\ & + \tau_{yz}[k_{24}(\sigma_x-\sigma_y) + k_{34}(\sigma_x-\sigma_z)] + \tau_{xz}[k_{35}(\sigma_y-\sigma_z) + \\ & + k_{15}(\sigma_y-\sigma_x)] + \tau_{xy}[k_{16}(\sigma_z-\sigma_x) + k_{26}(\sigma_z-\sigma_y)] + \\ & + k_{45}\tau_{xz}\tau_{yz} + k_{56}\tau_{xz}\tau_{xy} + k_{64}\tau_{xy}\tau_{yz} + k_{44}\tau_{yz}^2 + k_{55}\tau_{xz}^2 + \\ & + k_{66}\tau_{xy}^2 = \text{const}. \end{aligned} \tag{3.20}$$

This form is invariant with respect to a transformation of the coordinate system to a different equivalent crystallographic system and an applied hydrostatic pressure. It involves 15 unknown constants. For an isotropic medium, condition (3.20) becomes the well-known Mises condition. The deformation rates are determined by the associated flow law

$$\dot{\varepsilon}_{ij} = \lambda \frac{\partial f}{\partial \sigma_{ij}}. \qquad (3.21)$$

A special case of anisotropy involving 6 constants was studied in detail in 1948 by R. Hill:

$$k_{12}(\sigma_x - \sigma_y)^2 + k_{23}(\sigma_y - \sigma_z)^2 + k_{31}(\sigma_z - \sigma_x)^2 + k_{44}\tau_{yz}^2 + \\ + k_{55}\tau_{xz}^2 + k_{66}\tau_{xy}^2 = \text{const.} \qquad (3.22)$$

A series of studies were made in the USSR and abroad which were based on the conditions that were presented. We mention here the studies of Ye. V. Makhover (1947), M. Sh. Mikeladze (1951), V. O. Geordzhayev (1900), R. Hill, V. Ol'shak and other authors, which include torsion, the bending of plates and shells and other special problems.

The extension of the Tresk-St.-Venant condition to the case of an anisotropic body is more complex. The most general results along these lines were obtained by D. D. Ivlev (1959, 1966). For an anisotropic body, piecewise-linear conditions often not only lead to simpler boundary value problems, but also possibly have advantages from the physical standpoint (at least for crystals).

When the yield condition is formulated, it is assumed that it is independent of the hydrostatic pressure and that it is determined in the space of the principal stresses by a surface which is not concave. In the given case, this surface will be a hexagonal prism whose faces are parallel to tne hydrostatic axis $\sigma_1 = \sigma_2 = \sigma_3$. The yield limits in the plasticity condition are regarded as functions of the direction cosines of the principal axes of the stresses relative to the principal axes of the anisotropy. For example, in the case of a plane deformation, the yield condition has the form

$$(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2 = 4k^2(\psi),$$

where ψ is the angle between the direction of the largest principal stress and the x axis, and the x, y, axes are oriented along the principal axes of anisotropy. The relation $k(\psi)$ is assumed to be known.

Studies of torsion, the plane problem and some other problems were made by D. D. Ivlev (1959), G. I. Bykovtsev (1961) and M. S. Sarkisyan (1960, 1961).

With regard to plastic properties, real bodies are always heterogeneous to some extent. This heterogeneity may be due to various causes: The dependence of the yield point on the temperature field, the variable hardening, the effect of neutron radiation, etc. Sometimes the bodies consist of different materials (discontinuous heterogeneity). The use of the heterogeneity of the plastic properties makes it often possible to increase the strength of the bodies, which gives rise to certain special problems in the problem of optimal design.

As a result of the heterogeneity in the rigid-plastic body scheme, the yield point is no longer constant but becomes a given function of the coordinate (which is either continuous or discontinuous). This introduces considerable complications in the flow pattern.

The theory of plasticity of heterogeneous bodies was developed most intensely in Poland in the studies of the W. Olschak School.[1] In the Soviet Union a number of studies in the theory of rigid-plastic heterogeneous bodies were made by B. A. Druyanov (1959), A. I. Kuznetsov (1958, 1960), M. A. Zadoyan (1962), Yu. R. Lepik (1963) and other authors. An exhaustive bibliography is given in the survey that was mentioned above.

3.9. Elastoplastic Body

In many cases it is important to know the stresses and strains in the body which made a partial transition to the plastic state. There are many reasons for the interest in such elastoplastic problems. Thus, in the neighborhood of holes, cut and other "stress concentrators" local plastic deformations are formed. The corresponding fields must be known in order to evaluate the strength and the local deformations. Local plastic deformations occur in the majority of contact problems. The determination of the residual stresses and strains which are the basis for the calculation of the radial expansion in structures, also require a study of the elastoplastic state.

---

1. See the Survey: W. Olschak, Y. Rychlevski and W. Urbanovski, "Theory of Plasticity of Heterogeneous Bodies," 1962 (Russian Translation: Moscow, 1964).

The problem of temperature stresses also leads to elastoplastic problems. Finally, the solution of elastoplastic problems makes it possible to evaluate the rate of growth of the deformation approximations to the limiting state, and the applicability of the rigid-plastic solutions can be evaluated on concrete examples.

The suitable basis for the study of elastoplastic problems are the equations of flow theory (the Prandtl-Reiss equations)

$$d\varepsilon_{ij} = d\varepsilon^e_{ij} + d\lambda s_{ij}. \tag{3.23}$$

The elastic deformation increments $d\varepsilon^e_{ij}$ are calculated from Hooke's law. The stresses satisfy the Mises plasticity condition (3.3). In the plastic zones, equations (3.23) are valid, and in elastic zones $d\lambda = 0$ and equations (3.23) become Hooke's law. On the boundary of these zones, the plastic deformations are zero and the continuity conditions for the stresses, deformations and displacements are satisfied. The solution of such mixed problems is exceptionally difficult and it can only be obtained with the aid of computers. The usual technique which is used is to study the ("step by step") development of the elastoplastic state as the loading parameter increases, and various variants of the grid method or variation methods can be used to determine the current state.

Methods for the solution of a number of important engineering "one-dimensional" problems have been developed to a considerable extent (axisymmetric deformation of pipes, rotation of discs, bending of a straight and circular rod, etc.). In relatively few cases an analytical solution can be obtained.

When the external loads, the support conditions and the configuration of the body are sufficiently simple, we can expect that the load in the plastic zone approaches a simple load, and we can start out with the equations from deformation theory of plasticity, the Hencky equations

$$\varepsilon_{ij} = \varepsilon^e_{ij} + \varphi s_{ij}, \tag{3.24}$$

(instead of (3.23)) which considerably simplify the solution of the problem. Usually important engineering "one-dimensional" problems (pipes, discs, circular plates, etc.) are studied on

the basis of the deformation theory. This approach has been used extensively in the Soviet Union. A comparison of the available solutions based on both theories shows, as a rule, slight discrepancies.

Because of the difficulties of constructing the solutions in the plastic zones, the total plasticity condition is rarely used in the latter. This technique introduces distortions which usually cannot be estimated in this case.

The theorems on the adaptability of elastoplastic bodies under the action of cyclical loads are a slight digression from the problems that were touched on. Under certain known conditions, plastic deformations can occur in each cycle even when the loads are below the limiting loads. Plastic deformationof this type leads to fracture. Adaptability theorems give the boundaries for the change in the loads within which repeated plastic deformations do not occur due to the propitious field of residual stresses formed during the first loads. Although the use of these theorems does not require that the elastoplastic problem be solved, they are based on the model of an elastoplastic body in which residual stresses can be formed.

In conclusion we mention another difficult and still not very well formulated but important practical problem, the problem of elastoplastic vibrations.

3.10. Elastoplastic Torsion

This comparatively simple elastoplastic problem was studied in the early work of A. Nadai (1923), who developed a method for determining the solution experimentally on the basis of the membrane analogy. The first analytical solutions were obtained by E. Treftz in 1925 which dealt with the determination of the plastic zone formed near the reentrant angle during the torsion of a rod with an angle profile. Treftz applied the method of conformal mapping to the elastic zone of the section. F. S. Shaw applied successfully the method of grids in 1944 to the solution of the same problem and some other problems (using the R. Southwell relaxation techniques).

L. A. Galin (1944) developed a direct method for the solution of the problem of elastoplastic torsion of rods with a polygonal cross section.

Various variants of the semireversed method used for the first time by V. V. Sokolovskiy (1942) for the solution of the problem of torsion of an oval body which was nearly elliptical, turned out to be effective. First the form of the elastic kernel is given and then the plastic region for it is constructed in the appropriate manner. Examples of such solutions are given in

the studies of L. A. Galin (1949) and R. Mises. The same technique was used by V. Freiberger (1956) in the solution of the problem of the elastoplastic torsion of a circular ring with a nearly circular cross section. Variation methods are also useful in the solution of the elastoplastic torsion problem.

The existence and uniqueness theorem for the solution of the elastoplastic torsion problem of a rod with an oval cross section has been proved in the studies of B. D. Annin (1968) who also developed the algorithm for the numerical solution.

For a simply connected profile, unloading does not occur at any point of the cross section as the torque increases as shown recently by F. G. Hodge. However, during the torsion of a rod with a multiply-connected cross section, unloading can occur under certain known conditions. Naturally, this fact makes the analytical solution of elastoplastic problems in the class that was mentioned much more difficult.

Mathematically the problem of an elastoplastic nonplanar deformation is close to the elastoplastic torsion problem. Here a pure shear state also occurs but the stresses are given on the contour of the body. The studies of G. P. Cherepanov (1962) studied the elastoplastic problem for an arbitrary cutout in an infinite plane with the aid of methods of the theory of functions of a complex variable. The stresses on the contour of the cutout are given and it is assumed that the plastic zone contains the hole completely.

3.11. Plane Problem

The elastoplastic problems under plane deformation conditions are considerably more difficult. To reduce the difficulties, the formulation itself is usually simplified. In particular, it is assumed that in the plastic zone, $\sigma_z = 1/2\ (\sigma_x + \sigma_y)$, i.e.,(if the Mises yield condition is used) that the material is incompressible. Next, the state corresponding to finite values of the loads is considered and the "step-by-step" development of the elastoplastic state is not considered as the loads increase. Meanwhile, unloading in various parts of the plastic zones may occur during the process. Therefore, it cannot be said which constraints must be satisfied by the loads, since the finite state that is considered has been attained. Nevertheless, the few elastoplastic problems whose solutions have been constructed under the conditions that were mentioned are undoubtedly of interest.

An idea about the nature of the solution can be obtained on the basis of the analogy with the bending of a plate compressed on a rigid body which was given by L. A. Galin (1948).

Among elastoplastic problems that are not one-dimensional we should first mention the elegant closed solution of the problem of the tension of a plane with a free circular cutout found by L. A. Galin (1946). Tensile stresses p and q are acting at infinity in the direction of the x and y axes. It is assumed that the plastic zone completely contains the hole. This imposes certain constraints on the loading parameters p, q. The biharmonic property of the stress function in the plastic zone adjacent to the circular cutout is used in the solution.

The solution of L. A. Galin was generalized with certain additional conditions to the case of a plastic heterogeneous medium and to the case of a nonuniform temperature field. Some new results can be obtained using the method of a small parameter. However, these techniques cannot be used to extend the conditions for the problem considerably.

An approximate method for the solution of an elastoplastic problem for a plane with a cutout in the reversed formulation (the plastic zone is given) was developed by P. I. Perlin (1960).

Methods from the theory of functions of a complex variable are also used in the studies of G. P. Cherepanov (1963, 1964) but the assumption that on the unknown separation boundary of the elastic and plastic zones, the stresses are the corresponding second derivatives of the biharmonic function is no longer used. It is assumed that the stresses that were mentioned are known functions of the coordinates. The method developed for the solution has been applied to the analysis of the elastoplastic problem of the biaxial stressing of a thin plate with a circular cutout (plane stressed state) with the Tresk-St.-Venant plasticity condition in the case when $\sigma_\varphi \geq \sigma_r \geq 0$.

The studies of B. D. Annin (1968) that were recently published studied the problem of the elastoplastic distribution of the stresses in a plane with holes.

Finally, we will mention the class of problems connected with the generalized plane deformation which is relatively undeveloped. This class deals with the equilibrium of long cylindrical bodies subjected to additional axial stressing (unlike in the plane deformation when the displacement along the axis is equal to zero). For an elastic body, this problem reduces to the case of plane deformation resulting

from applying an appropriate axial stress. In elastoplastic problems, the state of the generalized plane deformation must be considered. From among problems of this type only the practical important problem of a thick-walled pipe subjected to the action of internal pressure and an axial force has been studied in detail.

3.12. Adaptability

Generally after the initial loading and unloading there will be residual stresses in the elastoplastic body. If the same load is applied again, new plastic deformations will occur under certain known conditions as a result of selfhardening.

Under the action of several independently varying loads, the question arises what are the safe boundaries for their variation which will guarantee that repeated plastic deformations do not occur.

Repeated loads may lead to two types of fractures: 1) fracture as a result of repeated plastic deformations with changing signs (plastic fatigue); 2) fracture as a result of an increased plastic deformation in one direction (progressive deformation).

When the field of residual stresses is selected appropriately, it can facilitate finding a region in which the loads can be varied in any manner without causing new plastic deformations. The body adapts itself within certain limits to the external forces. The adaptability region is determined by two theorems.

The first theorem (the static theorem) was proved in the general formulation in the well-known 1938 study of E. Melan. Suppose that the elastoplastic body occupies the volume V bounded by the surface S. The load F_n is given on a part of the surface S_F and the velocities on the part S_v are zero. We will denote by σ_{ij}^e the stresses in the body assuming ideal elasticity, and by ρ_{ij} some field of residual stresses which are independent of time.

According to Melan's theorem, the body adapts itself if the total stressed state $\sigma_{ij}^e + \rho_{ij}$ does not violate the yield condition, i.e.,

$$f(\sigma^0_{ij} + \rho_{ij}) \leqslant k^2. \qquad (3.25)$$

Conversely, adaptability cannot occur if a field of residual stresses ρ_{ij} which is independent of time and for which inequality (3.25) is satisfied does not exist.

The second theorem (the kinematic theorem) was proved by V. T. Koiter in 1956.

The body adapts itself if for all admissible velocity cycles of the plastic deformation $\dot{\varepsilon}_{ij0}$ and all possible changes in the loads F_{ni} within the given range, the inequality

$$\int_0^T dt \int F_{ni} v_{i0} \, dS_F > \int_0^T dt \int W(\dot{\varepsilon}_{ij0}) \, dV, \qquad (3.26)$$

holds, where v_{i0} is the kinematically possible velocity field which vanishes on S_v, T is some time interval, and $W(\dot{\varepsilon}_{ij0})$ is the depth of the plastic deformation at the admissible velocities. Conversely, adaptability does not exist if a cycle $\dot{\varepsilon}_{ij0}$ and a program for changing the loads can be found which violate inequality (3.26).

The determination of the adaptability boundaries is much more complex than the calculation of the limiting loads. Analytical solutions can only be obtained for the simplest problems. According to Melan's theorem, a field of residual stresses must be found which for the yield condition (3.25) makes the region in which the loads vary as large as possible. This formulation leads to mathematical programming problems. The application of mathematical programming methods to the adaptation problem is similar to their application used in finding the limiting load. As before, the apparatus of linear programming can be used in a number of important cases.

In individual cases, the cycle time can be long. Aging, creep and other phenomena may develop in this time, which have an effect on the yield point, the field of residual stresses and, hence, on the adaptability region. In a number of engineering problems, it is important that these effects be taken into account.

The adaptability problem during cyclic changes in the temperature field is of great applied interest. A plastic deformation which changes sign can occur as a result of heat transfer, which leads to fracture after a comparatively small number of cycles ("thermal fatigue"). A gradual dangerous increase in plastic deformations can also occur.

The extension of Melan's theorem to the case of variable temperature fields presents no difficulties. It was introduced by V. Prager in 1956 and independently by V. I. Rosenblum in 1957. Unlike in the isothermal case, here σ^{e}_{ij} must be interpreted as the solution of the corresponding thermoelastic problem.

Koiter's kinematic theorem has also been generalized to bodies subjected to variable heating (V. I. Rosenblum, 1965). Problems related to the adaptability of structures under variable heating conditions have been studied in detail in the studies of D. A. Hochfeld (1970).

Creep as well as a change in the yield point during the temperature cycles may play an important role. All this complicates considerably the calculations and requires a painstaking analysis even in the case of simple rod systems.

Finally, we note that frequently parts in heavily loaded structures are subjected to variable plastic deformations (i.e., they operate outside the adaptability region). In this connection, it becomes necessary to analyze the changes in the stresses and strains from cycle to cycle. When the fields are nonuniform, problems of this type are very difficult and their formulation and solution are only outlined.

3.13. Hardening Body

Modern structural metals harden considerably. The scheme of an ideal elastoplastic body is then unsuitable. Usually in these cases either the Prandtl-Reiss equations are used with the isotropic hardening condition, or the equations of deformation theory with the law of "a single curve" (the intensity of the tangential stresses is a function of the intensity of the shear). Solutions based on the equations of deformation theory have been developed extensively in the Soviet Union. Foreign studies are characterized by a certain amount mistrust with regard to the use of deformation theory, although its practical significance is not denied. The isotropic hardening law is only suitable for loading paths which are relatively simple. The scheme of a single curve is applicable even to a more narrow range of problems. Therefore,

the solution of boundary value problems on the basis of both theories is limited to the framework of a sufficiently simple load. It is not possible to formulate this condition more accurately. A comparison of the available solutions found on the basis of both theories usually shows small discrepancies.

In the theory of flow the fundamental relations may be given in the form

$$\dot{\varepsilon}_{ij} = (c_{ijkl} + h_{ijkl})\,\dot{\sigma}_{kl}, \tag{3.27}$$

where the c_{ijkl} are elastic constants, and the h_{ijkl} are some functions of the stresses, deformations and the deformation history. Outwardly these relations resemble the Hooke equations for a linear elastic anisotropic body, but the possible appearance of unloading (in which case $h_{ijkl} = 0$) in individual zones and the variability of the h_{ijkl} considerably complicated the solution. Within the comparatively narrow frame of reference of the isotropic hardening scheme that was mentioned above, the case of a load prolonged everywhere becomes important. Then when we consider the deformation process "step by step" and sequentially impart to the load small increments, at each loading stage we can assume that the coefficients in (3.27) depend only on the coordinates (not on the loading parameter or "time"). The solution of the elastoplastic problem reduces in this manner to the solution of a sequence of problems for an anisotropic elastic body with variable coefficients. The application of the scheme in complex (higher dimensional) cases is, of course, connected with overcoming great computational difficulties.

Several additional possibilities of constructing the solutions follow from the variational formulations. We can introduce at each stage quadratic forms of the deformation velocities $\Pi(\dot{\varepsilon}_{ij})$ and the velocities of the stresses $\Pi(\dot{\sigma}_{ij})$.

Then the velocity field v_i corresponds to the minimum of the quadratic functional

$$\int \Pi(\dot{\varepsilon}_{ij})\,dV - \int \dot{F}_{ni} v_i\,dS_F = \min. \tag{3.28}$$

The stress rates $\dot{\sigma}_{ij}$ minimize the quadratic functional

$$\int \Pi(\dot{\sigma}_{ij})\,dV - \int \dot{F}_{ni} v_i\,dS_\sigma = \min. \tag{3.29}$$

To solve the variational equations, various variants of direct methods can be used.

As we already mentioned, the application of deformation theory is justified in the solution of practical problems when the loads are not complex. Then the boundary value problems will involve finite values of the strains and stresses which is much simpler than in the theory of flow. It was possible to construct solutions of many special problems and to prove the existence of a solution (classical or generalized) in certain problems of elastoplastic equilibrium.

The solutions are obtained with the aid of various variants of the method of successive approximations (A. A. Il'yushin, 1948, I. A. Birger, 1951, et al.) or numerically. In the first case, the nonlinear terms are brought over to the right members of the equations or are included in the "elasticity coefficients" and subsequently the method of successive approximations is applied in one form or another. At each approximation stage, the linear elasticity theory problem must be solved with "additional" body forces (method of elastic solutions) or with variable elasticity coefficients ("method of variable elasticity parameters"). These processes are very laborious and in higher dimensional problems more than one or two approximations can seldom be constructed. The convergence of the greater part of the processes that are used has not been studied. The convergence of the method of elastic solutions under certain conditions has been established in the studies of A. I. Koshelev (1955) and S. G. Petrova (1957).

The equations of deformation theory can be represented with the aid of the work of the deformation Π and the additional work R in the form

$$\sigma_{ij} = \frac{\partial \Pi}{\partial \varepsilon_{ij}}, \quad \varepsilon_{ij} = \frac{\partial R}{\partial \sigma_{ij}}. \tag{3.30}$$

It can be easily shown that the real field of displacements u_i minimizes the total energy of the system:

$$\int \Pi\, dV - \int F_{ni} u_i \, dS_p = \min. \tag{3.31}$$

The minimum of the additional work (L. M. Kachanov, 1940, 1942) is attained for the real field

$$\int R\, dV - \int F_{ni} u_i \, dS_u = \min. \tag{3.32}$$

Variational equations are useful in the construction of approximate solutions using the Ritz method. The Ritz system will be nonlinear and its solution cannot always be obtained in practice, not to mention the difficulties connected with setting up the system itself. The modified Ritz method is more useful (L. M. Kachanov, 1959) . In it the coefficients are made more precise while considering a sequence of minimization problems for quadratic functionals. This technique eliminates the laboriousness of the solution and increases the number of approximations. A slightly different modification of the Ritz method was proposed by A. A. Il'yushin (1961). Generally, different direct methods based on variational equations (for example, the method of lines, the calculus of variations-difference method, etc.), can be used in the solution of nonlinear problems if these methods are properly combined with the method of successive approximations. This remark applies to the use of the B. G. Galerkin method.

The method of a small parameter which can be used to extend the range of application of the solutions that were found earlier stands somewhat apart. It can be applied both to the differential and variational equations of the problem. Thus, when the axisymmetric solutions are known, it is possible to study problems which are nearly axisymmetric with the aid of this method (with respect to the loads or contours of the body, inhomogeneity, etc.). This technique cannot be used to extend the range of the solution noticeably. Only a few problems of this type are of real interest in applications. Among these we may include the problem of slightly oval and eccentric pipes. the problem of the rotation of slightly eccentric disc, etc.

With regard to numerical methods, these must be used primarily for the solution of problems that are important in practice and involve a small number of geometric parameters. Here we must first mention the class of problems about the stress concentration beyond the elasticity limit.

It is also useful to mention certain possible experimental solutions in plasticity problems with hardening connected with the photoelastic method. We have in mind the application of the method of photoelastic coatings and the photocreep method.

Finally, the useful analogy between problems in steady state creep and problems for a hardening body based on deformation theory with the incompressibility condition should also be mentioned. This so-called "elastic analogy" makes it possible to replace the solutions by the experimental data and vice versa.

3.14. Conclusion

This section discussed only certain ("classical") boundary value problems. Problems in stability, dynamics, the theory of shells and plates were not discussed. In spite of the great effort and the undoubtedly rapid progress in the development of plasticity theory, the solutions of many problems are not known. Partially for this reason many important applied problems remain unsolved. Even the relatively simple problem of determining the load bearing capacity on the basis of the rigid-plastic scheme is connected with mathematical difficulties. The role played by numerical methods cannot be questioned in overcoming the latter. In particular, the use of extremal properties of the limiting load and the mathematical programming methods based on them have great promise. Here it will be useful to emphasize the value of these methods in determining the adaptability region. The optimal design problem based on the rigid-plastic scheme is not understood well, but undoubtedly it has great promise.

Elastoplastic problems with the ideal plasticity condition or isotropic hardening are much more difficult. Here success based on analytical methods should not be expected. Numerical methods with the aid of which the solutions of many important problems can be constructed occupy the first place. In particular, it is worthwhile to mention the class of problems dealing with stress concentrations which is of great interest in evaluating strength.

We should expect that the role played by elastoplastic problems under complex loading conditions, when it is necessary to utilize more fully the changes in the mechanical properties during the plastic deformation, will increase. Boundary value problems when complex loads are applied, during repeated (cyclic) plastic deformations, an analysis of elastoplastic oscillations are some of the problems that must be solved. Problems of this type, unlike the problems mentioned above, have still not been formulated unambiguously mathematically and the efforts of researchers must primarily concentrate in this direction.

Another class of problems of great interest is connected with plastic deformations with accompanying non-mechanical fields (thermoplastic problems, problems for an irradiated body, etc.). The most traditional and most important applied problem is the thermal plasticity problem. Here many approximate solutions based predominantly on the equations of deformation theory were obtained. However, the great variety of thermo-mechanical actions requires the construction and use of much more complex state equations.

Finally, viscosity effects must be taken into account on a greater scale. Viscoplastic media (in various variants) are most often discussed in journals. Next, problems with specific features arise in the study of elastoplastic deformations of new materials with a special heterogeneous structure. These problems, incidentally, are outside the scope of this survey.

CREEP THEORY

by

Yu. N. Rabotnov[1]

(continued)

1. In the compilation of this survey, the author used the published survey of N. N. Malinin dealing with creep problems in machine building. The author received a great deal of help in the selection and treatment of the material on viscoelasticity from L. Kh. Papernik, to whom the author is grateful.

§1. Introduction

The phenomenon of creep in the narrow sense consists of the fact that a body subjected to the action of constant loads deforms slowly with time. The problem in the mechanical theory of creep is to determine the defining equations which connect the mechanical parameters of the state, the stresses and strains. These relations must include in an essential manner certain time operators, either differential or integral operators. The creep process often terminates in the fracture of the body. Therefore, ideally, a mechanical theory of fracture must include elements which make it possible to predict the instant of the fracture.

The phenomenological approach which is characteristic of mechanics does not relieve the mechanics specialist from taking into account the physical process and taking into consideration those internal mechanisms which determine this process. The strain curves which are obtained as functions of time in tensile stress tests under a constant load for

steel at a high temperature, or any plastic (for example, polyethylene.at a normal temperature, concrete, ice, samples of rocks, etc., are very similar. Therefore, some hope exists that it is possible to construct general equations which will be suitable for the description of all materials under any conditions. However, the internal structure of the bodies that were mentioned is very different and the mechanisms which cause creep are also different. The great difference in creep processes in metals and polymers, which, for example, is connected with the difference in the determining mechanisms, can also be detected in a macro experiment. Thus, the creep deformation of steel is irreversible for all practical purposes; after the load is removed, the deformation that accrued does not return or only a small part of it returns. The creep deformation of a polymer when the stresses are not very high is almost completely reversible, and it vanishes when the load is removed after a certain time.

Different circumstances are also encountered in the study of creep in engineering alloys. Materials exist which are structurally stable in a given range of temperatures and time. The creep of such materials is described by relatively simple relations for which a mechanical theory can also be constructed. The matter is much more complex with those alloys which undergo phase conversions during the creep process at a high temperature. A description of the creep of such materials in the terminology of mechanics is connected with considerable difficulties. Different atomic-dislocation grid mechanisms are dominant in different temperature ranges, which follows from physical investigations. Therefore, the creep equations may differ considerably, depending on the field to which they are applied.

In this survey, the main attention will be on the theory of the creep of metals at a high temperature. However, certain theoretical results pertaining to the creep of polymer materials will also be discussed (viscoelasticity theory).

The creep phenomenon was encountered for the first time in connection with operating steam turbines and the problem of limiting the creep became very acute. Given the existing materials, the creep and the limited strength of the materials limited the possibility of increasing the working parameters of machines. The needs of electric power machine engineering led primarily to intense work aimed at producing new heat resistant materials and stimulated a great series of metalo-physical studies whose aim was to detect the creep mechanism

of metals. The contemporary state of the problem in this field is discussed, for example, in the survey article by V. L. Indenbom, A. N. Orlova and V. M. Rozenberg (1965) and in the book by V. M. Rozenberg (1968).

At the same time the designers were confronted with the problem how to use the available materials in the structures, how to evaluate the admissible temperature range and stresses to ensure a given life. To do this, it became necessary to construct a mechanical theory of creep. The testing of materials for creep using certain standard methods was carried out by industry on a large scale. The aim of these tests was to develop certain conventional criteria for the resistance of the alloy with respect to creep, and a comparative evaluation of the suitability of particular materials for use under certain conditions on the basis of these criteria. The large amount of experimental data that was accumulated as a result of tests of this type must naturally be used as the basis for the mechanical theory. However, this alone was not sufficient and special theoretical experiments were needed to create and justify a mechanical theory. The main problems which had to be clarified were primarily:

1. Are the laws for the creep known when the stress is constant? Is it possible to predict on the basis of these data the creep behavior under a variable load? In particular, is it possible to predict the relaxation law, i.e., the drop in the stress with time during a constant total strain?

2. All experimental data refer to uniaxial elongation Is it possible to make statements on the basis of these data about the creep during an arbitrary complex stressed state?

Experiments which were designed to clarify these problems were carried out systematically in various countries including our country. However, the final solution for these problems has not yet been obtained. This is due to the complexity of the experimental techniques, the high cost of such tests and the different properties of the materials.

With regards to an analysis of the creep tests during uniaxial elongation, a great deal of work was accomplished whose purpose was to find more or less universal relations which related the strain e, the stress σ and the temperature T (at σ = const). The goal of the search was to find ways of extrapolating the results that were valid for tests of short duration to long durations.

The corresponding formulas were proposed by I. A. Oding and other authors. In practice, various so-called temperature-time parameters were applied (see, for example, Yu. N. Rabotnov, 1966) for the extrapolation of the data on creep. However, each of these formulas is only useful for a particular class of materials and no reliable extrapolation methods exist.

The contemporary development of the mechanical theory of creep is characterized by a considerable extension of its applications. On one hand the appearance of transport gas pipes led to a considerable increase in the operating temperatures and, hence, to the use of new materials and on the other hand it required that serious attention be given to the analysis of creep processes during which the loads and temperatures are not stationary.

Creep problems were also encountered in jet technology and in supersonic aviation. Therefore, the creep theory of thin-walled structural elements, plates and shells was developed. Special stability problems arose for these elements on which investigators expended a great deal of effort in the last few years.

The wide use of polymer materials in engineering, in particular, reinforced plastics, made it necessary to study creep problems as they applied to these materials. Here the characteristic feature is that when the stresses are small the relation between the stresses and strains are linear. Therefore, creep can be regarded as lagging or hereditary elasticity (a term introduced by Volterra).

In the modern literature the term "viscoelasticity," which we will use is the more widely used term, although the Volterra term is more appropriate. The development of linear viscoelastic theory is mainly based on the Boltzmann-Volterra idea and its development concentrated rather on the engineering aspect than on the conceptual aspect.

For higher stresses relatively weak nonlinearity occurs. The general tendency in the last few years which represents a return to the old Volterra-Freche idea was to describe this type of relations with the aid of certain special operators. The application of this idea is connected with considerable difficulties and a number of new results were obtained along these lines.

Creep theory as a branch of the mechanics of a deformable solid was formulated relatively recently. The first studies in this field go back to the 20's and 30's. Their general character was determined by the fact that the problem

of creep was extremely important in power machine building and that the engineers were forced to seek simple methods which gave quick results in the solution of practical problems. The foundations of creep theory owe a great deal to the authors who made substantial contributions to the formulation of modern plasticity as a result of which many ideas and approaches have great generality. In our country the first studies in mechanical creep theory go back to N. M. Belyayev (1943), K. D. Mirtov (1946), and the first studies at the end of the 40's were the studies of L. M. Katchanov, N. N. Malinin, Yu. N. Rabotnov.

Linear viscoelastic theory is based on one hand on the fundamental concepts of Boltzmann and Volterra and on the other hand on the theory of viscoelastic rheological models which go back to G. Maxwell and V. Voigt. Combining the properties of elastic bodies and viscous fluids in a more general relation, this theory uses linear differential or integrodifferential equations. Therefore, it opens up great possibilities for the application of effective mathematical methods. Interest in the theory always existed but the absence of real engineering applications did not stimulate its intense development. The earlier studies in this field (A. Yu. Ishlinskiy, A. N. Gerasimov, A. R. Rzhanits, Yu. N. Rabotnov, and others), in fact, did not deal with the solution of specific engineering problems, but aimed to derive certain mathematical results from the assumed models.

A real field of application of linear viscoelastic theory opened up when the theory was applied to aging materials (N. Kh. Arutyunyan, A. A. Gvozdev, G. N. Maslov) and to creep problems in concrete and other structural materials. These studies were developed on a wide scale and the special survey by N. Kh. Arutyunyan is devoted to them in this collection (see pp. [these pages (155-202) missing in Russ. text]).

The behavior of polymer materials during moderate stresses which are usually tolerated in structures is described satisfactorily by linear visocelastic theory even when the kernels have a relatively complex form (forms which do not correspond to the simples rheological models of a Maxwell body or a standard viscoelastic body). The preceding theoretical studies furnished ready-made apparatus for the construction of a viscoelastic theory of polymers and considerable successes were achieved in this field in a short time. Many investigations were carried out by scientific teams in which A. A. Il'yushin, A. K. Malmeister, M. I. Rozovskiy, G. N. Savin, and other participated.

§2. Development and Justification of Theory

2.1. Creep as a Flow Process

We will first consider the one-dimensional case, for example, the tension of a cylindrical rod. The general point of view with regard to the creep process will be that creep is a viscous flow process accompanied by certain structural changes. This means that the creep rate for the given structural state is uniquely determined by the stress and the temperature:

$$\dot{p} = v(\sigma, T) \tag{2.1}$$

It is assumed that the complete deformation is the sum of the instantaneous deformation e_0 and the creep deformation p:

$$e = e_0 + p.$$

In turn the instantaneous deformation consists of the elastic and plastic components so that

$$\dot{e}_0 = \frac{\dot{\sigma}}{E} + g'(\sigma)\dot{\sigma} \quad (\dot{\sigma} > 0),$$

$$\dot{e}_0 = \frac{\dot{\sigma}}{E} \quad (\dot{\sigma} < 0).$$

If the creep is not accompanied by structural changes, or if the structural changes do not have an effect on the rate as a function of the stress and temperature, equation (2.1) defines the steady state creep process, and the bodies in the creep state can be treated as a nonlinear viscous fluid.

Extending this point of view to the general case of the triaxial stressed state, we must assume that the components $\dot{p}_{ij}$ of the creep deformation rate tensor are functions of the components of the stress and temperature tensor:

$$\dot{p}_{ij} = v_{ij}(\sigma_{rs}, T). \tag{2.2}$$

The only condition imposed on the functions v_{ij} is that the dissipation power be positive:

$$v_{ij}\sigma_{ij} > 0.$$

It is usually assumed that the relations (2.2) are of the potential type, i.e., that a creep potential $\Phi(\sigma_{ij})$ exists:

$$\dot{p}_{ij}=\frac{\partial\Phi}{\partial\sigma_{ij}}. \tag{2.3}$$

From the above, using the Legendre transformation, we obtain the inverse relations

$$\sigma_{ij}=\frac{\partial U}{\partial\dot{p}_{ij}}, \quad U=\sigma_{ij}\dot{p}_{ij}-\Phi. \tag{2.4}$$

Strictly speaking, the existence of the potentials Φ and U does not follow from the general laws of mechanics or thermodynamics. However, some justifications of the hypothesis that was adopted can be obtained from the thermodynamics of irreversible processes by generalizing the Onzager principle. A number of studies along the lines of constructing a thermodynamic theory of plasticity and creep were made by A. A. Vakulenko (1958, 1961, 19XX), who also considered more general rheological relations).

The next step in making the theory more precise and more complete is to take into account the structural changes accompanying the creep. The structural state of the material can be characterized by a choice of structural parameters q_i which are scalars in the one-dimensional case and tensors in the general case. The one-dimensional creep equation can be written in the form

$$p = v(\sigma, T, q_1, q_2, \ldots, q_s).$$

The equations describing the changes in the structural parameters over time must also be given. In the general case, it must be assumed that the creep potentials depend on the structural parameters:

$$\Phi = \Phi(\sigma_{ij}, T, q_m), \quad U = U(p_{ij}, T, q_m).$$

2.2. Creep Accompanied by Aging

The simplest assumption will be that the structural parameter which determines resistance to creep varies monotonically with time. Clearly **time** can be taken as such a parameter. If it is assumed that the instantaneous deformation is elastic, which is usually done, and the creep deformation is not accompanied by a change in volume, the equations of flow theory with aging will take on the following form:

$$\dot{e}_{ij}-\frac{1}{2G}\dot{\sigma}_{ij}=\frac{\partial\Phi(\sigma_{ij},t)}{\partial\sigma_{ij}}. \qquad (2.5)$$

Creep theory based on equations of this type was developed by L. M. Kachanov. The equations of the theory will have a particularly simple form in he case when the creep curves are similar. In this case the potential can be represented in the form of a product of a function of the stresses and a function of time $\Phi(\sigma_{ij})\ \tau'(t)$ and the equations coincide in **form** with the steady state creep equations (2.3) when we replace in the latter differentiation with respect to time by differentiation with respect to the modified time τ. Since the elastic deformations are described in terms of the stresses by relations of the potential type, relations (2.5) can be rewritten in the following form:

$${}_{ij}=\varepsilon_{ij}=\frac{\partial}{\partial\sigma_{ij}}\left(\Phi+\frac{\partial\Pi}{\partial t}\right) \qquad (2.6)$$

Here Π is the additional work. It is not necessary to assume in equations (2.6) that the elastic deformation is a linear function of the stresses. In addition to this, it can be assumed that the instantaneous strain is elastoplastic and that it can be described by equations of the type used in deformation theory. It can be shown (L. M. Kachanov, 1960)

that for a body in the creep state, a variational principle of the Castigliano type follows from (2.6) which is described by the fact that the functional

$$I = \int_v \left(\Phi + \frac{\partial \Pi}{\partial t} \right) dv$$

has a minimum for the true distribution of the stresses. The necessary condition is that the power of the variation of the external forces at the true rates be zero.

2.3. Steady State Creep.

Sometimes it is possible to ignore the steady state deformation in applications. Letting $p_{ij} = e_{ij}$, we obtain, instead of (2.3)

$$\dot{e}_{ij} = \frac{\partial \Phi}{\partial \sigma_{ij}} \tag{2.7}$$

Equations (2.7) are called the steady state creep equations. In fact, these equations are the equations for the flow of a nonlinear viscous fluid. Their form coincides completely with the equations of nonlinear elasticity theory or deformation theory of plasticity. Assuming that the potential Φ is a positive-definite convex function of its arguments, the uniqueness theorem was proved for steady state creep and variational principles of the Lagrange and Castigliano type were formulated.

With regard to the form of the function Φ, it is usually assumed that it can be represented in the form $\Phi = \Phi(s)$, where s is a homogeneous function of σ_{ij} of the first degree. The condition for the absence of three-dimensional creep implies that s must not depend on the hydrostatic component of the stress tensor. For isotropic materials, it is usually assumed that $s = \sigma_0$, where σ_0 is the intensity of the stresses, or $s = \sigma_1 - \sigma_3$, where σ_1, σ_3 are, respectively, the largest and smallest principal stresses. The creep equations (2.7) are now rewritten as follows:

$$e_{ij} = \Phi'(s) \frac{\partial s}{\partial \sigma_{ij}} \tag{2.8}$$

Formulas (2.7) or (2.8) are transformed using the Legendre transformation. Suppose that relations (2.8) are valid. The derivatives $\partial s/\partial \sigma_{ij}$ are homogeneous functions of σ_{ij} of degree zero. Hence, the six derivatives which are only functions of five independent arguments satisfy an identity relation. It can be shown that this identity relation can be written in the form

$$\omega\left(\frac{\partial s}{\partial \sigma_{ij}}\right) \equiv 1, \tag{2.9}$$

where ω is a homogeneous function of its arguments. We set $v = \omega(\dot{e}_{ij})$. Similarly, s is the equivalent stress and v the equivalent deformation rate. Now it follows from (2.8) that

$$v = \Phi'(s) = v(s).$$

According to formula (2.4) the stress potential is

$$U = s\Phi'(s) - \Phi(s) = U(\dot{v})$$

and the relations which are the inverse of (2.8) have the form

$$\sigma_{ij} = U'(v)\frac{\partial v}{\partial \dot{e}_{ij}} = s(v)\frac{\partial v}{\partial \dot{e}_{ij}}. \tag{2.10}$$

2.4. Creep with Hardening

By hardening are meant those structural changes in the material which occur as creep deformations accumulate and lead to a reduced creep rate for a given stress and temperature. In the one-dimensional case, the simplest assumption is that the magnitude of the accumulated creep deformation serves as a measure of the hardening. Thus,

$$\dot{p} = v(\sigma, T, p).$$

The extension of the hardening hypothesis to the three-dimensional case is connected with certain difficulties. Generally, the hardening caused by the creep is highly anisotropic which is evident, for example, from the experiments of V. S. Namestnikov. However, the majority of authors restrict

themselves to the assumption of hardening anisotropy which, in this case, is characterized only by one scalar parameter p. Assuming, as before, the existence of the creep potential Φ, which depends on a homogeneous function $s(\sigma_{ij})$ of the first degree, we will write the basic equation in the form (2.8), taking into consideration that Φ depends also on the parameter p.

A hardening measure can be naturally selected as follows. The quantities $\partial s/\partial\sigma_{ij}$ which are homogeneous functions of their argument of zero order satisfy identity (2.9). Now the hardening measure is defined in a natural way by the following formula:

$$p = \int \omega\,(dp_{ij}), \qquad (2.11)$$

and it follows from (2.8) that

$$p = \frac{\partial\Phi(s,\,p)}{\partial s} = v\,(s,\,p).$$

When $s = \sigma_0$,

$$\dot{p} = \int \left(\frac{2}{3}\,\dot{p}_{ij}\dot{p}_{ij}\right)^{1/2} dt;$$

If $s = 2\tau_{max}$, $p = p_1 - p_3 = 2\gamma_{max}$, where p_1 and p_3 are, respectively, the largest and smallest principal creep deformations.

When the creep curves are similar, the potential can be represented as a product of a function of s and a function of p. A power hardening law is often used in the applications

$$p_{ij} = \frac{1}{1-\alpha}\, p^{-\alpha} s^{\cdot\,(1-\alpha)}. \qquad (2.12)$$

When equation (2.12) is written down the dimensions are selected so that dimensional constants do not occur in the equations. For hardening laws of the form $\dot{p} = p^{-\alpha}f(\sigma)$, the function $f(\sigma)$ as S. A. Shesterikov has shown (1959) must satisfy the condition

$$f''(\sigma) \geqslant \frac{\alpha-1}{\alpha}\frac{[f'(\sigma)]^2}{f(\sigma)}.$$

In particular, it follows from the above that in formula (2.12) $n > 1$. Sometimes, by analogy with steady-state creep $f = \exp|\sigma| - 1$ is used (when the function f is selected in this way, it does not satisfy the Shesterikov condition). This condition may lead to strange and physically unacceptable behavior of the solution in the range of small σ after the exact solutions are found. In approximate methods, this fact does not play an important role and any approximation which is suitable for those values of the stresses which interest us is acceptable.

2.5. More General Hardening Laws

The selection of the quantity p as the hardening parameter is not unique. A more general hypothesis is that the structural parameters q_s are related to the stress, the creep deformation, the temperature and the time by certain differential relations which are generally not integrable. Some variants of these relations were discussed in the book by Yu. N. Rabotnov (1966). In particular the amount of the irreversible work dissipated in the creep process can be taken as a measure of the hardening

$$q = \int \sigma_{ij}\, dp_{ij} \qquad (2.13)$$

The selection of this measure was proposed in the studies of A. A. Vakulenko and I. I. Bugakov (19XX), and also by Yu. N. Rabotnov (1963). Experiments have shown that tests under variable loads are described better in this manner than with the aid of the hardening parameter (V. S. Namestnikov and N S. Vilesova, 1964).

2.6. Theory of Aging

The application of a physically well founded hardening theory in a particular variant and also of any equations such as the flow equations is connected with great difficulties. Therefore, in practice, in plants and design bureaus a theory which completely coincides in form with deformation plasticity theory has been used on a wide scale. This theory introduces explicitly in the equation time as a parameter. The initial data on the creep are conveniently represented in the form of

so-called isochrone curves. A series of creep curves in the $e - t$ coordinates for various values of σ describes graphically the relations between the three variables. This relation can be represented in the $e - \sigma$ coordinates in the form of a series of curves, each of which corresponds to a given time t. The creep calculations based on aging theory are reduced to a series of calculations based on the usual deformation plasticity theory, where every time the isochrone creep curve is identified with the stress-strain diagram of the material.

2.7. Deformation-Type Theories

The application of deformation plasticity theory in special problems turns out to be considerably simpler than the application of theories of the flow type. Therefore, in creep theory, many authors constructed equations on the basis of the following principle. It was assumed that the stress and strain tensors were related by the relations of the Nadai-Hencky-Il'yushin deformation theory:

$$\sigma'_{ij} = \frac{2\sigma_0}{3e_0} e'_{ij}$$

(σ_0 and e_0 are the intensities of the stresses and strains respectively and the primes denote the deviators). Further, it was assumed that the quantities σ_0 and e_0 are related in the same way as the stresses and strains during uniaxial tension. Thus, N. N. Malinin (1948), developing further the ideas of N. M. Belyayev assumes

$$e_0 = \sigma_0 \left[\frac{1}{E} + \int_0^{\tau} \frac{S(\sigma_0)}{\sigma_0} d\tau \right]. \tag{2.14}$$

Here, $S(\sigma_0)$ is a function which is determined from the experiment, $\tau(t)$ is a function of time which takes into account the form of the creep curves (N. M. Belyaev took $\tau = t$). F. S. Churikov (1949) started out with the hardening hypothesis and assumed that the intensities of the stresses and strains in the creep are related by equation (2.8) in which p is replaced by p_0 and σ by σ_0.

2.8. Fracture during Creep

The elastic fracture scheme proposed by N. Hoff as applied to the stressing of a beam under a constant load consists of the following. It is assumed that the creep rate is a function of the true stress which is equal to $\sigma_0(1 + e)$, where σ_0 is the stress referred to the initial area, and e is the final deformation. The deformation rate is $\dot{e} = e/(1 + e)$. The creep equation is obtained in the following form:

$$\frac{\dot{e}}{1+e} = v[\sigma_0(1+e)]. \tag{2.15}$$

The quantity σ_0 is given as a function of time (in the special case σ_0 = const). It turns out when (2.16) is integrated, that the deformation e may tend to infinity at a finite time $t = t_k$ which is taken as the fracture time. When th[illegible]ep obeys a power law $v = \sigma^n$, the variables can be sep[illegible]d the time until the fracture is found from the rel[illegible]

$$\int_0^{t_K^{**}} \sigma_0^n \, dt = \frac{1}{n}. \tag{2.16}$$

In fact, N. Hoff's analysis leads to a higher estimate of the time before fracture and after a certain value of the deformation, the stress becomes so large that plastic flow occurs in the body. V. I. Rozenblyum (1963) assumes an ideally plastic material with yield point σ_s and takes as the fracture instant the time until the condition $\sigma_0/(1 + e) = \sigma_s$ is satisfied. For σ_0 = const and a power law for the creep, we obtain

$$t_K = \frac{1}{n\sigma_0^n}\left[1 - \left(\frac{\sigma_0}{\sigma_s}\right)^n\right]. \tag{2.17}$$

In many engineering alloys, the fracture occurs as a result of the development of a system of cracks on the boundaries of the grains during a small deformation. The brittle fracture scheme was studied by L. M. Kachanov (1958) and Yu. N. Rabotnov (1959). The degree to which the material is damaged is characterized by the parameter ψ which varies from $\psi = 1$ for an undamaged material to $\psi = 0$ at the fracture

instant. It is assumed that the quantity ψ varies at each point in accordance with the equation

$$\dot{\psi} = -h\left(\frac{\sigma_1}{\psi}\right). \tag{2.18}$$

Here σ_1 is the largest tensile stress. When $h = (\sigma_1/\psi)^k$ is a power function, the time at which the quantity ψ attains the value $\psi = 0$ is determined from a condition which has the same form as (2.16);

$$\int_0^{t_k} \sigma_1^k \, dt = \frac{1}{1+k}. \tag{2.19}$$

A similar scheme which differs only in small details was studied by Yu. N. Rabotnov.

L. M. Kachanov also studied the case of mixed fracture when the condition $\psi = 0$ is attained during a sufficiently large creep deformation and when it is necessary to take into account the variable cross sectional area in determining the acting stress.

A series of studies by Swedish authors (F. Odquist, Ya. Hult, et al.) developed further the ideas of L. M. Kachanov which take into account the effect of the instantaneous plastic deformation and the deformation in the first creep phase. According to the well known Odquist scheme the deformation on a sector which is not in the stationary state is added to the instantaneous deformation.

Yu. N. Rabotnov (1963) assumes that the damage parameter $\omega = 1 - \psi$ enters the creep equations as a structural parameter. Thus, when hardening is ignored, the creep equations have the following form:

$$\dot{e} = v(\sigma, \omega), \quad \dot{\omega} = \varphi(\sigma, \omega). \tag{2.20}$$

Both the case of finite and small deformations can be analyzed. The analysis was carried out for a sufficiently general relation of the form

$$\dot{e} = \sigma^n (1-\omega)^{-q}, \quad \dot{\omega} = c\sigma^k (1-\omega)^{-r}.$$

This approach makes it possible to describe one-third segments of the creep curves.

If we assume that the functions v and φ in (2.20) differ only by a constant factor, $\omega = e/e_*$, where e_* is a uniform deformation which precedes fracture which is assumed to be independent of the stress. Then the creep equation has the form

$$\dot{e} = v\left(\sigma, \frac{e}{e_*}\right),$$

i.e., its form coincides with equation (2.8) describing the hardening (the instantaneous deformation is not taken into account). Thus, it becomes possible to describe the entire creep curve with the aid of only one structural parameter, the size of the creep deformation. This idea was applied by G. F. Lepin who used an equation of the form

$$\dot{e} = e^{-\alpha} \exp\left(\frac{\sigma}{1 - e/e_*}\right) \tag{2.21}$$

and showed on a large amount of experimental material its suitability for the description of creep and relaxation.

2.9. Shortlived Creep

When the temperature and stresses are sufficiently high, there is practically no hardening. Thus, the creep rate is initially determined by equation (2.1) which is valid and independent of the loading prehistory. Under these conditions, the instantaneous plastic deformation which is determined by equation (2.2) plays an important role. The elasticity modulus and the function $g(\sigma)$ depend on the temperature. Creep is usually accompanied by an intense formation of cracks as a result of which the creep rate increases and fracture occurs. The structural parameter ω must be introduced and the equations used for the short-term creep in the one-dimensional case are:

$$\dot{e} = e_{**}\left(\frac{\sigma}{1-\omega}\right)^n, \quad \dot{\omega} = \left(\frac{\sigma}{1-\omega}\right)^k. \tag{2.22}$$

For the majority of materials, we can take n = k. Then e_{**} is the uniform deformation at the instant of rupture (S. T. Mileyko, 1962, 1963, S. T. Mileyko and Yu. N. Rabotnov, 1966).

2.10. Linear Viscoelasticity

The creep of many non-metallic materials is described with the aid of the equations of linear viscoelasticity. One way of constructing the relations in the theory is to combine the elastic and viscous properties. Rheological models consisting of a set of springs and elastic resistances are used to represent combinations of this type. The relations between the stresses and strains for the one-dimensional case have the form

$$P(\sigma) = Q(\varepsilon). \tag{2.23}$$

Here P and Q are linear differential operators with constant coefficients. Relations of type (2.23) are used to describe both solids and fluids. The many studies dealing with the description of viscoelastic properties of fluids, helium, etc., belong to the field of rheology (these studies are not discussed in this survey).

Different apparatus for describing the viscoelastic properties of solids was proposed by L. Boltzmann, which was further developed in detail in the studies of V. Volterra, dating back to the beginning of the century. The linear Volterra operator is defined as follows:

$$\tilde{K}f = K\left[f(t) - \int_{-\infty}^{t} \varkappa(t-\tau) f(\tau)\, d\tau\right]. \tag{2.24}$$

This equation can be written in the form

$$\tilde{K}f = \int_{-\infty}^{t} K(t-\tau) f(\tau)\, d\tau,$$

if we assume that the kernel has a singularity of the delta-function type.

In V. Volterra's theory difference kernels $\varkappa(t, \tau) = \varkappa(t - \tau)$ are used which follows from the requirement for invariance with respect to the time origin.

Under certain constraints characterizing solids (2.24) is equivalent to (2.23), and the kernel $\varkappa(t - \tau)$ is a linear combination of exponential functions.

The general equations of linear viscoelasticity for an arbitrary anisotropic material can be represented as follows:

$$\sigma_{ij} = \tilde{E}_{ijkl} e_{kl}. \tag{2.25}$$

Here $\tilde{E}_{ijkl}$ is the matrix of linear Volterra operators corresponding to the matrix of elasticity moduli in the usual theory.

The approximation of the kernel $\varkappa(t - \tau)$ using exponential functions makes it possible to transform relations (2.25) by simple means, i.e., to find the resolvent kernels of the corresponding kernels. The principle formulated by V. Volterra that the solution of the usual elasticity theory problem can be transformed into a solution of the corresponding viscoelasticity theory problem when the elastic constants are replaced by operators is used in the solution of problems in viscoelasticity. In principle the values of the functions which depend on the operators can always be found when the functions are rational. If this is not the case certain difficulties arise. It should be noted that the Volterra principle can only be applied when the form of the boundary conditions does not change (for example, it is not suitable for problems dealing with a moving die).

In the modern literature a method based on the Laplace transform is also used in the solution of viscoelastic problems. For the images of the stresses and strains, equation (2.25) takes on the form of the usual Hooke law.

$$\bar{\sigma}_{ij} = \bar{E}_{ijkl} \bar{e}_{kl}. \tag{2.26}$$

The main difficulties are connected with the transition from the images to the original stresses and strains. For aging metals such as concrete, the kernel of the Volterra operator is not a difference kernel. The creep theory of aging metals which goes back to the studies of N. Kh. Arutyunyan underwent considerable development (a special survey is devoted to it in this volume, see below, pp (missing in Russian text).

For uniaxial expansion the linear viscoelasticity law takes on the form

$$e = \frac{1}{E}\left[\sigma + \int_{-\infty}^{t} K(t-\tau)\,\sigma(\tau)\,d\tau\right], \quad \sigma = E\left[e - \int_{-\infty}^{t} \Gamma(t-\tau)\,e(\tau)\,d\tau\right].$$

The function $K(t - \tau)$ is called the creep kernel and its resolvent kernel $\Gamma(t - \tau)$ the relaxation kernel. Sometimes, it is convenient to give not the kernels themselves but the spectra. When the function $\Gamma(t)$ is represented in the form

$$\Gamma(t)=\int_0^\infty \alpha A(\alpha)\exp(-\alpha t)\,d\alpha,$$

the function $A(\alpha)$ is called the relaxation spectrum. The creep spectrum is defined analogously. The complex modulus $E' + iE''$ is introduced in the solution of dynamic problems, which is expressed in terms of the relaxation spectrum according to the formula

$$E'-iE''=E\left[1-ip\int_0^\infty \frac{A(\alpha)}{\alpha+ip}\,d\alpha\right]. \qquad (2.27)$$

In the early studies in linear viscoelasticity theory (A. N. Gerasimov, A. Yu. Ishlinskiy, V. G. Gololadze, M. I. Rozovskiy, Yu. N. Rabotnov, et al.) the formal apparatus of the theory was developed and the qualitative effects that can be detected in particular cases were clarified. In studies along physical-chemical lines (G. L. Slonimskiy, et al.) the theory was applied to the description of those aspects of the behavior of different bodies which do not correspond to the usual models. The considerable development of the theory in the 50's is related to the substantial extension of its applications. When the stresses are not excessively large, the equations of linear viscoelasticity describe well the creep of concrete (taking into account aging) and also the creep of most polymer materials. This theory was successfully applied to the mechanics of rocks, ice surfaces, etc. The formulation of new applied problems stimulated the development of general methods and the search for many particular solutions.

2.11. Nonlinear Viscoelasticity

Already V. Volterra ("Fonctions de lignes," 1913) proposed that nonlinear viscoelasticity be described by relations of the form

$$e=\int_{-\infty}^{t} K_1(t-\tau_1)\,\sigma(\tau_1)\,d\tau_1+\int_{-\infty}^{t}\int_{-\infty}^{t} K_2(t-\tau_1,\ t-\tau_2)\,\sigma(\tau_1)\,\sigma(\tau_2)\,d\tau_1\,d\tau_2. \qquad (2.28)$$

In fact, this idea was forgotten for all practical purposes, and revived only recently in the studies of a number of American and Japanese authors. A simpler relation for a uniaxial elongation or pure shear was proposed by Yu. N. Rabotnov (1948):

$$\varphi(e)=\sigma+\int_0^t K(t-\tau)\,\sigma(\tau)\,d\tau. \tag{2.29}$$

M. I. Rozovskiy (1951) and N. Kh. Arutyunyan (1952) constructed for the same case a different equation, namely:

$$e=\psi(\sigma)+\int_0^t L(t-\tau)\,F[\sigma(\tau)]\,d\tau. \tag{2.30}$$

The function $\varphi(e)$ in the left member of (2.29) for the active load must be determined experimentally, and during unloading when applied to metals it must be linear and correspond to the elasticity law. Apparently, the main field of application of heredity relations of the type (2.29) or (2.30) is the mechanics of polymers; for metals the equations predict the observed increase, but exceed this effect (V. S. Namestnikov and Yu. N. Rabotnov, 1961). Combining nonlinear viscoelasticity and the hardening law, G. I. Bryzgalin was able to describe well the creep of celluloid under variable loads.

2.12. Experimental Studies

A very large number of experimental studies dealt with the investigation of long-term creep and fracture. The results made it possible to verify and improve creep theory and the computational methods used. These studies can be divided into the following three groups.

1. Obtaining the creep, relaxation and long-term strength characteristics of individual materials as applied to concrete engineering objects. Usually the goal of such investigations is to determine certain conventional comparable characteristics which can be used to select the material under given operating conditions. Along with the development of mechanical theory, these characteristics became the basis of calculations, even though by themselves they cannot be used to select a particular theory used as the basis for the calculation of a complex process.

2. Studies whose goal was to verify various creep theories.

3. Testing of natural objects or models and a comparison of the experimental data with the results obtained from calculations.

In this survey we will briefly dwell on studies dealing with the second and third groups.

A systematic verification of the hardening hypothesis is available in the series of studies by A. M. Zhukov, F. S. Churikov and Yu. N. Rabotnov (1953), V. I. Danilovskaya, G. M. Ivanova and Yu. N. Rabotnov (1955), V. S. Namestnikov and A. A. Khvostunkov (1960), V. S. Namestnikov (1957, 1960), Yu. P. Kaptelin (1962), G. M. Ivanova (1958). All these experiments were carried out during stressing and the results of tests under constant and variable loads were compared. As a result of these studies and also the work of foreign authors, it was established that the hardening hypothesis of type (2.8) describes satisfactorily, at least in first approximation, the first creep phase.

Creep tests in a complex stressed state are few in number and technically difficult. We note the studies of I. A. Oding and G. A. Tulyakov (1958) which validate for steady state creep the validity of the Tresk-St.-Venant criterion. The results of the experiments of V. S. Namestnikov (1957), which apply to the transient creep phase, did not confirm generally the simple relations described above and forced the author to use more complex constructions. V. S. Namestnikov (1957) has also shown that the hardening during creep is highly anisotropic.

A number of studies deal with determining fracture criteria in the complex stressed state. We note the experiments of B. V. Zver'kov (1958), Sh. N. Katz (1955), V. P. Sdobrev (1958, 1959), I. I. Trunin (1967), I. N. Laguntsev and V. K. Svyatoslavova (1959), I. N. Laguntsev and L. I. Fedotova (1959).

Systemmatic studies of the creep in models of elements of turboengines and also of full-scale discs were carried out at the Central Scientific Research Planning and Design Boiler and Turbine Institute, Im. I. I. Polzunov (D. P. Varshavskiy, P. Ya. Boguslavskiy, I. G. Polumordvinova, 1955) and the Central Scientific Research Institute of Machinery Manufacture and Metal Working (V. P. Rabinovich, 1959, 1960). The result of these studies is that the computational methods based on the simplest aging theory give results which are satisfactory in practice in predicting the magnitude of the deformations and residual stresses. The time until fracture can also be predicted with a satisfactory degree of accuracy.

§3. Steady State Creep in the Theory of Aging

The equations in the theory of steady state creep and the equations in the theory of aging are basically identical with the equations of deformation plasticity theory. The only difference is that in the theory of steady state creep the deformations are replaced by the deformation rates, and time appears as a parameter in the equations of aging theory. The methods used for the solution of the problems based on these two theories are essentially analogous. Usually, some simple analytical approximating function $v(s) = \Phi'(s)$ is selected for the steady state creep, for example $v = \epsilon_n (s/\sigma_n)^n$ or $v = \epsilon_e \exp(\sigma/\sigma_e)$, where ϵ_n, σ_n, n, ϵ_e, σ_e are constants. The main advantage of aging theory which is responsible for its wide use in engineering calculations is the possibility of using the initial creep curves without any analytic approximations which entail an inevitable distortion. Therefore, numerical methods are used most extensively, which represent a development or modification of the methods that were developed for the theory of small elastoplastic deformations.

3.1. Variational Principles

The Lagrange and Castigliano principles for creep problems are evidently a simple reformulation of the corresponding principles for a nonlinear elastic body, since the initial hypothesis assumes a relation of the potential type between the stresses and strains or deformation rates. A systematic development of approximate methods based on the Castigliano principle is due to L. M. Kachanov. When the steady state creep obeys a power law as the exponent n increases, the distribution of the stresses differs little from that corresponding to the limiting state of an ideal rigid-plastic body in a number of cases. Thus, the concept of a limiting creep state is introduced, and the stresses σ_{ij}^0 for this state are found on the basis of the scheme of a rigid-plastic body, where the yield point depends on the character of the load. The approximate values of the rates are found by applying directly the Castigliano theorem. The most accurate results are obtained when the stress components are represented in the form

$$\sigma_{ij} = \sigma_{ij}^0 + K(n)(\sigma_{ij}' - \sigma_{ij}^0).$$

Here σ_{ij}' is generally any statically possible distribution of the stresses corresponding, for example, to the elastic state. The multiplier K(n) is found from the condition that the corresponding functional be a minimum.

The method was illustrated on a large number of examples, dealing with trusses, beams and frames, torsion problems and other problems.

3.2. Methods of Successive Approximations

A natural technique for the solution of nonlinear problems in the mechanics of a body is the method of successive approximations where a linear problem is solved in each stage. In the method of elastic solutions due to A. A. Il'yushin, an elasticity theory problem with fictitious body forces and modified boundary conditions is solved in each approximation.

The method of variable elastic parameters (I. A. Birger, 1961) is based on the fact that the equations of creep theory coincide with the equations of linear elasticity theory, in which the elastic constants are functions of the coordinates. These functions are not known in advance, since they are nonlinear functions of the unknown quantities, the stress or strain components. Each successive approximation is found by integrating linear equations with variable coefficients which are expressed in terms of the parameters found in the preceding approximation. I. A. Birger developed techniques which can be used to obtain the fastest convergence of the successive approximations. The schemes are conveniently calculated on electronic digital computers.

The idea of variable elasticity parameters is also useful when variation principles of the Lagrange or Castigliano type are applied (L. M. Kachanov). Instead of finding the stationary values of a complicated nonquadratic functional, a sequence of quadratic potentials of the same type as in the corresponding theory of elasticity problems with a variable modulus is studied. Each functional is minimized using the Ritz method, and the values of the parameters that are found are used to calculate the elasticity moduli in the next approximation.

3.3. Calculation of Turbine Discs using the Method of Successive Approximations

The calculations of creep in a rotating disc of variable thickness with a variable temperature field is one of the most important applied problems in creep theory which is still significant to this day. One variant of the method of successive approximations for this problem is as follows. The radial stress σ_r is determined from the equilibrium equation as a functional of the circular stress:

$$\sigma_r = F_1(\sigma_\theta). \tag{3.1}$$

Next σ_θ is determined as a functional of σ_r, σ_θ involving the constant C from the consistency equation and the creep equations:

$$\sigma_\theta = F_2(\sigma_r, \sigma_\theta, C). \qquad (3.2)$$

The method of successive approximations is used to solve the system of equations that was obtained, and some distribution of the stresses σ_θ is used as the starting point.

When the functional F_1 is determined uniquely, the functional F_2 can be represented in differrent forms and the rate at which the successive approximations converge depends considerably on this. Various variants of the methods are available in the studies of P. Ya. Boguslavskiy (1950), A. G. Kostyuk (1953), N. N. Malinin (1959) and other authors. R. M. Shneyderovich uses as the unknown functions not the stresses but the strains and the circular deformation is expressed in terms of the radial deformation by means of a simple relation obtained from integrating the consistency equation. The functional representing the radial deformation is constructed with the aid of the equilibrium equation and the creep equations.

When the discs are manufactured using forging or pressing, the material acquires inevitably a texture causing anisotropy of the creep properties. O. V. Sosnin (1963) studied the steady state creep of an anisotropic disc. It became evident that the effect of the real anisotropy on the distribution of the stresses was not great.

3.4. Exact Solutions

Relatively few exact solutions are available for one-dimensional problems. Thus, the problem of a rotating disc is reduced to the integration of a system of two quasi-linear equations with given boundary conditions at the end points of the interval (N. N. Malinin, 1959). The use of electronic digital computers makes these calculations relatively simple. Piecewise linear potentials can sometimes be used in certain cases to obtain a solution of the problem in closed form. The problem of the stress concentration around a circular hole in a uniformly stressed plate was solved by V. I. Rozenblyum (1959) and for a Tresk-type criterion by Yu. V. Nemirovskiy (1964) using the largest reduced stress criterion. A. G. Kostyuk (1950) studied a disc with a hole whose thickness

varied according to a power law and depended on the radius when the creep law was a power law. A uniform expanding load was applied to the external contour. An exact solution of the problem was obtained in parametric form.

3.5. Plates

In the absence of forces the relation between the curvatures and the moments is completely analogous to the relation between the stresses and strains in the plane stressed state and the calculation of the corresponding potential does not lead to any difficulties. For circular symmetrically loaded plates, closed solutions are obtained when the creep laws are piecewise linear. L. M. Kachanov uses the Ritz method for these problems. A number of special cases were studied by N. N. Malinin with the aid of the Galerkin method.

The problem of a semicircular plate is of interest in connection with creep calculations of a turbine diaphragm. A relatively simple approximate solution based on the Ritz method was developed by V. I. Rozenblyum (1954). P. Ya. Boguslavskiy (1950) solved this problem using the method of successive approximations constructed on the principle which was described above and applied to the calculation of discs. The selected scheme corresponded sufficiently closely to the structure of the real diaphragm.

3.6. Shells

The creep theory of shells is usually constructed on the basis of Kirchhoff-Loew hypothesis in which terms of order h/R are ignored, where 2h is the thickness and R is the characteristic radius. The equations which relate the forces and moments on one hand and the deformation rates on the middle surface and the rates for the change of the curvature on the other hand in the theory of steady state creep are expressed in the following manner:

$$\varepsilon_{ij} = \frac{\partial \Phi}{\partial T_{ij}}, \quad \varkappa_{ij} = \frac{\partial \Phi}{\partial M_{ij}}, \tag{3.3}$$

or

$$\left.\begin{aligned} T_{ij} = \frac{\partial U}{\partial \varepsilon_{ij}}, \quad M_{ij} = \frac{\partial U}{\partial \varkappa_{ij}}, \\ \Phi = \varkappa_{ij} M_{ij} + \varepsilon_{ij} T_{ij} - U. \end{aligned}\right\} \tag{3.4}$$

The main difficulty is calculating the potential $U(\varepsilon_{ij}, \varkappa_{ij})$. If the Mises criterion is adopted, the potential U is obtained by integrating over the thickness a function of the intensity of the deformation rates H, which, in the given case has the form

$$H = [\varepsilon_0^2 - 2(\varepsilon\varkappa)z + \varkappa_0^2 z^2]^{1/2}. \tag{3.5}$$

Here ε_0 and $\varkappa_0$ are the intensities of the deformation rates on the middle surface and the rate at which the curvatures change, respectively, $(\varepsilon\varkappa)$ is a bilinear form constructed from the ε_{ij} and $\varkappa_{ij}$. To find the potential U, a function of H must be integrated over the thickness of the shell. It is not possible to obtain an explicit expresion for U.

One technique used is to average sectionally the functions H over the thickness of the shell. I. G. Teregulov (1962) proposes that H(z) should be regarded as a piecewise-constant function, namely:

$$H^2 = \varepsilon_0^2 \mp 2\lambda_1 (\varepsilon\varkappa) + \lambda_2^2 \varkappa_0^2. \tag{3.6}$$

The upper sign refers to the region $z > 0$ and the sign at the bottom to the region $z < 0$. It is proposed that the constants λ_1 and λ_2 be selected from the condition that the exact result be obtained from special cases selected in an appropriate manner.

Sometimes a two-layer model is studied instead of the real shell (Yu. N. Rabotnov, 1951). If the thickness of the layers is sufficiently small compared to the distance $2h_1$ between them, the distribution of the stresses along the thickness of each layer can be considered to be uniform, and the quantity H in each layer is obtained from formula (3.5) when $z = \pm h_1$. Hence, we must take in formula (3.6) $\lambda_1 = \lambda_2 = h_1$. The quantity h_1 for the model shell is related to half the thickness of the real shell by an equation which describes the equivalent behavior of the real and model shell in some special case, for example, during pure flexing. When the creep law is a power law with the exponent n, this condition implies

$$h_1 = h\left(2 + \frac{1}{n}\right)^{-\frac{n}{n+1}}. \tag{3.7}$$

Another simplification of relation (3.6) is obtained by setting $\lambda_1 = 0$, i.e., by assuming simply that H is constant along the thickness. Such a scheme was proposed by V. I. Rozenblyum; however, it was based on different concepts. For n = 1 the calculation of the potential U reduces to the integration of the expression for H^2 over the thickness. Thus, U turns out to be a linear combination of ϵ_0^2 and $\varkappa_0^2$. By computing the function Φ, we find that it depends on the expression

$$S = \left(t_0^2 + \frac{3}{4} m_0^2\right)^{1/2}. \tag{3.8}$$

Here t_0 and m_0 are the intensities of the dimensionless forces and moments that were determined in an appropriate manner. On the other hand, V. I. Rozenblyum obtained earlier the approximate limiting state condition for an elastoplastic shell in the form

$$t_0^2 + m_0^2 = 1. \tag{3.9}$$

The statement can be made on the basis of the well-known theorem of Ch. R. Kelladine and D. Ch. Drucker that in the space t_{ij}, m_{ij} the appropriately normalized surface S = const will lie between the surfaces $t_0^2 + 3/4\, m_0^2 = 1$ and $t_0^2 + m_0^2 = 1$. From this it follows that for $1 \leq n < \infty$, the approximation of the potential Φ will consist of letting it be a function of the quantity

$$S = (t_0^2 + km_0^2)^{1/2}. \tag{3.10}$$

Here $3/4 \leq k \leq 1$. A comparison with the exact result for the pure flexed state leads to the following expressions for k:

$$k = \frac{1}{4}\left(2 + \frac{1}{n}\right)^{\frac{2n}{n+1}}. \tag{3.11}$$

3.7. Axisymmetric Deformation of a Circular Cylindrical Shell

The application of a two-layer model gives especially simple results for the axisymmetric deformation of a circular cylindrical shell in the absence of an axial force. The curvature does not change in the transverse direction. Denoting by ε the rate of the circular deformation and by $\varkappa$ the rate at which the curvature of the generatrix changes we find that

$$H = \left(\frac{3}{4}\varepsilon^2 + \frac{16h^2}{k}\varkappa^2\right)^{1/2}, \tag{3.12}$$

where k is given by formula (3.11).

The problem of an infinitely long cylindrical shell under the action of a circular load was solved with the aid of the Lagrange variational principle by V. I. Rozenblyum. The shape of the deflection was assumed to be the same as in the solution of the corresponding problem in the theory of elasticity and the magnitude of the deflection was varied in the section to which the load was applied and the wavelength was also varied.

Yu. N. Rabotnov (1966) reduced this problem to the integration of the system of equations

$$\left.\begin{aligned} u'' + 2m\omega &= 0,\\ m'' - \frac{2u}{\omega} + 2p &= 0.\end{aligned}\right\} \tag{3.13}$$

Here u is the dimensionless rate of the circular deformation, m is the dimensionless longitudinal moment, and p is the loading parameter. The function ω is defined by the equation

$$v^2(\omega) = u^2 + m^2\omega^2, \tag{3.14}$$

where $v(\sigma)$ is the creep rate during uniaxial tension by the stress σ (with an accuracy up to a constant multiplier). The primes denote differentiation with respect to the dimensionless coordinate ξ. Equations (3.13) are the Euler equations for the variational problem for the functional

$$N = \int_0^l \left[u'm' + \chi(\omega) + \frac{u^2}{\omega} - m^2\omega - 2pu \right] d\xi, \tag{3.15}$$

where $\chi(\omega)$ is a known function. The arguments $m(\xi)$ and $u(\xi)$ of the functional are given and are varied independently, which is the advantage of the functional (3.15) compared to the Lagrange functional. When the latter is applied the deflection can be determined with a good approximation but when the moment is calculated, the accuracy is lost.

Problems dealing with the boundary effect for a semi-infinite shell with a supported edge and an edge supported on hinges were studied.

Yu. N. Rabotnov has shown in a subsequent article that a variational equation of type (3.15) can be obtained from the general Reissner variational principle, as well as for other problems in the theory of shells in which the force in one direction can be assumed to be known, on the basis of concepts of one type or another in which the rate at which the curvature changes in the orthogonal direction is zero. This occurs, for example, in the theory of cylindrical shells of medium length.

V. N. Mazalov and Yu. V. Nemirovskiy (1966) studied the problem of the symmetric deformation of a circular cylindrical shell using a two-layer model and the largest reduced stress criterion. The case of a shell of finite length supported on hinges under the action of a circular load was studied and the solution was obtained in closed form.

3.8. Boundary Effect in Shells

When the stressed state in the shell is predominantly torqueless and the intensity of the stresses is sufficiently large, the stressed state of the boundary effect near the supported edge can be calculated as a correction for the principal stressed state. This idea was applied by I. G. Teregulov, who used in the boundary effect zone, equations that were linearized in the neighborhood of the principal stressed state, which is assumed to be torqueless and, hence, known. The theory of the boundary effect under these assumptions

is similar to the theory of the boundary effect in elastic shells. The problem of the boundary effect in a cylindrical circular shell compressed in the axial direction was considered as an illustration. The boundary effect in a cylindrical shell was also studied by I. V. Stasenko (1962, 1963).

§4. Transient Creep

4.1. Variational Method in Flow Theory

The simplest theory of transient creep is flow theory accompanied by aging. As mentioned above, the L. M. Kachanov variational principle

$$\delta \int_{v} \left(\Phi + \frac{\partial \Pi}{\partial t} \right) dv = 0. \tag{4.1}$$

is valid for this theory. Here Π is the corresponding elastic potential. In principle the potential Π need not be quadratic, it can correspond to nonlinear elasticity, or, equivalently, to the instantaneous elastoplastic deformation described by equations of the deformation type. In this case, difficulties arise which are connected with the occurrence of unloading zones.

A natural approximate method for solving transient creep problems with the aid of variational equation (4.1) when the external forces do not change is as follows. Let σ'_{ij} be the distribution of the stresses corresponding to the elastic state, and σ''_{ij} the distribution of the stresses during transient creep. We set approximately

$$\left.\begin{array}{c} \sigma_{ij} = \sigma'_{ij} + \theta(t)\,(\sigma''_{ij} - \sigma'_{ij}), \\ \theta(0) = 0, \quad \theta(\infty) = 1. \end{array}\right\} \tag{4.2}$$

Condition (4.1) leads to a differential equation for the function $\theta(t)$. The method is only applicable in the form that was described when the creep curves are similar, since, only in this case we can speak about transient creep which is characterized by the stress distribution σ'_{ij}. However, a slight modification of the method removes this constraint.

For problems of the relaxation type, when fixed displacements are given on a part of the surface and the remaining part of the surface is free of forces (body forces are also absent), it can be assumed that all stresses vary proportionally:

$$\sigma_{ij} = \rho(t)\,\sigma'_{ij}.$$

The application of variational equation (4.1) leads in the given case to the conclusion that the function $\rho(t)$ is independent of the shape of the body and the manner in which it is secured, and it coincides with the function which describes the relaxation of the stresses, for example, during uniaxial stressing.

The application of this method has been illustrated on many examples that can be found in the book of L. M. Kachanov, and also in publications by other authors. These examples apply to rod systems, torsional and bending problems.

4.2. Bending of Rods on the Basis of Hardening Theory

Even the simplest transient creep problem of a rod with a rectangular cross section does not have an exact solution during pure bending. N. N. Shchetinin studied this problem with the aid of a creep equation of the form

$$\dot{p} = p^{-\alpha}(\exp|\sigma| - 1). \qquad (4.3)$$

During the analysis it became evident that the stress changes sign near the neutral axis. This is a consequence of the fact that equation (4.3) is not suitable for small σ (the right member must have the form σ^n, where $n > 1 + \alpha$ as shown by S. A. Shesterikov). To remove the singularity that was mentioned a correction was introduced into the equation for the problem. The technique of linearizing the original equation made it possible to construct the solutions in series and to investigate their convergence. To calculate the bending of rods with an arbitrary cross section and arbitrary temperature distributions, B. F. Shorr (1959) developed a numerical integration method. The following creep law was selected:

$$\dot{p} = h(p)\,q(T)\left(\exp\frac{|\sigma|}{A} - 1\right). \qquad (4.4)$$

To take into account both the first and second sectors on the creep curve

$$h(p) = p^{-\alpha} \; (p < p_c), \quad h(p) = p_c^{-\alpha} = \text{const} \quad (p > p_c)$$

were used in those regions where the stress (and, hence, also the creep rate) do not change sign. But the neutral axis is shifted in the creep process. Therefore, at some points the rate changes sign. For these regions, it is proposed that the function h(p) be replaced by the function h(q), where q is defined as follows:

$$\left. \begin{aligned} q &= p^{+} + \lambda p^{-} \quad (\sigma > 0), \\ q &= p^{-} + \lambda p^{+} \quad (\sigma < 0). \end{aligned} \right\} \tag{4.5}$$

Here p^+ is the creep deformation that accumulated during tension, and p^- is the creep deformation that accumulated during compression. The calculation itself is carried out numerically, in steps over time.

4.3. Application of the Tresk Criterion. Axisymmetric Problems.

When the principal axes of the stresses in the body are fixed and the inequality $\sigma_1 > \sigma_2 > \sigma_3$ always holds, the hardening law combined with the Tresk criterion leads to the following results. The magnitude of the equivalent stress is $s = \sigma_1 - \sigma_3$, $\dot{p}_1 = -\dot{p}_3 = \dot{p}$, $\dot{p}_2 = 0$. From here it follows that $2p = p_1 - p_3$, after which the creep equation is expressed in the same way as in the one-dimensional case:

$$\dot{p} = v(p, s).$$

If two principal stresses, for example, σ_1 and σ_2 are equal, the associated law gives a rate distribution which is not unique, namely,

$$\dot{p}_3 = -\dot{p}, \quad \dot{p}_1 = \lambda \dot{p}, \quad \dot{p}_2 = (1 - \lambda) \dot{p} \quad (0 \leqslant \lambda \leqslant 1).$$

The magnitude of the equivalent creep deformation p defined in the above manner is only meaningful in the case when the condition $\sigma_1 > \sigma_2 > \sigma_3$ is preserved during the entire process. When the signs change in this inequality, the maximum displacements occur in other planes and the possibility of integrating the hardening effect which occurs according to different mechanisms is an additional hypothesis.

The simplified variant of hardening theory that was presented was applied by Yu. N. Rabotnov to problems in the creep of rotating discs and cylinders. The creep law used had the form

$$\dot{p}p^{\alpha} = \exp(s - \beta). \quad (4.6)$$

Here the quantities p and s are dimensionaless, β depends on the temperature and, hence, is a given function of the coordinates. The principle of solving axisymmetric problems consists of the following. By integrating the equilibrium and consistency equations, the radial, circular and axial stresses are found in the form of certain functionals of p which still contain the integration constants. For a disc $\sigma_r = 0$, σ_r and σ_θ depend on the two constants B and C. Equation (4.6) is reduced to the form

$$\dot{p}p^{\alpha} \exp(p) = \exp(S - \beta). \quad (4.7)$$

Here S is a functional of p, which depends on the constants B and C, the radial coordinate, the initial distribution of the stresses, and the temperature deformation. The form of the functional S depends on the relation between the stresses σ_r and σ_θ. Equation (4.7) is integrated in steps over time, and the value of the functional S is calculated on the basis of the magnitude of p in the preceding approximation. The constants B and C are determined in each step from the boundary conditions. The case of a cylinder is studied in an entirely analogous manner. If the disc does not have a hole, a finite region is formed in its central part, where $\sigma_r = \sigma_\theta$. This case was considered by O. V. Sosnin (1960). The numerical integration has shown that the radius of this central region varies considerably during the creep process. In another study, O. V. Sosnin (1963) developed another method for the solution of the problem of a rotating disc, namely, he derived a differential equation for the radial displacement whose solution was obtained in a series.

The problem of a rotating disc was studied by I. N. Danilova (1959) as applied to the calculation of stresses in the run-in period of a rotor of a gas turbine.

4.4. Variational Principles in Hardening Theory

Variational equation (4.1) does not presuppose any special hypothesis about the character of the dependence of the potential on the structural parameters, and it remains valid in the case when Φ is independent of time and of the hardening parameter p. This fact was noted by S. A. Shesterikov (1957), who formulated the corresponding variational principle for hardening theory and applied it to the solution of relaxation problems. Setting approximately

$$\sigma_{ij} = \sigma'_{ij}\rho(t),$$

a functional equation for $\rho(t)$ can be obtained from (4.1) which can be solved under certain special assumptions about the creep law. S. A. Shesterikov obtained a more general solution of this equation for a power creep law with power hardening and illustrated its application on the problem of the relaxation of stresses in a disc with a hole (1960).

Recently more flexible variational principles have been used widely in creep problems, in which not only the stresses or deformation rates are varied independently, but also the rates of change of the stresses or of some other parameters. Thus, for example, the Reissner type variational principle for the creep can be formulated as follows. Let us consider the functional

$$J = \int_V \left[\dot{\sigma}_{ij}\dot{e}_{ij} - \Phi(\sigma_{ij}) - \frac{\partial \Pi(\dot{\sigma}_{ij})}{\partial t} - F_i \dot{u}_i \right] dv - \int_{\Sigma_u} \dot{\sigma}_{ij}\nu_j (\dot{u}_i - \dot{u}_i^*)\, d\Sigma - \int_{\Sigma_T} \dot{T}_i^* \dot{u}_i \, d\Sigma. \qquad (4.8)$$

It is assumed that $e_{ij} = 1/2\,(u_{i,j} - u_{i,j})$, σ_{ij}, $\dot{u}_j$ are varied independently since $\dot{\sigma}_{ij}$ and the structural parameters which may be included in the potential Φ are not varied. The rates $u_i = u_i^*$ are given on a part of the surface Σ_u and the forces T_i^* are given on a part of the surface Σ_T. The equilibrium equation follows from the condition that the variations of the functional (4.8) vanish, from the boundary conditions and from the creep equation.

The application of the variational equation (4.8) is connected with certain technical difficulties. For example, some of these difficulties are related to the fact that when the distribution of the stresses are given in the form of functions of the coordinates containing free parameters, when the volume integral of the potential Φ is calculated, the result cannot be expressed in the form of an explicit function of these parameters. To circumvent this difficulty, I. G. Teregulov (19XX) proposed a modification of the variational principle. Suppose that $\Phi = \Phi(q_i, s)$, where q_i are any structural parameters, s is a homogeneous function of degree one of σ_{ij}, and it is assumed that the elasticity varies linearly with the ductibility tensor B_{ijrs}. We let $\partial\Phi/\partial s = v(q_i, s)$ and consider the following functional:

$$J = \int_v \left[\varphi z - \int_0^z v(q_i, s)\, ds + \sigma_{ij}\dot{e}_{ij} - \varphi s - B_{ijrs}\sigma_{ij}\dot{\sigma}_{rs} - F_i\dot{u}_i \right] dv - \\ - \int_{\Sigma_u} \sigma_{ij}\nu_j(\dot{u}_i - \dot{u}_i^*)\, d\Sigma - \int_{\Sigma_T} T_i^*\dot{u}_i\, d\Sigma. \qquad (4.9)$$

Here, in addition to σ_{ij} and $\dot{u}_i$, φ and z are also independent arguments of the functional. Varying φ, we find z = s, varying z, we find $\varphi = v\ (q_i, z)$. The equilibrium equations, the boundary conditions and the creep equations are obtained from the above in the usual manner.

The advantage of functional (4.9) is that both the equivalent stress s = z and the function $v(q_i, s) = \varphi$ are not expressed in terms of the stresses but are given independently.

4.5. Power Creep Law with Power Hardening

A convenient analytical form for the creep law with hardening is

$$\dot{p}_{ij} = \frac{q^{-\alpha}}{1+\alpha} s^r \frac{\partial s}{\partial \sigma_{ij}}. \qquad (4.10)$$

Here, all magnitudes are reduced to dimensionless form, so that there are no dimensional constants in the equation. The hardening measure can be selected in various ways, namely:

a) $$q = \int \omega(dp_{ij}),$$

b) $$q = \int \sigma_{ij}\, dp_{ij}.$$

When constant loads are acting on the body, in the limiting case, when the elastic deformation is negligibly small, equations (4.10) become the steady state creep equations in a different time scale $\tau = t^{1/(1+\alpha)}$. The corresponding state can be called the quasistationary creep state (Yu. N. Rabotnov, 1966). Yu. N. Rabotnov (1966) proposed the following method for the approximate solution of problems dealing with the redistribution of the bonding reactions in statically indeterminate systems and finding the displacements of certain points. Suppose that the generalized forces Q_i to which the generalized displacements q_i correspond are acting on the body. We take $p_i = q_i - \beta_{ij}Q_j$, where β_{ij} is the matrix of the elastic coefficient for the effect. The solution of the quasistationary creep problem has the form

$$\dot{p}_i = Q^n \frac{\partial Q}{\partial Q_i},$$

where Q is a homogeneous function of the first degree of the forces Q_i. The approximate solution of the transient creep problem is obtained from the following system of equations:

$$\dot{p}_i = \frac{P^\alpha}{1+\alpha} Q^r \frac{\partial Q}{\partial Q_i}. \tag{4.11}$$

The quantity P is determined as follows:

a) The quantities $\partial Q/\partial Q_i$ are related by the identity $\Omega(\partial Q/\partial Q_i \equiv 1$, where Ω is a homogeneous function of the first degree, and

b)

$$P = \int \Omega(dp_i);$$

$$P = \int Q_i\, dq_i.$$

The case when one active constant force and one reaction are acting on the body was studied.

T. G. Mustafayev (1968) developed this method and applied it to the solution of statically indeterminate problems In the neighborhood of the minimum of the function Q which corresponds to the limiting quasistationary creep state, this function is well approximated by a quadratic relation, which simplifies considerably the computations. Examples of beams, frames, the relaxation of a composite pipe, the relaxation of a disc mounted on a shaft were also studied.

4.6. Plates and Shells

The most important and interesting problems in creep theory as applied to shells deal with stability problems and will be considered separately. An approximate method for the solution of geometrically linear problems on the basis of flow theory was proposed by V. I. Rozenblyum. It is assumed that the creep curves are similar, which makes it possible to introduce the modified time $\tau(t)$. The deformation rates of the middle surface and the rates at which the curvature changes with respect to the modified time are defined as follows:

$$\frac{\partial e_{ij}}{\partial \tau} = \frac{\partial}{\partial t_{ij}} \left(\Phi + \frac{\partial \Pi}{\partial \tau} \right), \quad \frac{\partial k_{ij}}{\partial \varepsilon} = \frac{\partial}{\partial m_{ij}} \left(\Phi + \frac{\partial \Pi}{\partial \tau} \right). \tag{4.12}$$

Here Π is the elastic potential of the shell expressed in terms of the forces and moments. The structure of equations (4.12) shows that for them the equivalent variational formulation is (4.1). Therefore, the approximate technique developed by L. M. Kachanov in which the forces and moments are given in the form

$$t_{ij} = t'_{ij} + \theta(t)(t''_{ij} - t'_{ij}), \quad m_{ij} = m'_{ij} + \theta(t)(m''_{ij} - m'_{ij}). \tag{4.13}$$

can be used to solve individual problems. Here one prime denotes quantities that refer to the initial elastic state and two primes the quantities that were found from the solution of the steady state creep problem. Analogously, the solution of the relaxation problem is sought in the form

$$t_{ij} = \rho(t)\, t'_{ij}, \quad m_{ij} = \rho(t)\, m'_{ij}.$$

Problems in the relaxation of stresses in a cylindrical plate during pure flexure, cylindrical bending of a rectangular plate, the creep in a freely supported circular plate under the action of uniform pressure were solved in this manner.

Yu. N. Rabotnov (1966) used the technique described in Section 4.5 to analyze creep problems of shells for power hardening. If the quasistationary creep potential is taken in the form $S^{n+1}/(n + 1)$, where S is a homogeneous function of t_{ij} and m_{ij} of first degree, taking the expression for S in the form (3.10), the hardening measure P is defined by the formula

$$P = \int_0^t (\varepsilon_{0n}^2 + g\varkappa_{0n}^2)\, dt. \tag{4.14}$$

Here ε_{0n} and $\varkappa_{0n}$ are the intensities of the creep deformation rates and the changes in the curvature resulting from creep and g is a constant which depends on n. For large values of the index n we can expect that the following approximation will be suitable for S

$$S = m_0 + t_0^2, \tag{4.15}$$

which corresponds to the limiting state of the rod subjected to the torque and the transverse force. Expression (4.15) can be transformed into a homogeneous expression of first degree, and the hardening measure is defined according to the general rule.

To estimate the errors of approximate methods, G. V. Ivanov (1966) considered the simplest case of an element of a plate to which a transverse force and a torque in the same direction were applied. The calculation was based on the variational equation of D. L. Sanders, et al., in which the

stress and strain rates are varied. For the problem under consideration, it was sufficient to vary the stress rates. The standard solution which was used was the solution obtained by replacing the integrals over the thickness by a 15 point Gaussian quadrature formula, and the result obtained, using the V. I. Rozenblyum method with a linear distribution of the stresses over the thickness which was approximated by four terms in the Legendre polynomial expansion, was compared. The last approximation always gives a good result, and for other approximations, regions can be found for the values of the parameters in which they are satisfactory.

§5. Stability during Creep

5.1. Formulation of the Stability Problem

The use of the term "stability" in the many problems which will be considered below is arbitrary to some extent. The creep of metals as a rule is not limited. This means that no matter how small the load, the deformation can be arbitrarily large after a sufficiently long time. Therefore, any creep process is not stable. This is illustrated by the well-known scheme of N. Hoff for determining the elastic rupture time. If we assume that the exponential creep law is valid for arbitrarily large deformations, under a constant load the deformation increases with time according to the law

$$e = -\frac{1}{n}\log(1 - n\sigma_0^n t). \qquad (5.1)$$

Here σ_0 is the stress referred to the original area of the cross section. It can be seen from expression (5.1) that $e \to \infty$ for $t \to t_{cr} = 1/(n\sigma_0^n)$. It is also clear that if the quantity σ_0 varies so that it becomes arbitrarily small, a time t can be found for which the change in e will be arbitrarily large.

For certain creep problems in a geometrically linear formulation, unlike in the problem of N. Hoff, the formal solution yields an infinite displacement for a finite time. This time is called the critical time. It is obvious that the time defined in this manner does not have a real meaning, it vanishes when the same problem is studied in a rigorous geometrically nonlinear formulation. However, the character of the relation between the displacement and the time is such that the critical time is an estimate which does not overestimate excessively the real operational capacity of the element.

On the other hand the creep is accompanied by an elastic and plastic deformation. The continuous increase in the displacements with time as a result of creep can bring the system into a state in which the displacements change instantaneously by a finite amount. In geometrically nonlinear systems an elastic crack can occur, and in plastic elements an instantaneous bulging as a result of insufficient elastoplastic resistance. During the solution of creep problems, the instant at which the crack or bulging occurs is detected by the fact that the rate at which the displacement increases becomes infinite at some finite value of the displacements and some finite time which is now taken as the critical time. It is known that for a rod from an elastoplastic material which was bent initially, the magnitude of the critical compression force depends on the initial bending. Conversely, if the force is given, an initial bending can be found for which this force will be critical. The increase in the bending as a result of creep can be considered to be equivalent to the increase in the initial bending of an elastoplastic rod. Thus, for any magnitude of the compression force, the critical state is attained at some instant. However, creep leads to a redistribution of the stresses; therefore, as S. A. Shesterikov (1962) has shown, the simple system that was presented is only suitable for a one-parameter system. A study of the bulging of rods in the presence of plastic deformations using a numerical method is available in the study of V. I. Van'ko and S. A. Shesterikov (1967).

5.2. Stability of Linear Viscoelastic Systems

The study of A. R. Rzhanits (1946) considered the problem of the stability of a compressed rod from a viscoelastic material whose behavior is described by the model of a standard viscoelastic body:

$$\dot{\sigma} + \lambda\sigma = E(\dot{e} + \mu e). \tag{5.2}$$

In contrast to metals, the creep in the body described by equation (5.2) is finite: for $t = 0$ $\sigma = Ee$, for $t = \infty$ $\sigma = (\mu/\lambda)\, Ee (\mu < \lambda)$. The results of this study reduce to the following. If the force is greater than the critical force of long duration but smaller than the instantaneous critical force, the rod is not stable in the sense that any perturbation leads to an unlimited increase in the bending with time. If the force is greater than the instantaneous critical force, the perturbation causes instantaneous loss of stability.

This analysis was presented here in order to emphasize the difference between the creep in metals and bodies whose behavior is described by rheological models with limited creep.

5.3. Small Deviations from Fundamental States

In the study of geometrically linear problems of rods, plates and shells, it is natural to consider the torqueless stressed state as the basic state and to linearize the creep equations around the basic state. Considering the problem of a compressed rod from a material obeying the creep law with hardening, Yu. N. Rabotnov and S. A. Shesterikov (1956) determined that the variations in the stresses and strains are related by an equation of type (5.2), in which the constants are replaced by known functions of time. The deflection is a function of the coordinate multiplied by the function of time $\tau(t)$. If the rod was originally a straight line and at some instant t it was perturbed, for example, a transverse load was applied to it, a critical time t_{cr} can be found such that if the perturbation occurred at an instant $t < t_{cr}$, $\dot{\tau} < 0$, and if $t > t_{cr}$, $\dot{\tau} > 0$. It was proposed that the time t_{cr} be taken as the critical time in some agreed on sense. In fact, the system is not stable relative to a load applied at any instant, the function $\tau(t)$ first decreases, reaches a minimum and then increases without a bound. The critical time agreed on in the sense mentioned characterizes the relative rate at which the bending increases after the perturbation is applied. L. M. Kurshin (1961, 1963) proposed that this relative rate be characterized by the sign of the second derivative. Thus, according to Kurshin, the time when the sudden perturbation causes a motion with initial zero acceleration ($\ddot{\tau} = 0$) is taken as the critical time.

In the case of a constant acting perturbation, for example, in the presence of an initial eccentricity of the applied load, $\dot{\tau} > 0$, the acceleration changes sign, it is first negative and then becomes positive. S. A. Shesterikov (1959) proposed to adopt as the critical time the time when $\ddot{\tau} = 0$ for a constant acting perturbation.

Linearized creep equations for plates were obtained independently at the same time by S. A. Shesterikov (1961) and L. M. Kurshin (1961). A number of problems dealing with the stability of plates and shells on the basis of the linearized theory were studied by S. A. Shesterikov, L. M. Kurshin, A. P. Kuznetsov (1964), I. G. Teregulov (19XX) and other authors. The same criteria as those pointed out above that were applied to rods were used. G. V. Ivanov (1961) drew attention to the fact that when the stability criterion is generalized to the case of nonelastic systems, the transition from the fundamental state to the additional state plays an important role, and he

gave a generalization of the classical criterion. The critical value of the loading parameter adopted is the smallest value for which a nontrivial equilibrium state can exist with the condition that the transition from one state into the nontrivial equilibrium state take place when certain constraining conditions imposed on the additional deformations are satisfied. Time is used as the loading parameter in creep problems.

5.4. Variational Methods for Bulging Problems

The study of stability problems with the aid of variational methods required that these methods be extended to the geometrically nonlinear theory. While the Lagrange principle can be naturally extended to the nonlinear case, the Castigliano principle in its usual form can no longer be applied, since the equilibrium equations contain the displacements. Therefore, for the problems that were mentioned, more general variational principles were developed in which the quantities characterizing the stressed state as well as the quantities related to the deformations can be varied independently. V. I. Rozenblyum (1954) obtained an approximate solution of the bulging problem for a compressed initially twisted rod from the stationarity condition for the functional

$$\mathcal{J} = 2\int_v \left(\Phi + \frac{\partial \Pi}{\partial t}\right) dv - \frac{P}{2}\frac{\partial}{\partial t}\int_0^l \left(\frac{\partial u}{\partial z}\right)^2 dz. \tag{5.3}$$

Here the volume integral has the same meaning as in (4.1), P is the compression force, u is the deflection, and z is the coordinate along the rod axis. The stress σ and strain u are varied independently.

The well-known Reissner variational principle that was formulated for the theory of elasticity can be naturally extended and applied to transient creep problems. In particular, a variational equation of type (4.8) can also be obtained for the case when a transverse force is present. Thus, it can be used to study bulging problems.

The variational equation of D. L. Sanders, G. D. MacComb and F. R. Schlechte (NACA Techn. Note., No. 4003, 1957), is convenient for the numerical calculations. The corresponding functional has the form

$$J=\int_v \left[\dot{\sigma}_{ij}\dot{e}_{ij}-\frac{1}{2}\sigma_{ij}\dot{u}_{k,i}\dot{u}_{k,j}-\frac{1}{2}(\dot{e}^M_{ij}+2\dot{p}_{ij})\dot{\sigma}_{ij}\right]dv- \tag{5.4}$$
$$-\int_{\Sigma_u}\dot{T}_i(\dot{u}_i-\dot{u}_i^*)\,d\Sigma-\int_{\Sigma_T}\dot{T}_i^*\dot{u}_i\,d\Sigma.$$

Here

$$\dot{e}_{ij}=\frac{1}{2}(\dot{u}_{i,j}+\dot{u}_{j,i}+u_{k,i}\dot{u}_{k,j}+u_{k,j}\dot{u}_{k,i}),$$

$\dot{e}^M_{ij}$ is the rate of the instantaneous plastic deformation which is expressed in terms of $\dot{\sigma}_{ij}$ or by Hooke's law, or by equations from the theory of plasticity of the flow type, $\dot{p}_{ij}$ is the creep rate which depends on σ_{ij} and any structural parameters, and $\dot{\sigma}_{ij}$ and $\dot{u}_i$ are varied. If σ_{ij} and u_i are given in the form of linear combinations of appropriately selected functions, the condition that the functional (5.4) be stationary leads to a system of ordinary differential equations which are linear in the derivatives.

G. V. Ivanov (1963) constructed for geometrically nonlinear problems a variational equation in which the stresses and displacements are varied independently and in which those states are compared for which both the equilibrium equations and the equations obtained from them by differentiation are satisfied.

The variational equation (4.9) proposed by I. G. Teregulov remains valid also in the case when the bends are not small and the components of a small deformation are expressed in terms of the displacements by means of nonlinear formulas. For steady state creep I. G. Teregulov (1962, 1966) also constructed another variational equation. The corresponding functional has the form

$$J=\int_v \Big\{\psi(v-H)+\int_0^H s(v)\,dv-$$
$$-\sigma_{ij}\Big[\varepsilon_{ij}-\frac{1}{2}(\dot{u}_{i,j}+\dot{u}_{j,i}+u_{i,k}\dot{u}_{j,k}+\dot{u}_{i,k}u_{j,k})\Big]+F_i\dot{u}_i\Big\}\,dv+ \tag{5.5}$$
$$+\int_{\Sigma_u}\sigma_{ij}\nu_j(\dot{u}_i-\dot{u}_i^*)\,d\Sigma-\int_{\Sigma_T}T_i^*\dot{u}_i\,d\Sigma.$$

Here the stresses σ_{ij}, the deformation rates ϵ_{ij}, the displacement rates $\dot{u}_i$ and also the functions ψ and H are varied independently.

5.5. Stability of Shells

For sufficiently thick shells the stability problem can be formulated in the same way as for rods. If the problem of the increase in the flexure with time is solved in a geometrically linear formulation, it turns out that the flexure becomes infinite for the finite value of time which is taken as the critical time. Thus, Yu. M. Volchkov (1965) studied the bulging of a compressed cylindrical shell of finite length supported along the edges and of a semi-infinite shell with a supported end. Yu. M. Volchkov and Yu. V. Nemirovskiy (1966) extended the method to shells reinforced by stringers and frames. The characteristic feature of such problems is that as a result of the securing conditions there is no initial torqueless state in the shell so that it is not necessary to introduce initial flexure in the analysis.

The situation is different for thin shells. Creep leads to an increase in the flexures and a redistribution of the stresses in the shell, so that at a particular instant of time the shell becomes unstable with respect to instantaneous perturbations which obey the elasticity law, and an elastic loss of stability of the crack type occurs. The study of A. S. Vol'mir and P. G. Zykin (1962) gives an approximate solution of the problem of the stability of a compressed cylindrical panel. It is assumed that the form of the flexed surface is preserved but that the flexure increases as a result of creep. The change in the flexure as a result of creep is assumed to be equivalent to the change in the initial flexure. On the other hand an initial flexure exists for each value of the compression force for which this force is critical. The time at which the magnitude of this equivalent initial flexure is reached is taken as the critical time.

In fact, creep leads to a change in the shape of the flexure and a redistribution of the stresses. Therefore, to determine the critical time it is necessary to solve the creep problem which is accompanied by the elastic deformation. In one-dimensional problems, the application of variational equations of one type or another leads to relatively simple approximate solutions. V. N. Shepelenko (1965) studied the stability of an arc with latched ends on the basis of variational equation (5.4), and I. G. Teregulov applied variational equations (4.9) and (5.5) to an infinitely long cylindrical panel and to a spherical segment.

Among two-dimensional problems the greatest attention was given to a cylindrical panel and a cylindrical shell. Here the studies of G. V. Ivanov (1966), L. M. Kurshin (1964), L. M. Kurshin and A. P. Kuznetsova (19XX), E. I. Grigolyuk and Yu. V. Lipovtsev (1965, 1966) should be mentioned. Various creep theories were used and simplifying assumptions of one type or another were made during the calculations, which, as a rule, were carried out on an electronic digital computer. Thus, in the studies of L. M. Kurshin and also of E. I. Grigolyuk and Yu. V. Lipovtsev, creep equations were used that were linearized around the basic torqueless state.

5.6. Fracture during Creep

V. I. Rozenblyum (1957) obtained a solution of the problem of determining the time until the fracture of a disc of constant thickness with a hole. The steady state creep equations extended to the case of finite deformations were used as the basis. The viscous fracture scheme was studied in this manner. L. M. Kachanov (1960) studied on the basis of his theories certain problems about the fracture time of rod systems and obtained the general formulation of the problem of the motion of the fracture front and determined the fracture time of a twisted shaft. Yu. N. Rabotnov (1963) solved the problem of the fracture of a disc with a hole using the brittle fracture scheme. The effect of the accumulated damage on the creep rate and consequently on the distribution of the stresses was also studied. Later Yu. N. Rabotnov (1968) studied the problem of the effect of the stress concentration on long-term strength. It was assumed that the distribution of the stresses differs little from the distribution of the stresses in the rigid-plastic body, but a variable magnitude of the degree of damage ω does occur in the plasticity condition which becomes similar to the equilibrium condition of an inhomogeneous loose medium.

§6. Linear Viscoelasticity

6.1. Rheological Models and Differential Relations

In early studies on viscoelasticisty, differential relations of type (2.23) were used as the basis, from which, in particular, the well-known Maxwell and Foigt models were obtained. A. N. Gerasimov (1938) generalized the Maxwell equations to the three-dimensional case and obtained equations of type (2.25) with an exponential kernel. In another study, A. N. Gerasimov (1939) studied the problem of small vibrations of viscoelastic membranes. A. Yu. Ishlinskiy (1940) studied a model which was named the standard viscoelastic body model for which the relation between the stresses and strains is given by equation (5.2). Longitudinal oscillations of the rod

were considered. In other studies A. Yu. Ishlinskiy added to model (5.2) dry frictional elements, and static models were studied which were constructed from a large number of viscoelastic elements with a certain distribution of the parameters. In 1945 A. Yu. Ishlinskiy proposed a generalization of equation (5.2) to the three-dimensional case.

A. R. Rzhanitsyn (1946) applied the model of a standard viscoelastic body to the solution of many problems, the motion of a load on a viscoelastic beam, a viscoelastic beam on a viscoelastic base, the stability of a viscoelastic rod and other problems.

A. Yu. Ishlinskiy (1946) studied the problem of the fracture of viscoelastic materials. An important generalization of the differential viscoelasticity laws is due to A. N. Gerasimov (1948) who proposed instead of the usual derivatives for the description of the viscoelastic properties derivatives whose orders are rational numbers in the sense of Liouville. The inversion of such relations leads to integral equations with an Abel kernel with a weak singularity. This idea played an important role in the further development of the theory.

6.2. Creep and Relaxation Kernels

The selection of kernels which reproduce sufficiently well the properties of real materials is important in the application of viscoelastic theory. L. Boltzmann assumed that the creep kernel has a strong singularity of the type $(t - \tau)^{-1}$, which leads to a contradiction. Apparently, G. Duffing was the first man to apply kernels with a weak singularity namely, $(t - \tau)^{\alpha}(-1 < \alpha \leq 0)$. In the studies of Soviet authors a great deal of attention was given to the selection and study of special kernels with a weak singularity describing the exceptionally fast increase in the creep deformation in the beginning and its asymptotic tending to some limiting value. G. L. Slonimskiy (1939) and A. P. Bronskiy (1941) proposed a kernel of the following form:

$$\varkappa(t) = t^{\alpha-1} \exp(-t^{\alpha}) \quad (0 < \alpha \leqslant 1). \tag{6.1}$$

Using the kernel (6.1), A. P. Bronskiy described the aftereffect processes in rubber. The resolvent kernel of a kernel of type (6.1) could not be found, but A. P. Bronskiy has shown that a function of the same form can be taken as an approximation of the resolvent kernel. A. R. Rzhanits (1946) formulated the boundedness condition for the kernel and proposed a new singular kernel which was simpler than (6.1) and had similar properties

$$\varkappa(t) = t^{\alpha-1}\exp(-\beta t). \tag{6.2}$$

Yu. N. Rabotnov (1948) constructed a class of functions which were the resolvent kernels of the Abel kernel $(t - \tau)^{-\alpha}$:

$$\mathcal{Э}_\alpha(\beta, t) = t^\alpha \sum_{n=1}^{\infty} \frac{\beta^{n} t^{(\alpha+1)n}}{\Gamma[(\alpha+1)(n+1)]}. \tag{6.3}$$

These functions were called fractional-exponential functions. If an Э-function is taken for the creep and relaxation kernel the essential singularities of the kernels (6.1) and (6.2) are preserved. However, operators with kernels constructed from Э-functions have a special algebra, and their resolvent kernels are formed from functions of the same class whose parameters can be computed according to simple rules. The properties of the Э-operators were studied in the work of M. I. Rozovskiy, I. I. Krush, N. N. Dolinina, Ye. S. Sinayskiy who proved a number of theorems about products of these operators and finding the inverse operators, etc. M. I. Rozovskiy (1959) derived the connection between the Э-functions and the Mittag-Loeffler functions. The asymptotic behavior of Э-functions was studied by B. D. Anin (1961), G. I. Bryzgalin (1963) and Ye. S. Sinayskiy (1965). S. Z. Wolfson (1961) established that the resolvent kernel of the Rzhanitsyn kernel has the form

$$\exp(-\beta t)\, \mathcal{Э}_\alpha(\gamma t). \tag{6.4}$$

V. G. Gromov (1967) has shown that operators with the kernels (6.4) have the same algebra as the Э-operators. At the same time he generalized the fundamental results in the algebra of exponential operators to any resolvent operators and studied analytic functions of the operators and a general method for their interpretation.

6.3. Determination of the Parameters of the Kernels from Experimental Data

M. A. Koltunov (1966) proposed a method for determining the parameters of the Rzhanits kernel and its resolvent kernel by means of a corresponding shift of the creep curve represented on semilogarithmic paper. Ye. N. Zvonov, N. I. Malinin, L. Kh. Papernik and B. M. Zeitlin (1966-1968) developed a method for

the selection of the fractional-exponential kernel using an electronic digital computer. The Laplace transform of the experimental creep curve is obtained and the best approximation for the image of the Э-function is sought. The studies of G. L. Slonimskiy and L. Z. Rogovina (1964) as well as the studies of G. L. Slonimskiy and his collaborators (1966) are devoted to the problem of determining the parameters of the kernel (6.1).

6.4. The Volterra Principle

The principle formulated by Volterra based on the fact that the linear operations of differentiation and integration with respect to the coordinates and multiplication by the time Volterra operator are commutative plays a fundamental role in the solution of static viscoelastic problems. Therefore any solution of the static problem in classical elasticity theory is transformed into a solution of the corresponding problem in linear viscoelasticity by replacing in the final result the elastic constants by the corresponding operators. If the elastic constants appear in the solution of the classical problem in the form of a multiplier which is a rational combination of the elastic constants, the determination of the rational function of the operators reduces to a successive solution of Volterra integral equations of the second kind. For exponential and fractional-exponential operators, these calculations are carried out according to standard rules. A more complex situation arises when in the solution of the problem in elasticity theory the elastic constants are not rational combinations and also if the types of boundary conditions change at various points of the surface.

The methods based on the direct application of the Volterra principle using Э-operators were developed by M. I. Rozovskiy (1962-1964) for various problems in viscoelasticity. With regard to reinforced and non-reinforced polymers, the studies of G. N. Savin and G. A. Van Fo Foi (1965), G. A. Van Fo Foi (1965-1967), G. I. Bryzglin (1965), F. Ya. Bulaves and A. M. Skudra (1964, 1965) should be mentioned. In a number of studies, the apparatus of linear viscoelastic theory was applied to the mechanics of rocks. In this field, the studies of Zh. S. Yerzhanov (1962, 1963), Sh. M. Aytaliyev (1964), V. P. Matveyeva, M. I. Rozovskiy and V. T. Glushko (1964), M. I. Rozovskiy and G. I. Bulakh (1964) should be mentioned. Various variants in the theory of viscoelasticity for the anisotropic materials were studied by M. A. Koltunov (1964), V. V. Bolotin (1966), A. A. Germelis and V. A. Latishenko (1967).

The Volterra principle was also applied to certain contact problems in viscoelasticity, namely to problems in which the contact region increases monotonically. Contact problems of this type were studied by A. B. Yefimov (1966), Ya. Ya. Rushitskiy (1967), M. I. Rozovskiy and N. N. Dolinin (1968). For those problems in which the viscoelastic operators are not rational combinations, M. I. Rozovskiy (1956) proposed a semisymbolic method which reduces the viscoelastic problem to the solution of an integro-differential equation. The problem of a moving die was studied by R. Ya. Ivanov (1964) and also by L. A. Galin and A. A. Shmatkova (1968).

6.5. Application of the Laplace Transform

By applying the Laplace transform to the system of equations and boundary conditions for the viscoelastic problem, the equations of classical elasticity theory are obtained for the images and after the solution of this problem in the final result the inverse images are transformed to the originals. The constraints that were mentioned above in connection with the application of the Volterra principle also hold in this case. The main difficulty consists of applying the Mellin transformation. The method that was mentioned was applied in the work of V. B. Zelenskiy (1963), M. A. Koltunov (1964), Ya. G. Skomorovskiy (1964), Ye. S. Sinayskiy (1964, 1965), V. N. Kukudzhanov (1963), T. Ya. Barinova (1965) and A. P. Khoroshun (1964). A. A. Il'yushin (1968) developed an approximate method for the solution of viscoelastic problems based on a special approximation of the solution of the problem in the theory of elasticity which depends on the Poisson ratio. The solution of the viscoelastic problem depends only on two functions which can be determined independently from the experiment. The studies of D. L. Bykov (1968) and also of V. S. Yekel'chik and V. N. Rivkind (1968) are also along these lines. The last authors studied the viscoelastic behavior of anisotropic plates and shells.

6.6. Dynamic Problems in Viscoelasticity

V. G. Gogoladze (1938) studied certain wave problems in the theory of viscoelasticity keeping in mind applications to seismology. Plane expansion and shear waves as well as Rayleigh waves were studied. The extension of the Volterra principle to free and forced oscillations was obtained in the studies of M. I. Rozovskiy (1963) and also in a number of studies made by M. I. Rozovskiy and I. I. Krush. The fundamental fact on which the theory is based is the commutative property established by M. I. Rozovskiy

$$\frac{d^m}{dt^m}\left(K^{*n}\varphi(t)\right) = K^{*n}\frac{d^m\varphi}{dt^m}, \quad \varphi(0) = \varphi'(0) = \ldots = \varphi^{(m-1)}(0). \tag{6.5}$$

For free oscillation problems M. I. Rozovskiy constructed a class of functions which are related to the fractional-exponential functions in a similar way as the trigonometric functions are to the usual exponential functions.

In the studies of internal frictional processes in metals when the amplitudes of the stresses and deformations are very small, the relations of linear viscoelasticity are valid. Until recently in the description of the frequency relations for the internal friction, rheological models were used predominantly which led to differential relations and also spectral representation of the kernels. In the studies of T. D. Shermergor and S. I. Meshkov (19XX), it was shown that kernels with a weak singularity of the Abel type describe well the relations observed in experiments.

6.7. Nonlinear Viscoelasticity

Relations of type (2.29) or (2.30) for the nonlinear viscoelastic behavior of the material in a uniaxial stressed state were applied on numerous occasions and improved on the basis of an analysis of the experimental data. N. I. Malinin and A. V. Dolgov (1964) described the results of their experiments by the relation

$$e = \frac{\sigma}{E} + \int_{\tau_1}^{t} \frac{f[\sigma(\tau)]\, d\tau}{(t-\tau)^{\alpha[\sigma(\tau)]}}. \tag{6.6}$$

M. A. Koltunov (1966-1968) proposed an equation which took into account directly the effects of the deformation rate and the loading:

$$\varphi(e, \dot{e}) = \psi(\sigma, \dot{\sigma}) + \int_0^t K(t-\tau)\,\psi(\sigma, \dot{\sigma})\, d\tau. \tag{6.7}$$

Various variants of nonlinear theories were studied by S. Z. Wolfson (1963, 1964) and A. A. Cizik (1964).

The relations of the type considered were extended to the complex stressed state in various ways. Yu. N. Rabotnov (1948) used the equation of deformation plasticity theory where it was assumed that the intensity of the stresses and strains were related by equation (2.29). I. I. Bugakov (1965, 1966) studied integral equations of the general form

$$e_{ij}(t) = e^{y}_{ij}(t) + \int_0^t Q_{ij}[t-\tau, \sigma_{\alpha\beta}(\tau), T(\tau)]\, d\tau. \tag{6.8}$$

Here e^{y}_{ij} is the elastic instantaneous deformation, and $T(\tau)$ is the temperature. In the studies of G. A. Teters (1965) and A. K. Malmeister (1965), the theory of local deformations is developed taking into account the time factor, which leads to nonlinear integral equations. A variant of nonlinear heredity theory as applied to frozen soils was constructed by S. S. Vyalov (1964). This theory was somewhat simplified in the study of Yu. K. Zaretskiy (1964).

The extension of the general relations (2.28) to the complex stressed state leads to very complex relations and it is doubtful whether the successive kernels can actually be determined experimentally. A. A. Il'yushin and P. M. Ogibalov (1966) developed a relatively simple variant of the theory for the case of weak nonlinearity which is typical of polymer materials. The basic requirement was that the relations obtained be linear in the tensors. Retaining in the expansions the third powers, the relations in the theory have the following form:

$$\left.\begin{aligned} \frac{1}{2g} e'_{ij} &= \int_0^t [K(t, \tau) + \varphi(t, \tau)]\, \sigma'_{ij}(\tau)\, d\tau, \\ \varphi(t, \tau) &= \int_0^t \int_0^{\xi} K_3(t, \tau, \xi, \eta)\, S(\xi, \eta)\, d\eta\, d\xi, \\ S(\xi, \eta) &= \sigma'_{ij}(\xi)\, \sigma'_{ij}(\eta). \end{aligned}\right\} \tag{6.9}$$

Here the primes denote the deviators of the corresponding tensors. The structure of the inverse equations of (6.9) is analogous. It is assumed that the material is isotropic but it is not assumed that the properties are invariant. For stable materials, it is possible to draw certain conclusions about the structure of the kernels K and K_3. Under certain

additional assumptions, simpler relations with kernels depending only on a single variable were obtained from (6.9).

The studies of B. Ye. Pebedri (1965), B. Ye. Pobedri and M. M. Soldatov (1966-1968), V. V. Moskvitin (1967) are devoted to the study of nonlinear relations of general form and also to the study of special cases.

6.8. Some Applications of the Theory of Viscoelasticity

Many applications in the theory of viscoelasticity deal with rods, plates and shells where, in addition to the usual viscoelastic relations, much more simple models of the Voigt or Maxwell type were also studied. Thus, in stability problems during creep the basic qualitative effect is related to the geometric nonlinearity as a result of which an elastic crack can occur. In the discussion of individual examples, the use of linear viscoelastic relations instead of the nonlinear creep law simplifies considerably the technique without changing essentially the qualitative results. On the other hand, depending on the properties of the kernel, and the character of the problem, the solution which corresponds to some initial perturbation may tend to a finite limit or it may increase without limit.

Problems dealing with the stability of viscoelastic rods were discussed by A. M. Datkayev and I. I. Krush (1966), A. M. Lokoshchenko and S. A. Shesterikov (1966), and I. Ye. Prokopovich (1967). The stability of shells was considered in the studies of V. I. Danilov (1966), I. G. Teregulov (1965), P. M. Ogibalov (1967), M. A. Koltunov (1966, 1967), A. M. Datkayev and I. I. Krush (1966), V. A. Kominar and N. I. Malinin (1966), I. Ye. Prokopovich (1967). The stability of an elastoplastic arc was studied by A. P. Kuznetsov (1965) and V. N. Shepelenko (1965).

Plates consisting of layers and shells with elastoplastic layers were studied in the work of E. I. Grigolyuk (1961), E. I. Grigolyuk and P. P. Chulkov (1964).

G. N. Savin and G. A. Van Fo Foi (1966) and V. G. Savchenko (1966) studied the problem of stress concentrations in viscoelastic structural elements.

PAGES 155 through 226 MISSING FROM RUSSIAN TEXT

THEORY OF ELASTIC SHELLS AND PLATES

N. A. Alumyae

§1. Introduction

Because of the many practical applications, the theory of shells developed in the Soviet Union rapidly keeping pace with the growth of military technology and the buildup of industrial power.

The application of the theory of shells expanded considerably with the passage of time and the center of gravity of the efforts shifted in the direction of the solution of concrete problems. While in the first decades construction technology served as the

impetus for the investigations, later the leading position shifted to aviation technology, shipbuilding and machine building.

Along with the continuous growth in the number of scientific investigations needed to ensure overall technological progress, the number of specialists engaged in the solution of problems in the theory of shells also increased. While earlier problems in the theory of shells were discussed within the framework of All-Union Conferences on Strength and Plasticity on the sectional level or in small meetings devoted exclusively to the calculation of shells (Moscow, 1952, Tartu, 1957), the subsequent All-Union Conferences in the Theory of Shells and Plates in Kazan' (1960), L'vov (1962), Yerevan (1963), Moscow (1965), Baku (1966) and Dnepropetrovesk (1969) where already very representative meetings with a continuously increasing number of participants (up to 700 people), reports and papers. The very complete publication of the papers from these conferences should be mentioned. Studies in the theory of shells were also represented fairly well in the All-Union Conferences on Theoretical and Applied Mechanics held in Moscow (1960-1968). The network of scientific centers engaged in the development of the theory of shells also expanded considerably in the USSR. Along with Moscow, Leningrad and Kiev strong centers were formed and strengthened in Kazan', Novosibirsk, Yerevan, Tbilisi, Khar'kov, Rostov-on-the-Don, and in other cities.

The number of published studies also attests to the developing scale of investigations in the theory of shells. The reference journal "Mekhanika" contained in 1955 in the section "Shells and Plates" about 60 studies of Soviet authors in the theory of plates and shells, and in 1967 the corresponding number was roughly 600 publications (many studies in the theory of plates and shells are referenced in other sections of the reference journal "Mekhanika," for example, in the sections "Elastic Waves," "Vibrations of Elastic Bodies."

However, it would be a mistake to assume that the main part of the results which makes up at the present time the contents of the theory of shells and plates was obtained in the last ten years. The intense development of the theory of shells became possible as a result of the fundamental studies of scientists-predecessors.

The results of the work of Soviet scientists in the field of the theory of plates and shells in the early development stages of the theory were often summarized, in particular, they were presented briefly in the survey articles of

Yu. N. Rabotnov (1950), I. G. Vasil'yev (1956), B. G. Korenev (1956) and O. D. Oniashvili (1957). In the present survey these results can only be touched on briefly.

We first note the studies of B. G. Galerkin (1932, 1935) as applied to the analysis of thick plates and the general solutions of the equations of elasticity theory expressed in terms of biharmonic functions, as well as the monographs of B. G. Galerkin (1934) and Yu. A. Shimanskiy (1934), dealing with the calculation of plates with various contours according to classical bending theory. The asymptotic integration method was applied for the first time to the calculation of plates by I. Ya. Shtayerman (1924), who also pointed out the analogy between the static calculations of a shell of rotation and a bent (plane) rod on an elastic base. The solution of a number of interesting problems in the torqueless theory of domes is given in the monograph of V. E. Novodvorskiy (1932) whose name is connected with one condition for the applicability of the torqueless theory: the tangential boundary conditions must not allow the bending of the middle surface (V. E. Novodvorskiy, 1933).

In the early 30's, cylindrical shells were studied intensely (the most important results were published in the articles of A. A. Gvozdev, 1932; P. L. Pasternak, 1932, and in the monographs of V. Z. Vlasov, 1933, 1936). In the studies of V. Z. Vlasov, the idea of combining the methods of the theory of elasticity and structural mechanics was realized gradually and very effectively. S. M. Feinberg (1936) proposed a simplified theory for the calculation of circular cylindrical shells with an open profile which reduces to the integration of a fourth order differential equation with complex coefficients. The problem of a girderless plating became very topical in those years. The exploratory study of L. S. Leybenzon was followed by the studies of S. A. Gershgorin (1933) and A. S. Maliyev (1935) in which the formulation of the problem was made more precise.

The first studies in the nonlinear theory of plates of the Karman type also go back to this period (P. A. Sokolov, 1932, B. I. Slepov, 1935, V. M. Darevskiy, 1936, P. Ya. Polubarinov-Kochin, 1936). In connection with these studies it is impossible not to note the outstanding achievement of N. V. Zvolinskiy (1940) who obtained a two-sided bound for the reduction coefficient of the plate after loss of stability with the aid of the variational method.

The early studies on the action of a shock on the plate are worthy of attention (A. I. Lur'e, 1934, A. P. Filippov, 1938).

The methods for the solution of linear and nonlinear problems in the statics and dynamics of plates that were developed towards the end of the 30's are presented in the outstanding monograph of P. F. Papkovich (1941) which played a very important role in the training of scientists and engineers in different engineering branches.

The first major studies in the general theory of elastic shells matured in the early 40's. The mastery and analysis of the theory of shells was facilitated by the application of tensor analysis by the leading scientists in the country, which was used to represent the fundamental relations in the theory. The consistency equations for the deformations were first derived by A. L. Gol'denveyzer (1939), A. I. Lur'e (1940) and A. L. Gol'denveyzer (1940) introduced in the theory of shells stress functions in terms of which the forces and moments which satisfy identically the equilibrium equations are determined. A. N. Kil'chevskiy (1940) developed methods for the theory of shells and for the solution of problems in it on the basis of the reciprocity theorem. The equations in terms of the displacements in geometrically nonlinear theory were published by Kh. M. Mushtari (1939). The variant of the theory discussed by him generalized the simplified nonlinear theory of Karman plates to plates with an arbitrary contour.

Later, V. Z. Vlasov (1944) represented the simplified equations of the general linear theory in a form which was analogous to the classical form of the equations for the plates in the Karman theory. Here all unknown quantities were expressed in terms of a single stress function (for the plane problem) and the bending function for the middle surface. In this study, Vlasov also introduced the now well-known concept of a flat shell. The flat shell is calculated on the assumption that the principal curvatures of the shell are constant and that the middle surface can be given in terms of a Euclidian metric (we note that in fact this variant became, after the appropriate generalizations were made, also the most popular variant in the formulation and solution of geometrically nonlinear problems in the theory of shells).

A series of studies in the qualitative investigation of stressed states in shells was initiated by A. L. Gol'denveyzer (1945-1947). Subsequently, the methods presented in his articles were used in the analysis of problems in linear stability and oscillation theory and also in the nonlinear theory of shells.

The analogy between the static and geometric relations in special cases has been noted a long time ago. In the general linear theory, this analogy was pointed out by A. L. Gol'denveyzer (1940). This property of the fundamental relations in the linear theory of shells was used most fully by V. V. Novozhilov (1946) in the derivation of the equations for the general theory of shells by introducing complex unknowns obtained pairwise from analogous magnitudes. The first applications of this theory deal with the calculation of shells of rotation and cylindrical shells.

V. V. Novozhilov (1946) and L. I. Balabukh (1946) proposed the simplest elasticity relations which do not contradict the sixth equilibrium equation and ensure that Kirchhoff's uniqueness theorem (or variational principles) and the Betti reciprocity theorems are satisfied in the theory of shells.

Along with the development of the general theory, important results were also obtained in the solution of special problems in linear theory. The theory of torqueless shells was enriched by establishing the dependence of the general solution on the sign of the Gaussian curvature of the middle surface (V. V. Sokolovskiy, 1943) using the analogy between problems in the bending of surfaces and torqueless theory to derive results about the uniqueness of the solution (Yu. N. Rabotnov, 1946) and by applying in a number of studies the theory of functions of a complex variable to calculate shells representing second order central surfaces. A large number of studies was devoted to the calculation of cylindrical shells, most to the types of shells encountered frequently in practice (V. V. Novozhilov, 1946, A. L. Gol'denveyzer, 1947, A. I. Lur'e, 1946). V. Z. Vlasov (1944), who developed further the idea of combining the methods of structural mechanics and of elasticity theory developed the variational method for the calculation of multiply connected prismatic shells, in particular for the calculation of vibrations in these structures (V. Z. Vlasov, 1947).

A mathematical presentation of the state of the linear theory of shells in those years is available in the survey article by A. L. Gol'denveyzer and A. I. Lur'e (1947). At that time many new results were obtained in the theory of shells and the theory differed already in many respects from the classical O. Loew presentation. Therefore, it is not surprising that in a short time the monographs of A. I. Lur'e (1947), V. Z. Vlasov (1949), V. V. Novozhilov (1951), A. L. Gol'denveyzer (1953) appeared which are the foundation of the contemporary theory of shells. They are also well known abroad and have been translated into foreign languages (English, German, Spanish). These outstanding studies are devoted to linear theory. The monograph of V. Z. Vlasov which includes a presentation of the foundations of nonlinear theory of flat shells is an exception.

The general nonlinear theory was developed mainly in the studies of Kh. M. Mushtari and K. Z. Galimov, and the results that were obtained are presented in a systematized form in their companion monograph "Nonlinear theory of elastic shells" (1957). These results will be discussed in greater detail in subsequent chapters of this survey.

§2. Fundamental Relations in the Theory of Elastic Shells

It would seem natural to assume that during their lengthy development, the fundamental equations of the theory of elastic shells acquired a final form and that today they are no longer the subject of studies and discussions. In fact, the last decade shows an ever increasing interest in the problem of constructing the equations themselves or, more accurately, in developing a procedure for improving gradually the precision of the stressed state. It would be an error to assume that this interest is exclusively related to new problems; the calculation of homogeneous anisotropic shells made from new structural materials and multi-layer anisotropic shells, the determination of the acceleration field around the propagation front of stress waves. This problem continues to be, not without justification, also a problem in the linear equilibrium theory of isotropic shells. Its formulation is stimulated in linear equilibrium theory primarily by the importance of developing the foundations for the calculation of shells of "medium" thickness, and second by the needs for the analysis of the stressed states at singular points (for example, around the apex of a conical shell in the zone where the concentrated load is applied), third by the necessity of clarifying the problem of satisfying the boundary conditions (or in which sense they will be satisfied with the aid of a particular computational algorithm). Finally, the fundamental methods for reducing problems in elasticity theory to problems in the theory of shells when the dimensions of the objects investigated are reduced to unity are most easily developed on the example of the simplest problems (in linear equilibrium theory).

However, it does not follow from what has been said that present day theory of shells is based on a shaky foundation. There is no doubt that for a wide class of practical problems the classical variant of the Kirchhoff-Loew theory describes adequately the stressed state of shells. Like many other outstanding achievements of science, this variant of the theory was slightly modified with the passage of time (although the modifications were necessary) and it will continue to be valid when applied to the solution of many complex problems in the theory of shells.

The weak point in the Kirkhhof-Loew theory is the seeming contradiction of the initial hypotheses: (1) when the deformation is determined over the thickness of the shell it is assumed that the transverse shear is zero, but the transverse forces are retained in the equilibrium conditions, (2) when the deformation over the thickness of the shell is determined, it is assumed that the lengths of the segments on the normal to the middle surface do not change in the deformation process, but $\sigma_{zz} = 0$ is used in the elasticity relations. At the present time, we have learned to eliminate these contradictions in the majority of cases by means of an appropriate interpretation. The exceptions are stressed states with a large variability index and stressed states in multilayer shells with a soft filler in which the transverse shear must be taken into account. However, since exceptions exist, a revision of the fundamental equations in the theory of shells with the aid of new scientific research techniques is justified. For example, the numerical solution of problems obtained in the theory of elasticity with the aid of an electronic computer, which are similar to problems in the theory of shells, can fully clarify new reduction methods or even formulate the reduction problem in explicit form.

At the same time it is useful to keep in mind the possibility of applying practically the new results expected from carrying out the revision of the theory. As a rule a shell is only a structural element. To calculate the shell, it is generally necessary to determine the conditions for the elastic fixing of its end cross section. Often this problem can only be solved in first approximation by expressing the conditions for the fixing in terms of a limited number of rigidity coefficients (or pliability coefficients). The kinematic conditions for the scarfing of the shell which serves as the rim for the structural shell is formulated in terms of the same number of generalized displacements (referred to the line of intersection of the middle and contour surfaces of the shell).

The classical Kirchoff-Loew theory determines the kinematics on the edge of the shell in terms of four generalized displacements, and its contemporary modifications (the Reissner-Timoshenko theory) in terms of five displacements. In the last case, it is assumed that the tangential displacements vary in the direction of the normal according to a linear law, and that the normal displacements are equal for all points on one normal.

But even if five generalized displacements are used to represent the kinematics of the shell, it is necessary to introduce an additional assumption, namely that the absolute value of σ_{zz} everywhere on the normal is much smaller than the sum of the absolute value of the tangential stresses $\sigma_{\alpha\alpha}$, $\sigma_{\alpha\beta}$, $\sigma_{\beta\beta}$.

Nonlinear equations in a theory of the Timoshenko type for rigid multilayer shells were derived by E. I. Grigolyuk (1958). Here, we will give the fundamental relations of a geometrically nonlinear variant of the theory for an isotropic single-layer shell (L. Ya. Aynola, 1965).

Let $a_{\alpha\beta}$, $b_{\alpha\beta}$ be the tensors of the first and second quadratic forms of the middle surface, let ∇_α be the covariant differentiation symbol in the metric $a_{\alpha\beta}$, G be the shear modulus, ν the transverse expansion coefficient, ρ the density of the material, h the thickness of the shell, p_α, p the components of the vector of external forces, m_α the components of the vector of moments referred to a unit area on the middle surface, $v_\alpha + z\varphi_\alpha$, $w + z\psi$ the components of the displacement vector, and z the distance of the point from the middle surface. Then the fundamental relations reduce to the following system of equations of motion:

$$\nabla_\gamma [(a_\alpha^\beta + e_\alpha^{\cdot\beta}) T^{\alpha\gamma}] - b_\alpha^\beta N^\alpha - b_\gamma^\beta \omega_\alpha T^{\alpha\gamma} + \nabla_\alpha (\varphi^\beta N^\alpha) + \nabla_\alpha (M^{\alpha\gamma} \nabla_\gamma \varphi^\beta) - b_\gamma^\beta b_\alpha^\theta \varphi_\theta M^{\alpha\gamma} - \rho h \ddot{v}^\beta + p^\beta = 0. \quad (2.1)$$

$$b_\beta^\alpha (a_{\alpha\gamma} + e_{\alpha\gamma}) T^{\gamma\beta} + \nabla_\beta (\omega_\alpha T^{\alpha\beta}) + \nabla_\alpha N^\alpha + b_\alpha^\beta \varphi_\beta N^\alpha + \nabla_\beta (b_\alpha^\gamma \varphi_\gamma M^{\alpha\beta}) + b_\beta^\gamma \nabla_\alpha (\varphi_\gamma M^{\alpha\beta}) - \rho h \ddot{w} + p = 0, \quad (2.2)$$

$$\nabla_\gamma [(a_\alpha^\beta + e_\alpha^{\cdot\beta}) M^{\alpha\gamma}] - (a_\alpha^\beta + e_\alpha^{\cdot\beta}) N^\alpha - b_\alpha^\beta \omega_\gamma M^{\alpha\gamma} - \frac{1}{12} \rho h^3 \ddot{\varphi}^\beta + m^\beta = 0; \quad (2.3)$$

The elasticity relations are:

$$P_{\alpha\beta\gamma\theta} T^{\gamma\theta} = Gh (e_{\alpha\beta} + e_{\beta\alpha} + e^{\cdot\gamma}_{\alpha} e_{\beta\gamma} + \omega_\alpha \omega_\beta), \quad (2.4)$$

$$P_{\alpha\beta\gamma\theta} M^{\gamma\theta} = \frac{1}{12} Gh^3 (\varkappa_{\alpha\beta} + \varkappa_{\beta\alpha} + e_\alpha^{\cdot\gamma} \nabla_\beta \varphi_\gamma + e_\beta^{\cdot\gamma} \nabla_\alpha \varphi_\gamma + b_\beta^\gamma \omega_\alpha \varphi_\gamma + b_\alpha^\gamma \omega_\beta \varphi_\gamma), \quad (2.5)$$

$$k_{(\alpha)} N_\alpha = Gh (\omega_\alpha + \varphi_\alpha + \varphi^\gamma e_{\alpha\gamma}), \quad (2.6)$$

where

$$e_{\alpha\beta} = \nabla_\alpha v_\beta - b_{\alpha\beta} w, \quad \varkappa_{\alpha\beta} = \nabla_\alpha \varphi_\beta - b_{\alpha\beta} \psi,$$

$$\omega_\alpha = \nabla_\alpha w + b_{\alpha\gamma} v^\gamma, \qquad P_{\alpha\beta\gamma\theta} = a_{\alpha\gamma} a_{\beta\theta} - \frac{\nu}{1+\nu} a_{\alpha\beta} a_{\gamma\theta},$$

and ψ is determined from the non-differential relation

$$(1 + \nu) Gh (2\psi + \varphi^{\gamma}\varphi_{\gamma}) - \nu a_{\alpha\beta} T^{\alpha\beta} = 0. \tag{2.7}$$

The boundary conditions for this system are given in the study of L. Ya. Aynola that was mentioned above.

The variant that was presented is used in the solution of very general nonlinear dynamic problems. The necessity of taking into account the natural forces makes it necessary to solve such problems in the displacements. In the solution of equilibrium problems it is desirable to have a large set of fundamental relations. The consistency conditions for the deformations are of primary interest (the ten quantities $e_{\alpha\beta}$, $\varkappa_{\alpha\beta}$, ω_{α} are determined in terms of the six quantities v_{α}, w, φ_{α}, ψ). However, in nonlinear theory they have not yet been obtained.

The diversity of nonlinear dynamic problems is very great and includes problems in the propagation of elastic waves with a finite amplitude, and also problems dealing with stationary nonlinear oscillations around the equilibrium state. The general equations of nonlinear dynamics are very complex. Therefore, in the solution of concrete problems a preliminary qualitative analysis of the solution must be made in order to simplify (or even make partially more precise the quasi-linear equations that were derived purely formally. Often such an analysis leads to well-known nonlinear equations of the Karman type, which are supplemented by taking into account the natural forces of the normal oscillations.

A variant of the relations derived by K. Z. Galimov (1951) based on the Kirkhhof-Loew hypothesis may turn out to be more convenient in the solution of nonlinear equilibrium problems. These relations reduce to the following:

the equilibrium conditions

$$\nabla_{\alpha} (S^{\alpha\beta} - b^{\beta}_{\gamma *} M^{\gamma\alpha}) + a^{\beta\lambda}_{*} P_{\lambda,\, \alpha\sigma} (S^{\alpha\sigma} - b^{\sigma}_{\gamma *} M^{\gamma\alpha}) - b^{\beta}_{\alpha *} N^{\alpha} + p^{\beta} = 0, \tag{2.8}$$

$$\nabla_{\alpha} Q^{\alpha} + b^{*}_{\alpha\beta} (S^{\alpha\beta} + b^{\beta}_{\gamma *} M^{\gamma\alpha}) + p = 0, \tag{2.9}$$

$$\nabla_{\alpha} M^{\alpha\beta} + a^{\beta\lambda}_{*} P_{\lambda,\, \alpha\gamma} M^{\alpha\gamma} - N^{\beta} - m^{\beta} = 0; \tag{2.10}$$

the consistency conditions for the deformations

$$c^{\beta\gamma}c^{\sigma\alpha}\left(\nabla_\gamma\nabla_\alpha\varepsilon_{\sigma\beta}-\frac{1}{2}\varkappa_{\alpha\beta}\varkappa_{\gamma\sigma}-\frac{1}{2}P_{\pi,\alpha\beta}P^{\pi}_{\cdot\cdot,\sigma\gamma}\right)-$$
$$-(2Ha^{\alpha\beta}-b^{\alpha\beta})\varkappa_{\alpha\beta}-Ka^{\alpha\beta}\varepsilon_{\alpha\beta}=0, \tag{2.11}$$

$$c^{\beta\gamma}(\nabla_\gamma\varkappa_{\alpha\beta}+(b_{\lambda\gamma}+\varkappa_{\lambda\gamma})P_{\cdot\cdot\,\alpha\beta}^{\lambda})=0; \tag{2.12}$$

the elasticity relations

$$S^{\alpha\beta}=A^{\alpha\beta\pi\lambda}\varepsilon_{\pi\lambda}+B^{\alpha\beta\pi\lambda}\varkappa_{\pi\lambda}, \tag{2.13}$$

$$M^{\alpha\beta}=B^{\alpha\beta\pi\lambda}\varepsilon_{\pi\lambda}+C^{\alpha\beta\pi\lambda}\varkappa_{\pi\lambda}; \tag{2.14}$$

the relations between the deformation tensors and the components of the displacement vector of the middle surface

$$2\varepsilon_{\alpha\beta}=a^{*}_{\alpha\beta}-a_{\alpha\beta}=e_{\alpha\beta}+e_{\beta\alpha}+e_{\alpha}^{\cdot\lambda}e_{\beta\lambda}+\omega_\alpha\omega_\beta. \tag{2.15}$$

$$\varkappa_{\alpha\beta}=b^{*}_{\alpha\beta}-b_{\alpha\beta}=\frac{1}{2}c^{\gamma\rho}c_{\pi\lambda}(a^{\pi}_{\gamma}+e_{\gamma}^{\cdot\pi})(a^{\lambda}_{\rho}+e_{\rho}^{\cdot\lambda})(b_{\alpha\beta}+\nabla_\alpha\omega_\beta+b^{\gamma}_{\alpha}e_{\beta\gamma})+$$
$$+c^{\pi\nu}c_{\gamma\zeta}\omega_\nu(a^{\zeta}_{\pi}+a_{\pi}^{\cdot\zeta})(\nabla_\alpha e_{\beta}^{\cdot\gamma}-b^{\gamma}_{\alpha}\omega_\beta). \tag{2.16}$$

$$P_{\lambda,\alpha\beta}=\nabla_\alpha\varepsilon_{\beta\lambda}+\nabla_\beta\varepsilon_{\alpha\lambda}-\nabla_\lambda\varepsilon_{\alpha\beta}. \tag{2.17}$$

where H is the mean and K the Gaussian curvature of the middle surface.

The requirement on the physical relations must be that they allow for the existence of the potential energy of the deformation, and the kinematic relation must give zero values of the components of the deformation tensors during the motion of the shell as a solid. The "elegance" requirements can include the existence of an analogy between the equilibrium relations and the consistency of the deformations (a variant of such nonlinear theory was published by the author of the survey in 1957).

The so-called sixth equilibrium equation can also be the subject of discussion. Classical theory requires that it satisfy identically the physical relations. However, A. I. Lur'e (1950) has shown that this problem is eliminated when symmetric tensors of the forces and moments are introduced.

In spite of the large number of proposed variants, no "unified" general nonlinear theory of elastic shells exists so far which satisfies all requirements. With regard to the practical applications of nonlinear theory, the vast majority of studies use a simplified variant of nonlinear relations known as a system of equations of the Karman type. The monographs of A. S. Vol'mir (1956), Kh. M. Mushtari and K. Z. Galimov (1957), M. S. Kornishin (1964), the survey articles by A. S. Vol'mir (1958), Kh. M. Mushtari (1958, 1962), V. I. Feodos'eva (1966) attest to this.

The use of the simplified system of equations of the Karman type in the cases considered in practice is sufficiently well founded and useful. However, the integration of even this system is connected with great difficulties. At the present time, a natural means for the solution of problems in the nonlinear theory of shells is the use of computer technology which was initiated in our country by A. Yu. Birkgan and A. S. Vol'mir (1959). At the same time progress in this direction is not as great as one would expect. As an example, we point out the problem of the axisymmetric forms of equilibrium of a spherical dome which attracted the attention of many distinguished investigators (V. I. Feodos'ev, 1963, M. S. Kornishin, 1966, I. I. Vorovich and V. F. Zipalova, 1966). If the general mathematical software for numerical techniques will be improved considerably in the nearest future, which can be expected, many difficulties in the solution of nonlinear problems in the theory of shells will be eliminated by developing standard programs (which is the case at the present time in linear algebra). However, possibly in some cases, it will be useful to develop specific computational algorithms for problems in the theory of shells. One procedure was proposed by M. S. Kornishin and Kh. M. Mushtari (1959). A short survey of the application of numerical methods to the theory of shells was given by I. V. Svirskiy (1966).

In conclusion it should be mentioned that the integration of the equations of the theory of shells and plates in elementary or special (tabulated) functions is only possible in exceptional cases. Far-ranging results along these lines were obtained by A. D. Kovalenko, who developed the application of the theory of generalized hypergeometric functions to determining the stressed state in discs, circular plates of variable thickness and conical shells of rotation based on linear

equilibrium theory. These results are partially presented in the monographs and the survey article of A. D. Kovalenko (1955, 1959, 1963) and in the book by A. D. Kovalenko, Ya. M. Grigorenko and L. A. Il'in (1963).

§3. Variational Methods

The inclusion in this survey of a chapter on variational methods may seem to be unexpected; however, these methods have, with their complex relations, such a wide range and diversity of applications, that their significance should be emphasized. The general theory of shells or its simplified variants for the solution of any concrete problems, of course, may be constructed without using the apparatus of variational methods. However, attention must also be given to the opposite point of view. Once a set of relations for the calculations has been constructed, it must be verified whether the given model of the elastic system has a potential which allows a formulation of the problem under consideration using the calculus of variations. In the concluding stage of the development of the equations of linear theory of shells, this good rule was kept in mind (A. L. Gol'denveyzer, 1944).

Historically the high regard for variational methods used in the derivation of the boundary conditions for a system of differential equations which model a thin elastic body of complex configuration and structure is justified.

The monograph of L. S. Leybenzon (1943) can be regarded as a pioneering study in the application of variational methods to the linear theory of plates and shells. It presents the Lagrange, Castigliano and Treftz methods for the case of a plate, and it also opened up the possibility of generalizing these results without any particular difficulties to the linear theory of shells.

The genuine value of variational methods became apparent during the further development of the theory of shells in connection with the formulation of new problems in nonlinear theory, the development of the theory of anisotropic shells and shells in layers, and attempts to perfect the linear theory of shells.

The starting point in the formulation of new problems in the majority of cases is the origin of the possible displacements which leads to the Lagrange variational formula for the given object. While the problem is conveniently formulated in the displacements, the functions in the variational calculations in the solution of the problem under consideration do not end in this. In the nonlinear theory of shells, the most widely used variants are equations of the Karman type, formulated

in mixed form (in terms of the deflection and the stress function). It is obvious that different variational formulas correspond to different formulations. The derivation of such formulas is often of sufficient interest (sometimes even for a heuristic justification of the Bubnov-Galerkin method). For example, a great deal of attention was given to the generalization of the Castigliano variational principles to the nonlinear equilibrium theory of plates and shells (N. A. Alumyae, 1950, K. Z. Galimov, 1951, 1958).

The first variational formulations of the nonlinear theory of shells were constructed intuitively. Among these we will mention variational equations of the mixed type in the generalized Karman theory (N. A. Alumyae, 1950, M. A. Koltunov, 1952) and also the equations in general nonlinear theory (K. Z. Galimov, 1956).

Somewhat later, L. Ya. Aynola (1957) showed on the example of equations of the Karman type that a closed system of variational formulas can be derived with the aid of Lagrange multipliers from the variations formulas for the possible displacements (going back to the original system). In the case of the Karman equations, the number of different formulas turned out to be 181. In the case of nonlinear theory, this number may be greater.

K. Z. Galimov, who developed in his studies the variational formulas for the general theory (exact theory within the framework of the Kirkhhoff hypotheses), paid no attention to inter mediate results, the variational formulas, and attempted to obtain a closed chain of these formulas. The basic results of this study are presented in the monograph by Kh. M. Mushtari and K. Z. Galimova (1957).

A complete set of variational formulas which are well known in the theory of isotropic shells was generalized by N. K. Galimov (1965) to the nonlinear theory of three-layer shells.

As was already mentioned, variational methods are a reliable means for the derivation of boundary conditions. One of the more complex problems in nonlinear theory is the formulation of the geometric boundary conditions in terms of the forces and moments, which was solved by K. Z. Galimov with the aid of variational equations (1958, 1960).

The starting point in the definition of nonstationary deformation processes with the aid of variational methods is the Hamiltonian principle for elastic systems. However, this principle is applied to the derivation of equations of

motion, not to the direct construction of an approximate solution using the Ritz method, since the generalized coordinates of the system are unknown at the endpoint of the time interval during which the process is studied. In order to use the Ritz method, additional terms must be added to the energy functional which describes the state of the system at the final instant of time, but the resulting functional has no longer a potential.

Relatively recently, L. Ya. Aynola (1966) constructed a variational equation for the solution of nonstationary linear problems in the form of a convolution integral (over time) where the functional arguments must satisfy the initial conditions with respect to the coordinates, but not necessarily with respect to the velocities. At the final instant of time, the functional arguments are not constrained at all. In nonlinear problems, analogous variational equations in the form of a bilinear convolution type integral can be obtained by doubling the number of functional arguments (the introduction of additional unknowns).

In the study of short-time nonstationary processes (for example, in the case when elastic waves propagating from the source did not yet envelop the entire shell), the application of variational equations in the form proposed by L. I. Slepyan (1965) may turn out to be useful. In these variational formulas, the change in the boundary of the deformed region over time is taken into account, i.e., the functional arguments are only given in the region of essential deformations. It is natural to expect that the regions of essential deformation that were isolated will considerably improve in practice the convergence of the solution of a wide class of nonstationary problems, including problems described by equations of the parabolic type.

The fundamental basis for the reduction of two-dimensional problems in the theory of plates and shells to problems in systems with a finite number of degrees of freedom are the Ritz and Bubnov-Galerkin methods for the solution of variational equations. The vast majority of nonlinear problems in the theory of plates and shells was solved in this way. In the process, the following questions always arise: In what sense does the approximate solution, provided it exists, satisfy the initial boundary conditions for the problem? What is the error in the approximate solution? A series of studies by I. I. Vorovich (1955-1958) in the nonlinear static thermodynamics of flat shells deals with these problems. Vorovich gave the answers to the problems that were formulated in terms

of functional analysis. Unfortunately, it is not possible to present these results here even conceptually.[1] We will only mention that the trend that was mentioned was developed further in the fundamental studies of V. N. Morozov (1958, 1962), L. S. Srubshchik and V. I. Yudovich (1962, 1966).

§4. Qualitative Analysis of the Stressed State

It is hard to imagine a trend which would stimulate more the development of the theory and the application of computational algorithms to shells than the development of general methods for the qualitative analysis of the deformed and stressed states. The results of the qualitative analysis show that it is possible to decompose the general stressed states into elementary states, to simplify the relations, to determine these elementary states, to determine the error estimates for the errors which occur during the transition to the simplified relations, and finally outline iterative processes for finding the general stressed state with the required uniform accuracy in the entire region.

The development of all these problems has a long history. Thus, for example, I. Ya. Shtayerman (1924) pointed out the usefulness of the separate determination of the basic (torqueless) stressed state and the boundary effects in shells of rotation under an axisymmetric load more than forty years ago. In the early 30's, methods for the calculation of cylindrical shells were developed intensely, mainly due to the successful studies of V. Z. Vlasov (1933, 1936) which led to a computational variant (called today in the terminology of V. V. Novozhilov, 1951, "semitorqueless" theory), which describes the general effects around the asymptotic edge. Later, the generalizations for a simplified calculation of the boundary effect in the statics of shells with an arbitrary contour with zero Gaussian curvature and negative Gaussian curvature around the asymptotic edge were presented in the studies of A. L. Gol'denveyzer (1947, 1953). The results of these studies have shown that for nonlinear shells, the relations that were obtained are special cases of the so-called "engineering" moment theory

1. In the studies of I. I. Vorovich, the basic analysis is carried out in so-called "energy" spaces, in which first the strong compactness of the approximations obtained using the Bubnov-Galerkin method for static problems is established. Next, the author derives the conditions for the original problem (external load, middle surface of the shell, supporting contour) for which the approximations converge for any Hölder norms. In the case of dynamic problems, the weak convergence of the approximations is established. To present the individual results of I. I. Vorovich in a more concrete form, it would be necessary to reproduce a large part of his studies.

of shells (in the terminology of V. Z. Vlasov, 1944), developed for the calculation of stressed states with a large variability index. In tensor notation, the resolving equation in the theory in mixed form is represented as follows:

$$i\eta^2 (\nabla^\alpha \nabla_\alpha)^2 V + c^{\alpha\gamma} c^{\beta\sigma} b_{\alpha\beta} \nabla_\gamma \nabla_\rho V = 0, \qquad (4.1)$$

where $V = 2Eh\eta^2 w + i\varphi$, φ is a function of the stresses, w is the normal displacement of the middle surface, and $\eta^2 = h\sqrt{3(1-\nu^2)}$. In the case of generalized boundary effects, the variability indices of the stressed states in the directions along and across the edge are different. Therefore, the differential invariants in this equation are written in a simplified form in which inessential terms are ignored.

When the asymptotic edges are long, the generalized boundary effects degenerate: the variability index of the stressed state around the edge is no longer sufficiently large to apply the resolving equation (4.1).

It should be noted that the engineering theory of shells by itself does not postulate the problem of decomposing the stressed state into elementary states. It can be said that for it this is a secondary problem. Such a variant in the theory of shells has already been used for a long time, not only in linear static problems, but also in nonlinear static problems in equilibrium stability and in dynamics (Kh. M. Mushtari, 1939, V. Z. Vlasov, 1947). The problems of decomposing the stressed state and determining separately the elementary stressed states in the problems that were just mentioned have not been studied thoroughly (Kh. M. Mushtari, 1949, N. A. Alumyae, 1953, 1954, L. Ya. Aynola, 1965, A. L. Gol'denveyzer, 1966). We note that in these problems the fundamental stressed state of shells of zero curvature very often refers to a type of generalized boundary effect.

It is important to emphasize that the most difficult point in the qualitative analysis of the stressed state is not determining the possible existence of a particular elementary state but determining the boundary conditions for the concrete elementary state. For example, the fundamental stressed state of a plane plate under a large more or less uniformly distributed transverse load will be a membrane state. This state is described by the system of equations

$$(\nabla^\alpha \nabla_\alpha)^2 \varphi = 0, \qquad c^{\alpha\gamma} c^{\beta\rho} \nabla_\alpha \nabla_\beta \varphi \cdot \nabla_\gamma \nabla_\rho w = p. \qquad (4.2)$$

In order to integrate this system by the method of separating the stressed states, it is necessary to determine the boundary conditions for the elementary stressed states. This requires an analysis of the nonlinear boundary effects (it is known that when the deflections in the plates are large, these occur). It goes without saying that it is not always possible to impose boundary conditions separately on the individual states, i.e., in the final analysis to determine them separately. Incidentally, problems dealing with the existence of the membrane solution of plates and shells of rotation have been thoroughly studied recently using functional analysis in the studies of N. F. Morozov (1962), L. S. Srubshchik (1964), L. S. Srubshchik and V. I. Yudovich (1964, 1966).

Similar difficulties also occur in linear problems. An example is the problem of determining the boundary conditions which must be satisfied in torqueless theory. The studies of A. L. God'denveyzer (1948, 1960), K. F. Chernykh (1964) are devoted to this problem.

For linear problems, the most sophisticated apparatus for investigating elementary stressed states was proposed by A. L. Gol'denveyzer (1953), who developed further this apparatus (1959) which represents a generalization of the asymptotic integration method[1] in ordinary differential equations to partial differential equations with a small parameter (the relative thickness of the shell)

$$V(x^1, x^2) = e^{kr}\left(v_0 + \frac{1}{k} v_1 + \dots\right), \tag{4.3}$$

where r, v_0, v_1, . . . are functions of the coordinates x^1 and x^2 only and k is a sufficiently "large" constant whose order is determined by the character of the oscillations in the boundary conditions.

1. This method is sometimes called briefly the V.K.B. (Ventzel-Kramers-Brillouin method). For brevity the generalized method of A. L. Gol'denveyzer will henceforth be also called the V.K.B. method and it will not be mentioned every time that the equations discussed are partial differential equations.

The following three procedures are the simplest procedures for determining recurrently r, v_0, v_1, . . . :

(1) The constant k is a relatively small quantity

$$c^{\alpha\gamma}c^{\beta\rho}b_{\gamma\rho}r_\alpha r_\beta = 0 \quad (r_\gamma = \nabla_\gamma r),$$
$$c^{\alpha\gamma}c^{\beta\rho}b_{\alpha\rho}(2r_\alpha\nabla_\beta v_0 + v_0\nabla_\alpha r_\beta) + i\eta^2k^3(r^\alpha r_\alpha)^2 v_0 = 0$$

etc.;

(2) The constant k determines the variability index of the same order as in the simple boundary effect

$$ik^2\eta^2(r_\alpha r^\alpha)^2 + c^{\alpha\gamma}c^{\beta\rho}b_{\gamma\rho}r_\alpha r_\beta = 0,$$
$$2[2ik^2\eta^2(r^\alpha r_\alpha)r^\beta + c^{\alpha\gamma}c^{\beta\rho}b_{\gamma\rho}r_\alpha]\nabla_\beta v_0 +$$
$$+ [2ik^2\eta^2\nabla_\gamma(r^\alpha r_\alpha r^\gamma) + c^{\alpha\gamma}c^{\beta\rho}b_{\gamma\rho}\nabla_\alpha r_\beta]v_0 = 0$$

etc.;

(3) The constant k is a relatively large quantity:

$$(r^\alpha r_\alpha)^2 = 0,$$
$$i\eta^2\{4r^\alpha r^\beta\nabla_\alpha\nabla_\beta v_0 + 4(\nabla_\alpha r^\alpha \cdot r^\beta + r^\alpha\nabla_\alpha r^\beta)\nabla_\beta v_0 +$$
$$+ [2r^\alpha\nabla_\alpha\nabla_\beta r^\beta + (\nabla_\alpha r^\alpha)^2]v_0\} + c^{\alpha\gamma}c^{\beta\rho}b_{\gamma\rho}r_\alpha r_\beta v_0 = 0;$$

Here, due to the multiplicity of the characteristic lines of the operator $(\nabla^\alpha\nabla_\alpha)^2$, the function v_0 must be determined from a second-order linear equation with variable coefficients. Clearly, to integrate the last equation, again the VKB method can be applied. Hence, v_0 must be represented in the form

$$v_0 = e^{\sqrt{k}\rho}\left(u_0 + \frac{1}{\sqrt{k}}u_1 + \ldots\right), \tag{4.4}$$

where $\rho = \rho(x^1, x^2)$, $u_0 = u_0(x^1, x^2)$, . . . and the VKB method leads to the following equation used to determine the function ρ:

$$r_\alpha r_\beta \left(i\eta^2 k \rho^\alpha \rho^\beta + c^{\alpha\gamma} c^{\beta\rho} b_{\gamma\rho}\right) = 0 \quad (\rho_\alpha = \nabla_\alpha \rho). \tag{4.5}$$

Thus, the solution in the case under consideration is sought in the form

$$V = \exp\left(kr + \sqrt{k}\rho\right) \cdot \left(u_0 + k^{-1/2} u_1 + \ldots\right). \tag{4.6}$$

This is a successful case in the sense that it was possible to find a procedure for finding the solution. This was possible since the multiplicity of the characteristics of the operator $(\nabla^\alpha \nabla_\alpha)^2$ is the same everywhere. A. L. Gol'denveyzer (1960, 1962) developed general techniques (for more general equations for constructing solutions by using small rational powers of k in the representation of the variability function and the intensity function.

Much more difficult problems in the construction of the solution may be encountered when the VKB method is applied. An example is the case when the recurrence procedure (1) is satisfied and the middle surface of the shelf contains a line where the sign of the Gaussian curvature changes. Incidentally, the determination of the function v_0 from a first order partial differential equation reduces to the integration of a system of ordinary equations. Therefore, the finding of singular points and determining the character of the solution in the neighborhood of these points should not be connected in each concrete case with theoretical difficulties. Problems of constructing a solution in the spirit of the VKB method is, in the presence of such singular points, the subject of research in modern mathematical analysis, even in problems which can be reduced to ordinary differential equations.

Having devoted in this survey a great deal of attention to the aspects of constructing a solution, to balance the discussion, it should be noted that the simple equation of the engineering theory of shells itself was obtained as a result of simplifying a system of more exact (?) equations on the basis of a qualitative analysis. Therefore, the determination of a particular stressed state is broken up into three stages: (1) finding the structure of the resolving equations for a given variability index for their stressed state and determining the range of applicability of the simplified

relations (here the VKB procedure is applied in an implicit form), (2) clarifying the possibilities of applying the VKB method in standard form to the integration of equations in a predetermined region when the solution has turning points in this region it is necessary, as a rule, (3) to generalize the VKB method at least to construct formally a solution in the region with a turning point.

At the present time almost all basic results were obtained by means of the first stage. The studies of A. L. Gol'denveyzer (1959, 1960) deal with the second stage. Problems in the third stage are still in the formulation stage when we speak about systems that are not reducible to ordinary differential equations.

It is unfortunate that only one study in the theory of shells can be pointed out in which the VKB method was applied to the actual (numerical) construction of a solution of a two-dimensional boundary value problem (A. Petrov-Denev, 1966). Hence, we can expect that the powerful qualitative analysis method is at least a satisfactory computational algorithm.

On the whole, the problem of the qualitative analysis of the solution of equations in the theory of shells does not differ from the corresponding problem in the theory of partial differential equations. For the time being real mathematicians, specialists in the theory of differential equations have not been attracted by the problems in the theory of shells. The participation of M. I. Vishik and L. A. Lyusternik (1957, 1960) was much too short to lead to a deep analysis of the mathematical theory of shells. At the same time it is felt that the theory of shells did not use everything that can be proposed for the "incorporation" of the theory of differential equations. Incidentally, it must be mentioned that also among specialists in the theory of shells recently there was a decline in interest in problems in the general theory, in particular, in problems dealing with the qualitative analysis of the stressed state of arbitrary shells. The limited capacities of computer technology are not responsible for this, which eliminate the need for a qualitative analysis, but rather the fact that many objects in the new technology, even though they operate under complex loading conditions, have a simple configuration (cylindrical panels, shells of rotation) and the problems for these are not so acute. Shells of a complex configuration are primarily encountered in modern architecture. The unique problems that arise there are solved there in one way or another without a substantial contribution to the theory.

§5. Reduction to Integrodifferential Equations

The natural striving to extend the arsenal of investigations and computational methods led to the formulation of boundary value problems in the theory of shells in the form of integral and integrodifferential equations. The studies along these lines deserve attention since a study of the properties of solutions of integral equations is a very powerful method of functional analysis which is considerably simpler than the study of differential equations. In addition to this, the numerical integration of functions is a more accurate operation than differentiation. Therefore, the required accuracy of the result is obtained with a smaller number of computations. However, we must say immediately that the kernels of integral equations in the theory of shells are not simple and are a hindrance to obtaining the final results without considerable effort. In addition to this, the results must be interpreted from the point of view of the theory of generalized solutions.

The reduction of differential equations in the theory of shells to integrodifferential equations is based on the work reciprocity theorem. To obtain the system of integro-differential equations, the actions of unit (concentrated generalized) forces on the shell are considered as the auxiliary states.

N. A. Kil'chevskiy (1940) was the first man to outline this approach to the derivation of the integrodifferential equations. In this and in later studies, Kil'chevskiy (1946, 1959) and also in many studies of his followers, the auxiliary states were defined in a plane plate. Therefore, a one-to-one correspondence was established between the metrics on the plate and on the shell that was studied. According to this concept, those terms in the equation which characterize the effect of the curvature of the shell must be considered as loads. Incidentally, it is not necessary to take the shell for the system on the basis of which the auxiliary states are constructed.

With the passage of time, this approach was developed considerably and the nomenclature of objects was also expanded as well as the sphere of forces acting on the shell. Here, we will only mention some studies devoted to problems in the equilibrium of cylindrical shells (N. I. Remizov, 1959), shells of rotation (G. I. Tkachuk, 1961) and flat shells (B. N. Fradlin and S. M. Shakhnovskiy, 1958) devoted to a study of the dynamics of shells using operational calculus (N. A. Kil'chevskiy, 1955) and representing the integrodifferential equations of shells in forces and moments (N. I. Remizova, 1962).

Without using the reciprocity theorem for the work of an elastic system, it is possible to derive with the aid of purely formal methods, the equilibrium integrodifferential equations even for finite displacements of a flat shell and a shell of rotation (A. A. Berezovskiy, 1959, 1960). This approach was applied earlier in the more simple cases on a number of occasions (V. N. Shanshmelashvili, 1955, I. A. Birger, 1956).

Because of the difficulties that were mentioned above, which arise in the construction of the solution of auxiliary problems, the kernels of the integral equations, the method of reducing the differential equations to integrodifferential equations and integral equations, on the whole have not been very successful in the theory of shells.

On the other hand, this method was used effectively to solve completely problems in the stressed state around holes which are far from simple (rectangular and elliptical openings) in cylindrical shells (D. V. Vaynberg and A. L. Sinyavskiy, 1961). This gives reason to the proponents of this school to express the hope that the method is advantageous in the solution of very complex problems whose class will be gradually outlined (N. A. Kil'chevskiy, 1964).

§6. Complex Representation of Equations in the General Theory of Shells

The analogy between the static and geometric relations in the theory of shells led V. V. Novozhilov (1946) to represent the equations in complex form, in which the unknowns are the complex displacements. This method can only be applied to linear equilibrium problems but these have obvious advantages in their solution. Already in the first development stage of the corresponding theory, the inessential terms in the equations were determined. The introduction of complex functions made it possible to reduce the order of the differential equations by one-half which made the system more tractable. This is very important in the solution of problems with variable coefficients. For example, when an axisymmetric or antisymmetric load for a shell of rotation is considered, the problem reduces to a second order equation, where the complications caused by the presence of turning points can be easily analyzed. A typical example of such a case is a toroidal shell (Ye. F. Zenova, V. V. Novozhilov, 1951, V. S. Chernin, 1955). This remark applies, however, to any shell with nonpositive curvature. In other cases, the method simply leads to a simplification of the qualitative analysis and the arguments that are necessary for the solution (R. M. Malkin, 1954). It is of interest to note that problems exist, for which the boundary conditions can be formulated in terms of complex forces or displacements; in this case it is not necessary to separate the real and imaginary parts before the solution is obtained (in analytic form). Problems of

this type were discussed in the monograph of K. F. Chernykh (1962, 1964) in which all fundamental results connected with the representation of the relations in the theory of shells in complex form are presented. Among these, we will note the following.

Variational equations have been constructed for which the equations in the complex forces, the equations in the complex displacements and the complex conjugate conditions are Euler equations (K. F. Chernykh, 1958). Methods for the determination of the displacements in the case of a multivalued function of the stresses have been developed and the Meisner equations for an antisymmetric load were introduced (K. F. Chernykh, 1959). An effective technique for the calculation of shells under concentrated forces (V. V. Novozhilov and K. F. Chernykh, 1963) was also developed. More precise equations for flat shells were obtained by V. Z. Vlasov, and the complex thermoelastic equations were also derived.

In conclusion it should be noted that in linear equilibrium theory many known variants of calculating elementary stressed states are represented in complex form without establishing the relations with the concepts introduced into the theory of shells by V. V. Novozhilov. This connection was studied by A. L. Gol'denveyzer (1957) from the general theoretical standpoint.

A short survey of the development of their complex representation is given by V. V. Novozhilov (1964).

§ 7. Stressed State Around a Hole

The development of the problem of the stressed state around holes was started by A. I. Lur'e (1946) on the example of a circular holes in a cylindrical circular shell under the action of uniform pressure. In the study that was mentioned which served as the conceptual source for a large number of subsequent studies, Lur'e starts out with a system which defines the quickly changing stressed state around a hole.

If the center of the hole is identical with the origin of a geodesic system of coordinates, the stressed state around the hole is described by the solution of an equation in complex form

$$\begin{aligned}\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2}\right)^2 V - i\chi^2\Big[(\gamma-\delta\cos 2\theta)\frac{\partial^2}{\partial\rho^2}+\\+(\gamma+\delta\cos 2\theta)\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{2\delta}{\rho}\sin 2\theta\frac{\partial^2}{\partial\rho\,\partial\theta}+\frac{1}{\rho^2}\frac{\partial}{\partial\theta}(\gamma+\delta\cos 2\theta)\frac{\partial}{\partial\theta}\Big]V=0\end{aligned} \tag{7.1}$$

for an arbitrary shell with curvature parameters γ, θ (the line $\theta = 0$ coincides with the line of curvature), where $\chi^2 = 2\sqrt{3(1-\nu^2)}\, r_0^2/hR$, and h is the thickness of the shell, r_0 is the radius of the circular hole ($\rho = 1$), R is the characteristic radius of curvature of the middle surface. Equation (7.1) is an approximation of the more general equation

$$\Delta^2\Delta^2 V - i\chi^2 c^{\alpha\gamma} c^{\beta\rho} b_{\alpha\beta} \nabla_\gamma \nabla_\rho V = 0 \tag{7.2}$$

in a geodesic coordinate system around a small hole.

A solution in trigonometric series in the coordinate, leads to a complex system of compatible equations. In the case of a cylindrical shell ($\gamma = 1/2$, $\delta = -1/2$) A. I. Lur'e was able, by transforming the function V to simplify the problem considerably and to reduce it to the solution of a Bessel equation. Numerical results were obtained in the case of a small hole ($\chi^2 \ll 1$) by expanding the solution in a series in powers of χ. Thus, in first approximation, the stressed state of a plate is studied, and the successive approximations are obtained as the solutions of the nonhomogeneous equations for the plane problem and the bending of the plate. The above makes it possible to apply to the solution of the problem the powerful apparatus of analytic functions which was developed by N. I. Muskhelishvili (Yu. A. Shevlyakov, 1953). The results of the studies on the application of the theory of analytic functions are described in a general survey on the theory of elasticity which is included in this collection (pp. 57-104). However, the application of the apparatus of the theory of analytic functions is not required in this class of problems. Lur'e and the majority of his followers carried out the studies within the frame of reference of mathematical analysis of a real variable.

After the publication of the fundamental article by A. I. Lur'e, gradually, studies dealing with the stressed state around holes appeared. At the present time the number of publications on this problem is rapidly increasing. Thus, G. N. Savin in his survey report (1962) at the L'vov Conference mentioned forty domestic studies on stress concentrations in shells and the number of such studies increased in the subsequent five years.

The method of a small parameter (which, in the case under consideration, is the normalized radius of the opening) can be applied to the solution of a large class of problems dealing with the determination of the stressed state around holes.

It can even be said that studies in which the approach for obtaining the solution proposed by Lur'e is used occur most frequently in this field of study. This is not surprising since there is enough room for generalizing the problem without basically changing the method. Thus, instead of a uniform internal pressure other types of loads on a cylindrical shell can be considered, for example, torsion (Yu. A. Shevlyakov and F. S. Zigel', 1954). Several studies considered spherical and flat shells orthotropic shells, shells with a hole secured in one way or another (with a ring with various rigid properties or with a gasket which is hard or elastic). A survey of the studies that were made along these lines until 1961 is given by G. N. Savin (1961, 1962). The fundamental studies in the field under consideration were carried out in the beginning by Yu. A. Shevlyakov (1953, 1955, 1956) and I. M. Pirogov (1956, 1962, 1963), who published several tens of studies on a relatively narrow topic. Later G. N. Savin and his school also joined these studies and extended the class of investigations to holes whose contours are a smooth curve without corner points. This generalization was achieved by means of the conformal mapping $z = \omega(\zeta)$ ($z = \rho \exp i\theta$, $\zeta = \rho_1 \exp i\gamma$) of the unit circle $\rho = 1$ into the contour Γ with the aid of the function $\omega(\zeta) = c(\zeta + e/\zeta^k)$. For example, an elliptical hole with semiaxes a and b is characterized by the coefficients $c = 1/2\,(a + b)$, $\epsilon = (a - b)/(a + b)$, $k = 1$. The stress concentration in a spherical shell around an elliptical hole was studied by G. N. Savin and G. N. Van Fo Fu (1960) and for a square and triangular hole by A. N. Guz' (1964, 1965). It should be mentioned that Guz' published in those years a large number of articles on the results of the studies of the stressed state around small holes with different contours in shells with different configurations. The results were obtained by expanding the solution in series in powers of a small parameter. The method of a small parameter was also used to study physically nonlinear problems on the concentration of the stresses around holes (I. A. Tsurpal, 1963). In addition to what has been said above, it should be mentioned that the results obtained from the application of the method of a small parameter are better, the smaller the hole, while the classical theory of shells cannot be used at all to study the concentration of the stresses around very small holes.

Recently interest was shown in the problem of the reinforcement of a hole during which the same stressed state that exists under the given load without a hole is preserved. (G. N. Savin and N. P. Fleyshman, 1964, V. I. Tul'chiy, 1965). Incidentally, this problem cannot always be solved.

Relatively few results are available on the concentration of the stresses around large holes ($\rho \gtrsim 1$). This can be explained by the fact that for $\rho \gg 1$ the calculation of the shell reduces to a typical homogeneous problem in the theory

of shells in a multiply-connected region, in which the presence of a hole in a shell is no longer a determining factor. Among problems of this type, the simplest problems are problems with a circular hole in a shell with positive Gaussian curvature and the most difficult problems are problems with holes whose contour is tangent at individual points to the asymptotic line (in the case of shells with a negative or zero Gaussian curvature). At these points, the simple boundary effect degenerates, which can be easily seen by studying, for example, the approximate equation for determining the simple boundary effect (for $\chi^2 >> 1$) around a circular hole

$$\frac{\partial^2 V}{\partial \rho^2} - i\chi^2 (\gamma - \delta \cos 2\theta) V = 0;$$

which shows that when $\theta = \theta_0$, where $\gamma = \delta \cos 2\theta_0$, the boundary effect degenerates. Also such a point on the contour is a singular point of the torqueless operator of equation (7.1). Therefore, the decomposition of the solution into the torqueless state and the boundary effect in this case does not have the usual qualitative properties. An analysis and solution of such problems is undoubtedly of interest.[1] The reader may acquire familiarity with the formulation of this problem by reading the article by L. B. Imenitov (1966), which considers the special case of a shell with a positive curvature.

The energy method for determining the stresses for large holes was applied by O. A. Frolov (1961). D. V. Vaynberg and A. L. Sinyavskiy (1961) used for this purpose on an experimental basis the work reciprocity principle.

A thorough review of the state of the problem on the stress concentration around holes was given by G. N. Savin (1966). The survey emphasizes that in practice the concept of a small hole can be applied in a wider interval (up to

1. For large holes the torqueless operator in equation (7.1) may not be adequate for the problem under consideration since, as a rule, only terms with the highest order derivatives are given.

$\chi^2 = 3$) which was established by theoretical and many experimental studies (A. Ya. Aleksandrov, et al., 1966).

§8. Calculation of Shells for a Concentrated Load

An elementary design rule requires that more or less large concentrated loads be applied to the edges reinforcing the shell which distribute the load on the shell along conjugate lines. In spite of this sometimes it is necessary to apply the load to the shell directly on a small area whose dimensions are commensurate with the thickness of the wall of the shell.

The determination of the stressed state of shells under a concentrated load has attracted the attention of investigators for a long time. A spherical shell was studied by A. G. Gol'denveyzer (1944), a freely supported flat shell by V. Z. Vlasov (1949), a cylindrical shell by V. M. Darevskiy (1952). In all these studies analytical expressions were obtained for singularities of the solution in the neighborhood of a point where the normal concentrated force was applied. Later the class of problems was extended to various types of acting forces (tangential and moment concentrated loads) and shells with various contours. The apparatus of the theory of generalized functions and polyharmonic equations was applied to the analysis of the stressed state. We note here the studies of V. V. Novozhilov and K. F. Chernykh (1963) and also of G. N. Chernyshev (1963) on finding singularities in an arbitrary elastic shell caused by concentrated forces and moments.

The knowledge of the analytical expressions for the singularities of the solution around a concentrated load is of great theoretical and practical importance. The latter makes it possible to improve the convergence of the series to which usually the calculations of a shell subjected to the action of a concentrated load leads. However, it should be mentioned that this possibility is used only rarely.[1]

1. For example, in the survey article by Yu. P. Zhigalko (1966) dealing with the calculation of cylindrical shells, formulas are derived for the singularities, but in the next article in the same collection (Yu. P. Zhikgalko and N. G. Gur'yanov, 1966) dealing with a freely supported shell, these formulas are not applied to obtain faster convergence of the solution which is represented in the form of a double trigonometric series.

The reason for this is clearly that the analytical expression for the singularity approximates the solution satisfactorily only in a small region around the point at which the load is applied. Formulas which will take into account the presence of a stressed state of the boundary effect type are of great interest. But only the first steps we made along these lines (V. S. Chernina, 1963, 1965, G. N. Chernyshev, 1966).

Further, the stressed state below or around the loading area is basically three-dimensional. Therefore, the equations of the theory of shells cannot describe the singularities of the solution (at best they can only model these singularities) which are only obtained on the basis of the leading operators in the equations of the Kirkhhoff-Loew theory. In this connection the question arises what corrections are introduced into the results on the singularities of the solution obtained until now by using equations of the Reissner (or Timoshenko) type or a higher order two-dimensional system, which take best into account the three-dimensional character of the real stressed state.

Recently studies dealing with the effect of the local reinforcement of the loading area by plates of different shapes on the stressed state of cylindrical shells began (V. M. Darevskiy, 1964, Yu. G. Konoplev and A. V. Sachenkov, 1966). Recommendations were made for the selection of the shape of the plate to reduce the stress concentration around the point where the load was applied.

Recently the particular characteristics of the stressed state were studied around a concentrated force during the free oscillations of a cylindrical panel (V. M. Darevskiy and I. L. Shmarinov, 1966). The survey article by V. M. Darevskiy (1966) is of interest.

§9 Torqueless Flexed Shells

Thin shells with very small flexural rigidity (often called "soft" shells) are calculated mainly on the basis of torqueless theory. They are characterized by a more or less uniformly distributed internal pressure. In the general case, in equilibrium, two zones are formed in the shell: "the expanded" zone and the "crumpled" zone. In the "crumpled" zone, one of the principal torqueless forces is zero (the crumples are formed due to the local loss of stability) and the second is positive. Of course, the boundary between the zones is not known in advance.

Until now the main objects of studies were shells of rotation. The development of a general theory of torqueless highly pliable shells is in the initial stage (S. A. Alekseyev, 1966). This is not surprising, since here the problem can only be linearized in special cases. The physical relations in studies on pliable shells vary over a wide range. On the one hand materials exist for pliable shells which can be considered as materials which do not expand. The assumption that the material does not expand simplifies the study considerably. This class of problems was studied by S. A. Alekseyev (1955). On the other hand, often it is necessary to take into account that the material can be deformed considerably when the mechanical characterization is based on the relation between the true stresses and the logarithms of the strains (A. S. Grigor'ev, 1957, 1961, I. V. Keppen, 1960, I. I. Fedik, 1962).

Torqueless boundary effects may exist when the tensile stresses are great (Yu. N. Rabotnov, 1946). Under favorable conditions, this makes it possible to apply to the torqueless shell a load which is distributed along a line (not an area). A number of problems of this type were solved by V. I. Asyukin (1964) including the case of a toroidal shell.

Recently considerable attention was given to the load bearing capacity of pliable torqueless shells (I. S. Mamedov, 1963, Yu. F. Fokin, 1965), which expresses itself in terms of the maximum load even when the assumed materials are ideal materials with unlimited strength.

By and large the problem of torqueless pliable shells would reduce to the determination of the basic stressed state in rigid shells in the case of small displacements. But, as a rule, the theory of soft shells is characterized by exact geometric relations. Further, the problem of the local stability in soft shells is not crucial for statements about the operating capacity; often the crumples that are formed can be tolerated. Nevertheless a clarification of the conditions for the existence of a biaxial stressed state is of interest, since local criteria of the type

$$a_{\alpha\beta}T^{\alpha\beta} > 0, \qquad c_{\alpha\gamma}c_{\beta\rho}T^{\alpha\beta}T^{\gamma\rho} > 0$$

are not very effective in the solution of complex problems. S. A. Alekseyev (1965) derived the condition for the non-negativity of the principal components of the tensor of the forces $T^{\alpha\beta}$, which imposes a constraint on the shape of the shell under the given load; for example, in the case of a

triaxial ellipsoid and uniform pressure, this is expressed by the requirement that it be possible to construct triangles on the segments which are inversely proportional to the square of the axes.

A concise presentation of the theory and the fundamental problems of soft shells is given in the articles by S. A. Alekseyev (1965, 1966) which also give partially a short survey of the work carried out so far.

§10. Ductile Shells in Instruments

A large series of studies deals with the determination of the main characteristics of an elastic corrugated membrane which is an important element in certain instruments. In first approximation such a membrane can be studied as an anisotropic plate, in fact, it is a membrane with a Gaussian curvature which changes sign (for example, in the case of a sine wave corrugation) or a set of short connected conical shells (with a saw-tooth profile of the membrane). The many parameters which determine the configuration of a corrugated membrane, which are necessary to calculate a ductile shell on the basis of nonlinear theory, present great difficulties which are a hindrance to obtaining general results on the operating characteristics that depend on the structural parameters. At the same time during the calculation of a corrugated membrane, the fundamental problem is not to determine the distribution of the stresses, but to find the flexure in the center of the membrane. This makes it possible to solve the problem using variational methods, which, until now, have been the main tool in the study of corrugated membranes.

The study of D. Ya. Panov (1941) was one of the first studies in the nonlinear theory of membranes with very slight crimping. Later, V. I. Feodos'yev (1945, 1946, 1949) joined the investigators engaged in this topic. With the passage of time the constraining assumptions of a gently sloping corrugation and its smoothness were removed and the crimping of flat shells was studied (L. Ye. Andreyeva, 1953 1958, 1962). Experimental studies were also carried out (V. Ya. Il'minskiy, 1955). In the calculation of corrugated plates and flat shells with the aid of variational methods, the selection of the coordinate functions, especially in the case when their number must be small, is of great importance, when the corrugation of the shell is dense. When the coordinate functions are selected, the step and depth of the crimp, its shape and the individual hardness characteristics must be taken into account in the selection of the coordinate functions (E. L. Aksel'rad, 1963, 1964).

A second type of thin-walled elastic elements widely used in instrument building are bellows. These objects were already analyzed in the monograph of V. I. Feodos'ev (1949) and later in the studies of V. I. Korolev (1954) and V. S. Chernina (1955). A bellow is a composite structure made from toroidal shells; therefore, the development of methods for the calculation of bellows followed progress in the theory of toroidal shells. S. A. Tumarkin (1959), V. S. Chernina (1961), A. N. Volkov (1962, 1963), A. R. Yaroshenko (1965), V. A. Sukharev (1966) contributed to the development of these mutually interrelated problems in the last decade. Studies dealing with the flexing of curved thin-walled pipes (E. L. Aksel'rad, 1961, 1965), are related to these studies.

§11. Natural Oscillations

A determination of the frequency spectrum and the associated forms of the natural oscillations present a wide-open field to investigators. It is an auxiliary problem in dynamic calculations of both forced oscillations and also of other quasi-stationary processes. With the exception of freely supported flat shells and cylindrical panels, any problem in this field is relatively difficult to solve even today.

Studies in the vibrations of plates and shells have a long history. The determination of the natural frequencies in plates, for example, can be treated as classical problems in mathematical physics. The same cannot be said with regard to oscillations in a spherical shell, even though the latter have been the subject of many studies at the present time (P. Ye. Tovstik, 1965).

Classical problems are not necessarily simple. The early investigations of the oscillations of shells are characterized by the solution of special problems (A. P. Filippov, 1937, V. Z. Vlasov and B. M. Terenin, 1947, O. D. Oniashvili, 1950, V. Ye. Breslavskiy, 1953, 1954, Z. I. Grigolyuk, 1956). The state of the initial development of this problem is described in the monograph of O. D. Oniashvili (1957) and some later results for special objects are given in the handbook by V. S. Gontkevich (1964). We mention the studies of R. L. Malkina (1958, 1960) on the vibrations of noncircular cylindrical shells, V. Ye. Breslavskiy (1959) on the effect of holes on the frequencies, I. I. Trapezin (1959), D. D. Ul'yanitskiy (1958) on small oscillations of a conical shell and hydroturbine blades. U. K. Nigul (1958) studied in detail the complete spectrum and forms of the oscillations of a cylindrical shell. The first studies on free oscillations in anisotropic and multi-layer shells go back to the same period.

A natural generalization of the problem of free oscillations was obtained during the analysis of the natural frequencies in a shell under a load (in some usually torqueless stressed state). V. Ye. Breslavskiy (1956) and M. V. Nikulin (1959) give results for concrete loads. It is known that the study of the oscillatory properties under a load is the main method used to investigate the equilibrium stability of the given system. Therefore, the center of gravity in this series of studies are problems dealing with the stability of elastic systems.

A comparatively large series of studies on the vibrations of plates and shells with finite deflections was started by Z. I. Grigolyuk (1955). The main method for investigating the vibrations of shells with finite amplitude is reduction to a system with one or two degrees of freedom with the subsequent application of the results developed in nonlinear mechanics. At the present time, this technique is used predominantly in the solution of more complex problems in the dynamics of shells. This includes problems on parametric oscillations, nonlinear flutter, dynamic stability during an impact load, etc.

In its form, the linear theory of the oscillations of plates and shells differs little from linear equilibrium theory. According to the D'Alembert principle, the effect of inertia can be taken into account as a load. The development of this theory could proceed in parallel with the development of equilibrium theory. However, so far we do not yet have studies in the theory of the oscillation of shells which are as general as those available in equilibrium theory. Without belittling the significance of O. D. Oniashvili's monograph and V. S. Gontkevich's handbook, it must be stated that they deal with special results and that their goal is not a systematic analysis of the great variety of oscillatory forms from the standpoint of the general theory of shells.

At the same time, it was discovered a long time ago that in problems on small oscillations of shells, the general state of motion (and the stressed state) can be decomposed into elementary states which are known from the general equilibrium theory of shells. These states were described in the survey article by N. A. Alumyae (1958). With the exception of the simplest objects, a qualitative analysis of the problem whose goal is to decompose the general state of motion into elementary states, leads to a considerable reduction in the amount of computational work. Using this procedure, L. Yu. Poverus and R. K. Ryayamet (1958) determined the principal vibration tones of a conical shell on the basis of torqueless theory.

The problem that is discussed was the subject of a thorough analysis by A. L. Gol'denveyzer (1961, 1966), who approached it from the standpoint of the general theory of shells, i.e., as it applied to an arbitrary shell. In his last article, Gol'denveyzer summarized the results of the qualitative study of free oscillations with a large variability index for the displacement state. The purpose of the investigation was to determine regions for the parameters characterizing the variability function in which the general displacement state could be decomposed into elementary states. The problems were classified taking into account the geometric properties of the contour line on which the character of the additional integrals used to satisfy the boundary conditions depend. The main attention in the article was given to torqueless transverse oscillations occurring at relatively low frequencies, accompanied only by small tangential oscillations. The resolving equation for these oscillations has a curious structure.

$$c^{\alpha\gamma}c^{\beta\delta}b_{\alpha\beta}\nabla_{\gamma}\nabla_{\delta}w \pm \omega a^{\alpha\beta}\nabla_{\alpha}\nabla_{\beta}w = 0 \qquad (11.1)$$

which deserves further study (ω is the reduced frequency). The same remark applies to a pure flexural deformation of shells with negative curvature where the bending of the middle surface with a large variability index is not localized as in shells with positive curvature.

Theoretical difficulties in the application of the VKB method may also arise in the solution of one-dimensional problems. The point is that the equations with variable coefficients have in a certain frequency band in the integration region so-called branch points, in the neighborhood of which the VKB method breaks down. The articles of N. A. Alumyae (1960) and P. Ye. Tovstik (1965, 1966) are devoted to the solution of the problems that arise in this case. In the case of two-dimensional problems, these problems, as applied to the theory of shells, have not been studied for all practical purposes.

Recently in the study of basically two-dimensional linear problems, the greatest attention was probably given to the asymptotic method for determining the frequencies (in the terminology of V. V. Bolotin, who proposed this method in 1960). The basic idea of the method consists of the following. During free oscillations of a rectangular plate with a large number of node lines, it is natural to expect that the deflection in the internal part of the plate is defined by the expression

$$w(x, y) = w_0 \sin k_1 (x - \lambda_1) \sin k_2 (y - \lambda_2),$$

where k_1 and k_2 are the variability indices of the stressed state in the directions x and y. Taking this into account the deflection can be sought in the form $w = w_1(x)w_2(y)$. The new unknown functions $w_1(x)$ and $w_2(y)$ are determined to some extent separately: when $w_1(x)$ is found it is assumed that $w_2(y) = \sin k_2(y - \lambda_2)$. The solution of the one-dimensional problems still contains the unknown parameter k_2. An analogous procedure is used to obtain an equation which determines k_2. An equation is set up for $w_2(y)$ and it is assumed that $w_1(x) = \sin k_2(y - \lambda_2)$. The joint solution of the characteristic equations that were obtained for the two one-dimensional problems (with the parameters k_1 and k_2) determines the compatible roots k_1, k_2 and at the same time the frequency. The basic idea assumes that all integrals in the function $w_1(x)$, except $\sin k_1(x - \lambda_1)$, introduce a correction into the stressed state only around the edges, i.e., that they are damped relatively fast. (V. V. Bolotin calls such integrals the "dynamic boundary effects"), but it should be noted that the variability index in these integrals is the same as in the principal integral.

The method that was described was developed rather rapidly by V. V. Bolotin and his school. After studies dealing with a rectangular plate (V. V. Bolotin, 1961, V. V. Bolotin, et al., 1960) the spectrum of the transverse oscillations of cylindrical closed shells and cylindrical panels was investigated (Yu. V. Gavrilov, 1961, 1963), of flat shells (V. V. Bolotin, 1960), and of plates on the basis of Timoshenko's theory (V. N. Moskalenko, 1961). According to Timoshenko's theory, the boundary effect degenerates at high frequencies, and the degeneracy consists of the fact that the basic stressed state is described by several terms of the form

$$w = w_0 \sin k_1 (x - \lambda_1) \sin k_2 (y - \lambda_2).$$

in which case the application of the method becomes somewhat more complex.

The method was studied on the example of a plate also from the variational point of view using the method of separation of variables (L. Ya. Aynola, 1963).

The generalization of the asymptotic integration method to shells for which the variability of the metric coefficients must be taken into account is undoubtedly of interest, but it requires, in the opinion of V. V. Bolotin (1961, 1962), the apparatus of the VKB method. First, shells must be analyzed whose middle surface is developed into a surface of rotation whose contour lines coincide with the line of curvature. Shells whose contours lie on the minimal surface in which the contour lines are the asymptotic lines of the middle surface (the calculation of theoscillations of turbine blades) are also undoubtedly of academic and practical interest.

With regard to the applicability of the method, the question remains whether all frequencies can be obtained in this way. For the time being it is difficult to answer this question in the affirmative, since it was shown on the example of a simple one-dimensional problem that when the asymptotic method is applied, certain frequencies are lost (Zh. K. Makhortykh, 1964). Nevertheless, the method proposed by V. V. Bolotin is very effective in determining a large number of frequencies and natural forms of free oscillations and the method was applied on a wide scale to the solution of various dynamic problems.

V. V. Bolotin (1963, 1966) studied, on the basis of the asymptotic method, the densities of the natural frequencies of plates and flat shells and showed the existence of the flexural oscillation spectrum, where shells with a negative curvature have one accumulation point and shells with a negative curvature two points. The accumulation points of the spectrum of natural oscillations are found at frequencies $\omega_1 = |c/R_\alpha|$ and $\omega_2 = |c/R_\beta|$ (for the latter only in the case of shells with negative curvature). In these expressions, c is the rate at which the compression- tension waves propagate in the shell and the coordinate grid on the middle surface is such that $|R_\alpha| < |R_\beta|$, where R_α, R_β are the principal radii of curvature. The empirical data that were obtained from an analysis of spherical and circular cylindrical shells validate the theoretical results. Nevertheless, it is interesting that at the frequencies that were mentioned, the characteristic lines of the equations for the torqueless flexural oscillations have multiplicity; however, multiple characteristics also occur in shells with positive curvatures at frequencies ω_1 and ω_2 (for a spherical shell these values coincide). The question of the relation between these phenomena has not yet been answered. We note here that the first studies on the asymptotic behavior of the natural frequencies of the oscillations of cylindrical and flat shells were carried out by S. A. Tersenov (1955).

The extension of the results on the density of the natural normal oscillations to other types of oscillations in shells (tangential oscillations, high frequency flexural oscillations, which occur when rotary inertia is taken into account on the basis of Timoshenko's theory) is undoubtedly of interest. These results are also applied in the study of plates and shells subjected to the action of random loads.

§12. Transient Deformation Processes

In modern engineering, problems often arise in which the history of the process is only of interest in a short time interval that is commensurate with the time during which the deformation waves covered a path which is equal to the characteristic dimension of the middle surface of the shell. Transient deformation processes are characterized by the presence of unperturbed regions in the shell during a substantial part of the entire history of the process. The boundary of the perturbed region is characterized (depending on the load) more or less by a pronounced front of the stress waves. When the motion of the shell is described by equations of the hyperbolic type, this fact which is clear physically, is also reflected mathematically. Generally the solution is discontinuous at the wavefront. Since in an elastic medium the ruptures spread at two rates, the pattern of the ruptures (not to mention the field of the displacements) may be very complex when it is taken into account that the ruptures are reflected from the lateral and contour surfaces. It is clear that during a slow (and long) increase of the loads the role of these ruptures in the stressed state is negligible. Transient processes which occur during the collision of a shell with another object or barrier and also during the flow of a shock wave past the shell are of primary interest in the applications.

The state of the studies of transient deformation processes in shells and plates is described in detail in the survey by L. Ya. Aynola and U. K. Nikula (1965) and certain supplementary details can be found in the survey report of the author (N. A. Alumyae, 1966). Here we will restrict ourselves to a condensed presentation of the fundamental results. Certain data on the shock on an arbitrary shell have already been presented in the monograph of N. A. Kil'chevski (1949), in which the shell was modeled on the basis of the Kirkhhoff-Loew theory. The displacements that occurred during the shock were determined by applying the work reciprocity theorem.

Studies of transient processes based on a model of the hyperbolic type were started by Ya. S. Uflyand (1947) who derived a new variant of the equation for the bending of a plate by generalizing to the plate the system of hypotheses proposed by S. P. Timoshenko for improving the accuracy of the equation of motion of a rod. Uflyand applied the Laplace transform method to the solution of the nonstationary problem and obtained some numerical results.

If we exclude the activity of geophysicists (see, for example, G. I. Petrashen', 1951, 1953) with a somewhat different sphere of interest, the work of Ya. S. Uflyand was followed almost by a decade of silence in publications dealing with the topic discussed. However, recently there was a considerable revival of interest and two fundamental problems began to take on shape.

The first problem is the development of methods for the analysis of rapidly changing fields based on equations of elasticity theory.[1] One of these methods is based on the application of double integral transformations and the use of the method of steepest descent, taking into account only a finite number of modes. Naturally the solution of concrete problems must precede the study of dispersion relations in the high frequency region. It was carried out, for example, for an antisymmetric deformation of a plate, relative to the middle surface (Yu. K. Konenkov, 1960, Y. K. Nigul, 1963, A. I. Myannil and U. K. Nigul, 1963) for which a great deal of information on the phase and group velocities at which the waves propagate is available. Already these results make it possible to estimate the accuracy of the dispersion relations obtained on the basis of simplified theories (I. T. Selezov, 1960). Along with these the analysis of concrete transition processes is of great interest. These include the study of U. K. Nigul (1963) on the wave bending process in a semi-infinite plate.

The application of the Kan'yar method to the inversion of double integral transformations leads, in essence, to the

1. Approximate theories (in particular, a theory of the Timoshenko type) does not model adequately the motion around fronts of stress waves, where it consists of high frequency oscillations.

construction of a system of elementary waves which are formed during the reflection of the primary waves from lateral surfaces. This method requires great analytical sophistication but even then it does not lead to simple computational algorithms (G. I. Petrashen', 1958). The available results determine mainly the character of the discontinuities at the fronts of the elementary waves.

Partially due to analytical difficulties, and partially in connection with the greater possibilities of applying computer technology, articles began to appear in which the dynamic equations of the theory of elasticity are integrated directly using numerical methods (U. K. Nigul, 1965, 1966). The results that were obtained give a sufficiently clear idea about the applicability of approximate theories, in particular, the assumption was confirmed that the solutions obtained using the approximate theory smooth the motion around the fronts and other discontinuity lines (this shortcoming may turn out to be very palpable when the accelerations must be determined).

The second problem arises as a result of the fact that the entire necessary volume of calculations cannot be carried out with the aid of the theory of elasticity, and, therefore, simplified theories must be perfected which are based on various reduction procedures. This problem is just as acute in nonstationary dynamics as in the theory of three-layer shells. The approaches to its solution have been presented briefly in the section devoted to the reduction of the equations of elasticity theory to equations in the two-dimensional theory of shells (§ 16). For the time being theories of the Timoshenko type have not been applied further to a concrete analysis and applications. Approximate models that can be used to describe the stressed state around discontinuities are not available.

A sufficiently large number of problems dealing with transition processes have been solved on the basis of a theory of the Timoshenko type. If those investigations are omitted in which the method of expanding the oscillations in eigenfunctions is used (which was developed for the solution of quasistationary problems), the publications of M. V. Dubinkin (1959), V. D. Kubenko (1965), N. D. Veksler, et al., (1965, 1966) dealing with the solution of one-dimensional problems during a load on the edges and also the article of A. V. Agafonov (1965) on the action of a concentrated force on a cylindrical shell (two-dimensional problem) can be mentioned. Next, the studies on the reaction of a plate or shell to the action of a moving load should be mentioned. D. Ye. Kheysin (1963) studied the stationary motion of a plate floating on the surface of a fluid, V. L. Prisekin (1961) determined the critical velocities of motion of the

load in the axial direction in a cylindrical shell, L. I. Slepyan (1966) determined the asymptotic laws for the increase in the amplitudes of the displacements under critical velocities of motion of the load on simple objects (rod, torqueless cylindrical shell) M. A. Il'gamov and A. A. Yabbarov (1965) studied the steady state motion of a cylindrical shell with a moving separation boundary between the gaseous medium and the solid elastic filler (taking into account thermal effects), A. N. Tyumanok (1965) determined on the basis of a theory of the Timoshenko type the transient oscillations of a cylindrical shell and obtained formulas for the discontinuities during the flow of a shock wave around a spherical shell (N. A. Alumyae, 1966). P. F. Sabodash (1965) studied the steady state action of a moving load on the plate on the basis of elasticity theory.

A great deal of attention was given to nonstationary problems in hydro- and aeroelasticity. A hydraulic shock, taking into account the deformability, was the subject of study in the articles by N. A. Kil'chevski, et al., (1962), A. S. Vol'mir and M. S. Gershteyn (1966). In the last study the model of the pipe was nonlinear both geometrically and physically. The action on a long circular cylindrical shell of an acoustical wave with a plane front parallel to the axis of the shell was studied by E. I. Grigolyuk and V. L. Prisekin (1963) in a linear formulation and later by A. S. Vol'mir and M. S. Gershteyn (1965) in a nonlinear formulation. These results refer to the initial shock stage.

With rare exceptions, the problems that have been solved so far are one-dimensional. This is understandable if the great power of computer technology is taken into account on one hand and the complexity of the discontinuity pattern in the stressed state on the other hand. The solution of two-dimensional linear problems is on the agenda in the coming years. Taking into consideration the needs of technology and the experience of geophysicists, the program that is being outlined cannot be carried out without applying computer technology which already furnished the solution for a number of complex nonlinear problems.

The application of the simplest computational algorithms to the solution of partial differential equations describing transition processes is only possible in the cases when the solution is sufficiently smooth. This can be achieved if the discontinuous part is isolated in the general solution up to the required order. Incidentally, such an expansion is used very often in the solution of linear one-dimensional problems. When the nonlinear process can be described by piecewise linear equations, known methods from linear theory can be used to isolate the discontinuous part of the solution.

The study of discontinuous solutions of quasilinear equations is, for the time being, a hiatus in the theory of shells. It can be expected that the results in the study of quasilinear equations will contribute additional aspects to the problem of the dynamic stability of nonstationary processes in plates and shells.

The prospects for the studies that were touched on here are clouded by one fact: in the zones adjacent to the discontinuities, the theory of shells does not describe adequately the phenomena that occur there. Therefore, the studies in this region must be carried out keeping close touch with the development of the first problem pointed out in the beginning of this section. Finally, we note that strong discontinuities can occur in practical problems (for example, during the flow of a shock wave around a cylindrical shell, whose front is parallel to the axis of the shell). Therefore, the analysis and determination of the qualitative and quantitative characteristics of the discontinuities is not an "academic" pastime, i.e., some hyperbolism in the "parabolic nature."

§13. New Problems in the Dynamics of Shells

In the last decade a number of interesting directions developed in the dynamics of plates and shells in which so far the fundamental results apply to the dynamics of systems with a finite number of degrees of freedom. These include parametrically induced oscillations, oscillations induced by a gas flow, oscillations of vessels partially or fully filled with a liquid, oscillations under random loads or structural properties.

The series of studies that were mentioned here were stimulated by practical needs. Therefore, there is no need to speak about underestimating the importance of the development of the studies that were started. At the same time the problems that were formulated are very complex when the plates and shells are considered as objects of a one-dimensional or two-dimensional continuum. As a result, the plate or shell is reduced already in the initial stage by some variational method to a system with a finite number of degrees of freedom. Therefore, the impression may be created that the new investigation directions had so far no bearing on the "internal" theory of shells, even though it is hard to deny that to reduce the shell to a system with one degree of freedom, it is necessary to have a clear idea about the work of the shell under the given load conditions. At any rate the closeness of these directions to the "internal" theory of shells must be verified by specialists in the theory of shells, and, therefore, we will dwell in this survey on the basic development stages in the fields that were touched on.

Parametric Oscillations. In certain linear problems of plates and flat shells it is possible to separate the variables (B. Z. Brachkovskiy, 1942, G. Yu. Dzhanelidze, 1955). In this case the structural oscillations are determined by the well-known Mathieu-Hill equation, whose coefficients determine the zones of the parameters in which the oscillations are not stable. These problems were first solved for plates by V. A. Bodner (1938) and Z. I. Khalilov (1942), for shells by A. N. Markov (1949) and O. D. Oniashvili (1950).

In shells, due to the large density of the spectrum of the free oscillations, the instability zones of the parametric oscillations cover a considerable region in the "force-frequency" plane. Therefore, for all practical purposes, it is necessary to determine the amplitudes of the oscillations with the aid of nonlinear theory and take into account the damping. This problem was formulated by V. V. Bolotin for plates (1954, 1956) and later also for a spherical shell (1958). In the study of nonlinear problems, as in the case of linear problems in which the variables are not separated (N. A. Alfutov and V. F. Razumeyev, 1955), the shell is modeled by a system with one or two degrees of freedom.

When the system has one degree of freedom, its dynamics are defined by an equation of the type

$$f'' + 2\varepsilon f' + \omega^2 [1 - T(t)] f - e(t) f^2 + d(t) f^3 = q(t),$$

where ω is the frequency of the free small oscillations of the system. The structure of the system does not change if instead of the homogeneous isotropic shell an inhomogeneous shell with anisotropic layers is considered in the nonlinear elasticity conditions in a nonstationary temperature field (S. A. Ambartsumyan and V. Ps. Gnuni, 1964, V. Ts. Gnuni, 1965). Most often, the problem consists of determining the regime for the stationary forced oscillations.

Oscillations of Shells and Plates in a Gas Flow. The first studies on the combined oscillations of plates and a gas deal with subsonic flow velocities (G. I. Kopzon, 1956, V. V. Bolotin, 1956) and also low supersonic velocities. The problems were considered in a linear formulation assuming a potential flow. It must be taken into account that in this formulation the dimensionality of the aerodynamic problem is larger by one unit than in the elastodynamic problem for plates. However, it was soon discovered that for large velocities of the flow, the aerodynamic interaction can be taken into account in a highly simplified form (with Mach number $M>2$)

This variant, which takes into account the aerodynamic forces, received the name "piston theory" and was already applied in the studies of A. A. Movchan (1957) and R. D. Stepanov (1957). It must be mentioned that in addition to the conditions $M \gg 1$ and the quasistationarity of the flow, there is an additional condition for the variability indices along and across the perturbed flow (V. V. Bolotin, 1961). "The piston theory" remained for the time being the basic computational model for the flow around a plate or shell. In spite of the simplification in the aerodynamic part, an exact solution of even linearized boundary value problems, can only be obtained in exceptional cases. One such case, the axisymmetric flow, through an infinitely long closed cylindrical shell was the subject of many investigations (B. I. Rabinovich, 1959), Yu. Yu. Shveyko, 1960, G. Ye. Bagdasaryan, 1962, Ye. P. Kudryavtsev, 1964), which represented various generalizations of the simplest problem formulated by V. V. Bolotin in 1956. To this incomplete list, studies must be added in which the oscillations of shells in a gas flow in which the temperature of the shell varies over time are studied (S. A. Ambartsumyan and G. Ye. Bagdasaryan, 1964).

In the case of plates and shells of finite dimensions, the Galerkin method is used to reduce the problem to a system with a small number of degrees of freedom. In the solution of nonlinear problems, for the time being, this is the only method for obtaining final results and the number of degrees of freedom is usually two.

The first nonlinear problems in aeroelasticity were solved by V. V. Bolotin (1958, 1960) and also by his collaborators (1959). We also note the studies of Yu. Yu. Shveyko (1961), Yu. N. Novichkov (1962), and G. Ye. Bagdasaryan (1963). The study of nonstationary flutter during the simultaneous change in the velocity and temperature was also initiated by V. V. Bolotin (1962). K. K. Livanov (1963) took into account the effects of tangential inertia on the critical velocity (usually only the normal acceleration is taken into account in the problems that are being considered). A survey of studies on the oscillations of plates and shells in a gas flow published until 1961 is available in the report by V. V. Bolotin (1962).

Since linear problems in the stability of plates and shells in a gas flow reduce in the final analysis to a study of a system with two degrees of freedom, various generalizations of the solution of "classical", i.e., apparently simplest problems, can be obtained without any theoretical difficulties. The object can be flat, anisotropic, multilayered, edged, nonlinear elastic shells, and the procedure remains basically unchanged when all these factors are taken into account.

Oscillations of Shells Filled with a Fluid. The free oscillations of partially or completely filled vessels have naturally two different spectral regions. At low frequencies the fluid oscillates and the shell is practically without inertia (quasistatic). Conversely, at high frequencies, the shell oscillates and carries with it during the motion together with the vessel a certain volume of the fluid. In spite of the possible simplifications (ideal fluid, small oscillations) problems in hydroelasticity are by far not simple even in the case of axisymmetric oscillations of shells of rotation, since even then, the motion of the fluid is determined by the two-dimensional wave equation.

For the time being the list of studies made on this problem is not long. We mention here the studies of B. N. Bublik and V. I. Merkulov (1966), Yu. S. Shkenev (1964), V. P. Shmakov (1964), F. N. Shklyarchuk (1965, 1966) who introduced for the simplification of the hydrodynamic part the hypothesis of the plane reflection of the fluid. The analysis of the natural frequencies made by Yu. N. Novichkov (1966) is unnecessarily complicated by the given contact between the shell and the diaphragm which is not constructive, and in the study of G. Ye. Bagdasaryan and V. Ts. Gnuni (1966) the problem is reduced to a linear system with one degree of freedom.

There is no doubt that the problems under consideration represent a wide field for further investigations.

A survey of studies on quasistationary problems in the aero- and hydroelasticity of plates and shells is available in the report of L. I. Balabukh (1966).

Oscillations under Random Loads. Often thin plates and shells are subjected to atmospheric turbulence, acoustical radiation from operating engines, etc., i.e., they are subjected to random loads which induce oscillations in a wide region of the spectrum. The large density of the natural frequencies of the oscillations under these conditions makes it impossible to apply the method of expanding the solution in eigenfunctions. On the contrary, an effective method for studying the oscillations which occur under random loads is to replace the discrete computational scheme by a distributed scheme, instead of summing the free oscillations over the frequencies, integration in the space of wave numbers is used (in different terminology, the variability indices of the basic stressed state in two characteristic directions). An effective means for studying the simplest objects under a wide-band load is the asymptotic method of determining the natural frequencies and eigenfunctions that was proposed by V. V. Bolotin (1961).

In the article that was mentioned this technique is used and applied to a fixed plate. The mean square of the normal stresses near the edge was calculated under certain conditions pertaining to the correlation properties of the load. Later, V. V. Bolotin (1963) has shown that for the mean squares and the spectral densities integral estimates can be obtained under sufficiently general conditions for the correlation functions of the load. Special forms of a random acoustical field used as a load were the subject of the studies made by M. F. Dimentberg (1961, 1962) and Yu. A. Fedorova (1963) which were based on correlation methods. Next, Yu. A. Fedorov (1964) studied the action on a freely-supported plane plate of plane acoustical waves with a random frequency and amplitude, taking into account the geometric nonlinearity of the deformation by the method of a small parameter. V. A. Pal'mov (1965) derived the spectral densities for the deflection and stresses at a point sufficiently far from the edges under a random load of the wave type. He did not use the method of expanding the solution on the basis of the shapes of the natural oscillations, and took into account only a particular solution (with a large variability index).

A survey of the studies that were carried out until 1964 can be found in the article by V. V. Bolotin (1964) in which, in addition to the correlation method, the possibility of obtaining results in the application of the quasistatic method and the method of kinetic equations to the investigation of the statistical properties of the oscillation of plates and shells under random loads is discussed. Bolotin notes that a very large number of studies is devoted to the application of mathematical statistics to various branches of physics and technology which can be interpreted in terms of the theory of waves and shells. The application of these results to problems related to the theory of shells consists of determining the common properties of the oscillation spectrum. In linear problems this can be done at the present time. With regard to the oscillation of shells with finite amplitudes, probably it will be necessary to restrict oneself to a study of concrete problems which are of immediate practical interest. From the standpoint of the theory of shells, the effort must be directed toward taking into account the continual character of the work of the shell (V. V. Bolotin, 1966).

§14. Anisotropic Shells

The development of studies in the field of homogeneous anisotropic shells proceeded generally along the lines of developing the corresponding branches in the theory of isotropic shells. This is natural since the elastic coefficients of an isotropic and anisotropic body rarely differ in order of magnitude. In this case, it is easily seen that

(1) the basic equations of a transverse isotropic shell differ from the equations of an isotropic shell only by a different coefficient for the transverse displacement; (2) the basic equations of the orthotropic shell preserve the same structure but instead of two elastic constants it has 6 elastic constants and at the same time (3) during anisotropy with one plane of symmetry (from here on called the total anisotropy) the structure of the basic equations of an anisotropic and an isotropic shell is already different. As one would expect, the majority of the results on the anisotropic deformation refer to orthotropic shells, and most often, it is assumed that the principal directions of orthotropy coincide with the lines of curvature of the middle surface. Nevertheless, it must be mentioned that the representatives of the theory of anisotropic shells were very active also in reviving the problem of improving the classical theory of shells based on the Kirkhhoff-Loew hypotheses (S. A. Ambartsumyan, 1958). The internal stimulus for bringing this problem to the fore was the inconsistency of the classical theory during considerable anisotropy of the material, and also the necessity of applying to such materials relatively thick shells (including multilayer shells).

The first studies on anisotropic shells were published a long time ago (I. Ya. Shtayerman, 1924, Kh. M. Mushtari, 1938), but a more or less thorough development of the corresponding theories started only about twenty years ago when the first articles by S. A. Ambartsumyan (1947, 1948) were published. Of course, also the monographs of S. G. Lekhnitskiy (1943, 1947) played an important role in the development of the theory of anisotropic plates and shells, although the apparatus of the theory of functions that was applied there cannot be applied to the calculation of shells (with the exception of certain special cases of torqueless shells).

In one decade the development of the theory was carried out on the basis of the Kirkhoff-Loew hypotheses, and under multilayer anisotropic shell conditions, for an entire packet of layers. This approach should be fully applicable to a wide class of problems in which the composite layers do not have essential anisotropy or pronounced different elastic properties.

In a short time the results obtained for problems of isotropic shells were generalized to anisotropic shells on the basis of torqueless theory, the basis of calculations of shells of rotation under symmetric cyclic loads which included the problem of simple boundary effects. Until then the main attention was focused on orthotropic shells (on the principal lines of curvature), and these studies did not lead to the discovery of important new phenomena. An exception is

the case in which the middle surface of the shell has an umbilical point (an isolated point). (This point, according to S. A. Ambartsumyan, should be called a physical-geometric singular point). However, at the present time, this problem is only of theoretical interest.

When more general anisotropic problems were considered, it became evident that torqueless and pure torque states could not be isolated. Instead of these elementary states, the complex stressed state with a small variability index is formed. Next, in a shell whose contours lie on the surface of rotation, the deformation will no longer have this property under a cyclic symmetrical load, and the anisotropy has a great effect on the intensity function of the boundary effect. The remainder term in the first approximation for the simple boundary effect determined by the VKB method increases in absolute value (L. A. Movsisyan, 1958, 1959).

The equations of orthotropic cylindrical shells were first derived by Kh. M. Mushtari (1939). The general anisotropic case was studied much later (S. A. Ambartsumyan, 1948). However, with regard to methods for the integration of the equations during general anisotropy, the first results were obtained relatively recently (V. S. Sarkisyan, 1963). The abundance of elastic constants during general anisotropy gives rise, in cylindrical shells, to a large number of possible variants of the relations describing elementary states (S. A. Ambartsumyan, 1954). Perhaps it should be mentioned that the states of an isotropic cylindrical shell reduce to the generalized boundary effect and a simple boundary effect only during the calculation of the stresses around a concentrated load or a small hole; the state with a large variability index in an arbitrary direction on the middle surface is added to these states.

Studies in the linear and geometrically nonlinear deformation of flat shells deal with the orthotropic case in which the main methods used to integrate the equations are borrowed from the field of isotropic shells (Kh. N. Mushtari, 1938, Ye. F. Burmistrov, 1955, 1956). Thermoelastic problems in a nonlinear formulation have also been studied (Ye. F. Burmistrov, 1960, S. P. Durgar'yan, 1962, S. A. Ambartsumyan, 1963), and variants of nonlinear theory were constructed which were based on hypotheses differing from the Kirkhhoff-Loew hypotheses (S. A. Ambartsumyan and D. V. Peshtmaldzhyan, 1958).

The monograph by S. A. Ambartsumyan (1961) is devoted to a systematic presentation of the linear theory of anisotropic shells. A wider class of problems on anisotropic shells is covered in his survey articles (1962, 1964) in which the main development trends and problems which will successfully advance

the theory of anisotropic shells have been formulated. In the opinion of Ambartsumyan at the present time problems in which the anisotropy of the deformation has a general character deserve the greatest attention. In this field the danger exists that effort and means will be dissipated in the solution of special problems. Instead, an effort should be made to solve fundamental methodological problems, to classify individual situations, to carry out a qualitative analysis of the stressed state for each class and to develop on this basis effective solution methods.

§ 15. Shells in Layers

The three-layer shell consisting of two thin outer layers from a strong material joined to a light middle layer of low strength, the filler, is the most widely used type of layer shell in modern engineering. (Two-layer and multi-layer shells are also used.)

The problem of reducing the three-dimensional problem to a two-dimensional problem manifests itself here in a more complex manner. The point is that the filler may not only not be from an anisotropic material, but may also have homogeneous general anisotropic properties, or be made from a corrugated sheet, etc., with structural anisotropic characteristics which are difficult to determine and the joint deformations between the individual layers must be determined along lines which lie at discrete distances.

The first publications in the country on the theory of plates in layers and shells go back to the end of the 40's (S. A. Ambartsumyan, 1948, 1949, A. P. Prusakov, 1949). In these and in many subsequent studies, the system of Kirkhhoff-Loew hypotheses for the entire packet was taken as the basis for the construction of the relations used in the calculations. The main attention in the beginning was given to three-layer plates and to stability problems (the general and local loss of stability of the supporting layers. A list of studies from this period can be found in the corresponding section of the survey article by L. M. Kurshin (1962). The problems and results of the calculation of shells in layers are also discussed in great detail in the monograph and surveys of S. A. Ambartsumyan (1961, 1962, 1964), and the rich reference material on these calculation formulas and experiments can be found in the book by A. Ya. Aleksandrov, L. E. Bryukner, L. M. Kurshin and A. P. Prusakov (1960).

To construct simple and universal equations for the calculation of three-layer plates with a light filler, it was necessary to use different hypotheses with respect to the filler. One of the pioneers in this direction was A. P. Prusakov (1951). From the methodological point of view, the justification of the working hypotheses sometimes entails internal contradictions, which can be seen on the example on which the basic ideas were illustrated in the article by L. M. Kurshin that was just mentioned (see pages 168-169 in that article).

Perhaps by developing the theory of layer shells, it was discovered for the first time that it was essential to abandon the usual Kirkhhoff-Loew hypotheses and take into account the effect of the transverse shear and also compression.

E. I. Grigolyuk (1957, 1958), who constructed the geometrically nonlinear theory of three-layer shells with a symmetric structure, started out with the assumption that the Timoshenko hypothesis can be used for the middle layer, and that the outer layer satisfies the Kirkhhoff-Loew hypotheses. It was assumed that the deflection in all layers is the same. As a result a 12-th order nonlinear system was obtained. A generalization of these results to shells with a nonsymmetric structure was obtained by Kh. M. Mushtari (1961). The weak point in this variant of the theory is the assumption that the rotation vector of the normal near the outer layers is the same and equal to the gradient of the deflection.

S. A. Ambartsumyan (1957) proposed that the distribution of the transverse shear be given along a parabola in order to make the classical theory of shells more precise. This assumption replaces the Kirkhhoff-Koew hypotheses that the normal to the middle surface is unchanged after the deformation (the remaining Kirkhhoff-Loew assumptions are retained). The construction of the theory on the basis of this hypothesis is somewhat more complex than that based on the energy method applied by E. I. Grigolyuk, but it only manifests itself more or less essentially in nonlinear problems.

In one study on three-layer shells, E. I. Grigolyuk and P. P. Shulkov (1963) took also formally into account the filler by introducing an appropriate coordinate. The deformation of the outer layers was assumed to satisfy exactly the Kirkhhoff-Loew hypotheses, which led to a 16-th order system. Later the same authors abandoned taking into account the compression (assuming $\sigma_{zz} = 0$) when they set up the physical relations for the layers) and derived a 12-th order system which in some cases can be reduced to a 10-th order system when one boundary effect of the St.-Venant type is ignored. At the same time

the simplified relations do describe one boundary effect of the St.-Venant type by an equation which has the same structure as the equation that was ignored. For the time being, it is not clear which of these boundary effects is physically more significant.

Recently the interest in multilayer shells increased. It is possible to construct a theory on the basis of the Kirchhoff-Loew hypotheses without any particular difficulties. and indeed in many cases acceptable results have been obtained with the aid of such a theory (S. A. Ambartsumyan, 1961). However, when the individual layers have different elastic properties, studies must still be made to set up an adequate computational model. The nonlinear equations of isotropic multilayer shells during arbitrary heating using the hypothesis of outer normals in each layer were obtained by E. I. Grigolyuk and P. P. Chulkov (1965). The system of hypotheses that was adopted reduces the calculation of an n-layer shell to a system of quasilinear differential equations of order (6 + 4n) whose solution must satisfy 3 + 2n boundary conditions on the contour of the truncated shell.

In the theory of multilayer anisotropic shells many problems must still be solved, even though the method used for the solution is based to some extent on the achievements in the theory of homogeneous isotropy. We will mention here only some of these problems which are most essential: 1) which equations can describe the slowly changing stressed states? 2) Do stressed states exist (and under what conditions) which are known as simple boundary effects in classical theory and what is their number on the edge? 3) Under what conditions do the simple boundary effects degenerate into generalized boundary effects with slower damping away from the edge? 4) What is the number of boundary effects of the St.-Venant type generated by the concrete theory of a multi-layer shell? Can they be grouped into individual classes by the properties of the stressed states, and is the given theory for the description of the boundary effects of the St.-Venant type adequate? The real possibilities of determining the 3 + 2 n elastic-fixing coefficients on each boundary must be kept in mind; it is precisely here where the gap between the theory and practice is great.

§16. Reduction of Problems in Elasticity Theory to the Theory of Shells

Clearly, there are many possible ways which can be used to reduce problems in elasticity theory to problems in the theory of shells for thin-walled objects, such as plates and shells. The main results in this field are discussed in the surveys by I. I. Vorovich (1966) with emphasis on equilibrium problems. The state of the reduction problem in the solution of dynamic problems is presented in the survey article by L. Ya. Aynoli and U. K. Nigula (1965).

The reduction methods can be classified arbitrarily into the following main groups: (1) analytical methods, (2) variational methods, (3) asymptotic methods.

The method of power series in which the coefficients in the expansion for the unknown quantities are determined recurrently from six basic functions (of the internal coordinates α, β of the middle surface) (along the coordinate z which is normal to the middle surface) have the oldest history. The latter are determined by the conditions on the lateral surfaces (N. A. Kil'chevskiy, 1939, 1963), which they satisfy with an accuracy up to terms of a certain order z^k, so that in practice only truncated systems (i.e., systems of differential equations of a finite order). It should be mentioned that the satisfaction of the boundary conditions (on contour surfaces) and of the initial conditions with a given accuracy requires the derivation of a system of differential equations with a high order of accuracy. The truncated systems obtained by expanding the solution in power series were obtained in a number of concrete cases by I. T. Selezov (1961, 1963) and U. K. Nigul (1962).

In the case of simplest objects (plates, circular, cylindrical and spherical shells) the power series algorithm can be reduced to the elegant formulas of the "symbolic" method developed by A. I. Lur'e (1942, 1955) or to V. Z. Vlasov's method of initial functions (1955). The symbolic method can also be applied to the derivation of simplified dynamic equations with small variability indices (Yu. K. Nigul, 1963). However, the boundary conditions for the equilibrium equations of thick plates were obtained using a variational formulation of the problem (V. K. Prokopov, 1965).

The method of "homogeneous" solutions is intimately related to the symbolic method in the study of the simplest objects that were described above. The solution of the problem in elasticity theory using this method is solved in the form of an infinite sum of particular solutions satisfying homogeneous boundary conditions on the lateral surfaces (which are parallel to the middle surface), but, generally, not the boundary conditions on the contour surfaces. The particular solution of the equations of elasticity theory satisfying the nonhomogeneous boundary conditions on the lateral surfaces is added to the set of solutions. The main steps in the solution of the problem are (1) determining the roots of the transcendental characteristic equation of the "homogeneous" solutions and (2) determining the procedure which defines the arbitrariness in the integration of the homogeneous solutions in terms of the given boundary conditions on the contour surfaces. Usually the principle of possible displacements is used for this purpose.

The method that was described was applied to the study of the equilibrium of circular plates by V. K. Prokopov (1952), O. K. Aksentyan and I. I. Vorovich (1963). The case of a closed circular cylindrical shell during an axisymmetric deformation was studied by V. K. Prokopov (1949) and also by N. A. Bazarenko and I. I. Vorovich (1965). The survey article by G. Yu. Dzhanelidze and V. K. Prokopov (1963) is devoted to the application of this method to the theory of elasticity.

It should be added that when the roots of the characteristic equations are determined, and the stressed state corresponding to each root, the unknowns are systemmatically expanded in powers of the small parameter, i.e., the relative thickness. Naturally, this is the most reliable method for studying the reduction problem, but, unfortunately, it can only be applied to a very limited class of problems. For example, in the study of the propagation of stress waves, the method of homogeneous solutions can be applied without complications only for certain boundary conditions which admit a sin-cos integral transformation with respect to the coordinate (U. K. Nigul, 1963).

The procedure for comparing the formal solutions in the form of contour integrals for problems in the theory of elasticity and the theory of plates should be included among the analytical reduction methods. According to the idea of G. I. Petrashen (1951); both theories should give the same expansion for the unknown quantities in the low frequency (complex) oscillations part. Since the lower dimensional approximate theory cannot ensure this completely, the condition for the applicability of the approximate theory are derived from the comparison.

Energy methods in which the unknown quantities are approximated as functions of z by some closed system (for example, Legendre polynomials) are very often used for the reduction, and the differential relations between the coefficients are obtained from the Lagrange or Castigliano variational formulas (or some other extended variational formula). In this way systems of differential equations of an arbitrary order can be constructed and the corresponding boundary conditions can be obtained without any difficulty (in many methods applied to the solution of the reduction problem, the formulation of the boundary conditions for the truncated system is the most vulnerable point in the theory).

Examples of the construction of such improved theories of shells are given in the studies of I. N. Vekua (1955, 1965), I. G. Teregulov (1962), N. A. Kil'chevskiy (1963), and V. V. Ponyatovskiy (1962). An analysis of the systems that were derived shows that when the order of the system of differential equations exceeds eight, boundary effects of the St.-Venant

type occur in the solutions. In addition an increase in the order of the system of equations (physically this corresponds to an increase in the number of degrees of freedom) leads only to new integrals with a high variability index, boundary effects of the St.-Venant type. Thus, if the St.-Venant boundary effects must be isolated which correspond to the torsion on the edge and to the plane deformation on the edge in the first approximation, the system of differential equations in the theory of shells must be a 14-th order system. However, so far there are no published results on the analysis of such expanded systems of equations in the theory of shells.

In connection with the attempts to solve the reduction problem by variational methods, the formulation of the problem of the best variant of a system of differential equations for determining the principal stressed states should be mentioned. Usually the structure of the equations is given (for example, in the case of the bending of a plate it is required that the solution equation be a fourth order equation, and what is sought is the best distribution of the displacements and stresses in the energy sense along the thickness which is constant and expressed in terms of one (unknown) function of z (L. Ya. Aynola, 1963). The problem reduces to the solution of a system of integrodifferential equations using the method of successive approximations.

The third way of solving the reduction problem is using the direct asymptotic integration method. Here, by a change of the coordinates (which is different for different stressed states which must be found, a parameter (say, ϵ) characterizing the thin wall of the shell is arbitrarily introduced into the equations of elasticity theory. Next, a particular intensity index must be associated with each unknown function, which makes it possible to determine recurrently the unknown terms in the expansion in powers of the small parameter ϵ. It is clear from the above, that to apply the method successfully, it is very desirable to have preliminary information about the basic properties of the stressed state that is being determined, since otherwise confusion may arise in finding noncontradictory intensity indices. However, when this "starting" point is overcome, the subsequent steps lead quickly to elegant procedures for determining the stressed state whose accuracy is improved successively for a wide class of problems.

In our country this approach was developed by A. L. Gol'denveyzer (1962, 1963, 1965) and his collaborators A. V. Kolos (1964, 1965), A. N. Volkov (1965) and M. I. Guseyn-Zade (1965). The results that were published show that the basic asymptotic integration process leads only in the first approximation to the computational relations known in the classical theory of plates and shells which describe the so-called basic stressed state (compression and bending of plates, torqueless and torque states, and states with a large

variability index). The relations from which boundary effects of the St.-Venant type are determined in the first approximation are essentially new (partial differential equations for the torque on the edge and the plane deformation on the edge) which undergo rapid damping at the edge.

Like in all reduction methods not based on energy relations, the determination of the boundary conditions which the differential equations that are integrated in a certain approximation stage must satisfy, presents certain difficulties. This problem was only solved partially for several variants of the boundary conditions for the plate.

It is of interest to note that the basic procedure reduces to finding the stressed states which vary quickly in the direction of the normal (z), but not in the directions (α, β) tangent to the middle surface. At the same time this assumption leads in the zero approximation to stressed states for which the displacements u_α, u_β, w and the stresses $\sigma_{\alpha\alpha}$, $\sigma_{\alpha\beta}$, $\sigma_{\beta\beta}$ are linear functions of z. In this case, of course, it is difficult to say that these quantities vary quickly with respect to z. Thus, a great deal must still be reformulated in order that the results which have been known for a long time be fully applicable in practice without contradicting the initial assumptions.

The application of asymptotic integration methods to the solution of the reduction problem is by and large in the initial development stage. A clear example of this statement is the problem of the stressed states of a closed shell of the solid spherical type (with positive curvature everywhere?) that was formulated by A. L. Gol'denveyzer. This problem is considered to be the most suitable problem for the classical theory of shells. The results of an analysis of the solution of this problem are very intriguing. Gol'denveyzer has shown that by making certain changes in the physical relations, the accuracy of the equations of the classical theory of shells can be improved. However, these relations cannot be derived on the basis of the Kirkhhof-Loew hypotheses. Therefore, all that can be said is that in the case under consideration, it was possible to represent the new content in an old form, which cannot always be done and which is not always useful.

The asymptotic integration method was also generalized to the derivation of dynamic equations of plates with large displacements (L. Ya. Aynola, 1965, 1966). The results have shown that the known equations of Karman's membrane theory, with a plane stressed state and purely linear theory are, under certain loading conditions, asymptotic approximations of the equations of geometrically nonlinear elasticity theory.

The studies that were mentioned above should be of methodological interest, the equations of motion and the boundary conditions are derived from the requirement that the variations of the corresponding functional be zero with the required asymptotic accuracy.

Among the reduction methods, the application of the work reciprocity theorem for an elastic system to the derivation of two-dimensional integrodifferential equations occupies a special position. The development of methods and techniques along these lines was begun by N. A. Kil'chevskiy (1940). The results of the studies were summarized by him on a number of occasions in survey articles and in a monograph (1962, 1964) in which bibliographic references referring to the fundamental studies in the problem that was touched on are given.

The work reciprocity theorem can be interpreted broadly, since the forces and displacements can also be considered in a generalized sense. It is well known that in this theorem, two states are compared. One of them is the sought main state and the other the auxiliary state. This theorem can be useful if the solution of the auxiliary problem is considerably simpler than the solution of the basic problem. One of the two possibilities is that the solution for the action of a concentrated force in an infinite elastic system is taken as the basis for the auxiliary state. But the shell has (at least in the direction of the normal to the middle surface) a finite length; therefore, the absence of the "medium" in this direction must be compensated by a load which is distributed on the boundary surfaces of the shell (and also on the contour surfaces which are usually present). In the reduction problem, generalized forces are considered instead of the concentrated force (for example, zero, first and higher order moments along the thickness) and the corresponding generalized displacements. This requires that relatively simple modifications be introduced into the procedure that was described above.

In the study of shells with zero curvature and flat shells whose middle surface is isometric to the plane plate, the state of the plate is often taken as the auxiliary state, which simplifies the construction of kernels, but at the same time changes their structure. Recently, the idea was proposed to use "focused" kernels, i.e., rapidly damped auxiliary states, to improve the convergence of the computational procedure (N. A. Kil'chevskiy, 1960, N. A. Kil'chevskiy, Kh. Kh. Konstantinov and N. I. Remizov, 1966). For the time being this class of problems is characterized by different formulations of the problems, by new methods that are being proposed and by the absence of concrete experience which is

obtained by solving the reduction problem to its logical end, i.e., to a certain system of two-dimensional equations. Of greatest interest is the solution of problems for which the stressed state of the shell must be found with the aid of the equations of elasticity theory (for example, boundary effects of the St.-Venant type, the state around the "concentrated" load around propagating perturbation fronts, etc.).

However, this remark applies equally to all directions engaged in the solution of the reduction problem. The main topic in the nearest future must be the problem of the stressed state around singular points and "distortion" lines of the stressed state. From the standpoint of the solution of this problem, all known methods have equal chances for success. Perhaps pure numerical methods used in the solution of the equations of elasticity theory (without an explicit formulation of the reduction problem) should be added to the methods that were discussed here.

§17. Conclusion

Any problem of elastic shells is characterized by a large number of initial conditions. As an example, we can point out the following:

--elasticity laws: linear and nonlinear;

--anisotropy of the material: isotropy, orthotropy, transverse isotropy, general anisotropy;

--the structure of the shell: one-layer, two-layer, three-layer, multilayer;

--connectedness of the middle surface: simply connected, doubly connected, multiply connected (for example, perforation;

--curvature of the middle surface: positive, zero, negative, changing sign (torus, corrugated plate);

--geometric properties of contour lines: not asymptotic, simple asymptotic, multiple asymptotic;

--form of load: distributed over the surface, distributed over a line, concentrated;

--deformation process: static, stationary, quasistationary, transient;

--definition of load: given, depending on the interaction of the shell and the external field, random (with given statistical characteristics);

--geometry of the deformation: linear, nonlinear;

--computational model: torqueless, Kirkhhof-Loew, Timoshenko type, various simplified variants of these theories, (for example, generalized boundary effect), elasticity theory;

--method of analysis: qualitative analysis, analytical solution, numerical method.

Often it is extremely difficult to draw an exact boundary line between the details in the individual characteristics.

The questions that were touched on in this survey are an incomplete and subjective selection from all problems in the theory of elastic shells. The list of studies that was mentioned which may not be very impressive due to its length has the same character.

The contribution of Soviet scientists to the world's treasures in the theory of elastic shells and plates is great. There is no doubt that the network of well-known scientific schools that was developed in this country will ensure a successful solution of problems in the theory of shells which arise in practice and also the needs of the science itself.

ONE TREND IN THE DEVELOPMENT OF A THEORY OF SHELLS

I. N. Vekua

§1. Introduction

1.1. This survey consists of two parts. The first part deals with the general problem of constructing a theory of elastic shells. The second part deals with a brief presentation of certain results in membrane theory of convex shells. We will deal only with those problems which were the subject of a study by the author which began approximately in 1950. Both parts are intimately related to the application of methods of the theory of analytic functions which began to find their way into the theory of shells in the 40's, mainly due to the well-known studies of N. I. Muskhelishvili on the plane problem in the theory of elasticity.

The theory of shells is obviously an applied science, but it is related to many branches of contemporary analysis, and it was the source for the formulation of a number of important and interesting mathematical problems. The study of the torqueless theory of convex shells led to the necessity to expand the classical theory of functions. A new branch of analysis was developed, the theory of generalized analytic functions, which is also intimately connected with the geometric problem of infinitely small bends in convex surfaces (I. N. Vekua, 1959).

1.2. The theory of shells is a branch of the mechanics of a continuous medium. It developed methods for the calculation of thin-walled shells which are widely used in modern engineering structures and in machine building. Typical examples of shells are walls and various covers, platings in ships, fuselages and airplane wings, the bodies of submarines, etc. We can distinguish elastic and nonelastic shells. This will depend not only on the material of the shell but mainly on the character of the distribution and the magnitude of the external load as well as the type of external kinematic (geometric) relations. If the distribution of the external load is piecewise continuous and it does not exceed some characteristic critical load, the shell can

be considered as an elastic shell. Below, when we speak about elastic shells, we will assume that they obey the generalized linear Hooke law. Such shells are used rather widely in engineering. Therefore, their study is of considerable applied interest. In the study of the torqueless stressed equilibrium of bent shells we can abandon the frame of reference of elastic properties. Here it is not necessary to use elasticity relations when we are dealing only with the determination of the stressed state. In this case we have a statically determinate problem. The elasticity relations must only be used in the case when it is also necessary to determine the deformed state. Such a problem arises when, for example, not only a physical boundary condition must be satisfied, but also a certain geometric (kinematic) boundary problem.

1.3. The calculation of shells on the basis of the equations of elasticity theory is connected with great mathematical difficulties. The science does not yet have at its disposal practical convenient methods for the solution of a comparatively wide class of applied problems. The theory of shells tries to simplify these problems by taking into account specific features of the shells. First of all, the fact that the thickness of the shell is small compared to its other two linear dimensions was taken into account. It can be easily seen that the pattern of the deformed and stressed state also depends to a considerable extent on the properties of [illegible] middle surface. In many engineering applications [illegible] e encountered whose middle surfaces are considerably [illegible] When this fact is taken into account, the problem can be s[illegible]lified considerably.

1.4. The first essential important step in the theory of shells is the reduction of the three-dimensional problem to a two-dimensional problem. For this purpose the shell is often represented as a sufficiently hard material surface with a very small finite thickness. Clearly such a model is a very coarse approximation of the real shell. Nevertheless, it allows us to simplify the mathematical problem and to recreate the pattern which is sufficiently close to the pattern observed in real shells. Such schematic representation of the problem requires that a number of hypotheses of a physical and geometrical character be adopted, whose formulation, as a rule, is based on intuitive concepts. At first glance these hypotheses seem to be very plausible, but their weakness is the absence of exact methods which would verify them experimentally for a wide class of problems and materials. Their applicability range can only be determined in some cases in a posteriori fashion by comparing the numerical results that were obtained with the experimental data or with the exact solutions of the corresponding three-dimensional problem. This rather fluid

situation gives rise to various variants of the theory of shells. The existing variants sometimes differ considerably from one another and it is difficult to make a judgment about the advantages of a particular variant. The common shortcoming of many existing variants of the theory of shells is the absence of internal consistency between the initial kinematic and physical assumptions. This lack of agreement can be seen, for example, in the fact that the system of differential equations of the theory of shells does not ensure that the boundary conditions which follow from the initial assumptions are satisfied.

1.5. The majority of the variants is based on a physical hypothesis which assumes that to describe the stressed state of a thin shell it is sufficient to determine in practice the forces and moments acting on the elementary transverse areas. In each transverse area five quantities must be determined: the normal and tangential forces, the shearing force, the torque and the bending moments, which clearly depend on the position of the point on the middle surface and also on the orientation of the area. This problem reduces to the determination of two tensors of rank two and one tensor of rank one which belong to the middle surface. Thus, altogether, ten functions which are the components of the unknown tensors must be found. These functions satisfy a system of five first-order partial differential equations. Therefore, generally, the problem is statically indeterminate. In order to eliminate this indeterminacy, it is necessary to adopt additional constraints on the character of the distributions of the stresses or deformations in the shell. An attempt can be made to make the problem statically determinate by adopting for this purpose certain constraints on the law for the distribution of the forces and moments. For example, assuming that the moments of the stress forces are zero everywhere, we obtain a statically determinate problem. However, the torqueless equilibrium state is a very special case of the general stressed state of the shell. This can be seen from the fact that the corresponding system of differential equations does not completely ensure that the natural physical boundary conditions for the problem are satisfied. Generally the value of only one of the two components of the force vector can be given arbitrarily on the boundary. For example in the case of a convex shell when the boundary values of the normal force are given, it is possible after solving the problem, to determine the corresponding boundary value of the tangential force. Therefore, it is not possible to state in advance arbitrarily the boundary values of both components of the force vector. In spite of this, the torqueless or, in different words membrane theory of shells, has important practical applications. This

is explained by the fact that as engineering practice and theoretical studies have shown, the discrepancy mentioned above gives rise to some boundary effect which has no important effect on the character of the distribution of the stresses away from the boundary for a wide class of problems. The great advantage of membrane theory is the comparative simplicity of the corresponding mathematical problem and also the fact that it does not use elasticity relations and, hence, can be applied to a very wide class of elastic and nonelastic shells.

A second way of eliminating the indeterminacy inherent in the problem of determining the forces and moments is to use relations which follow from Hooke's linear law. However, when this is done a number of additional assumptions must be made to obtain the correct mathematical problem.

When the physical hypothesis which was mentioned above is taken as the basis for the theory of shells, certain constraints are imposed on the character of the deformation of the shell. If the shell obeys the requirements of the physical hypothesis this means, in fact, that the elementary transverse areas must be considered as absolutely rigid figures (at least in first approximation). If this were not the case, strictly speaking, we could not replace the continuous distribution of the stress forces over the area by a statically equivalent set of forces and couples (force and moment). The complexity of the problem is constructing a kinematic model which is in complete agreement with the physical hypothesis that was adopted. The lack of agreement which occurs here follows from the fact that the system of differential equations that was obtained here is not compatible with the boundary conditions which follow from the initial hypotheses. For example, this kind of situation arises when the well-known Kirchhoff-Loew hypothesis is used. It imposes constraints on the deformation which are too stringent. As a result of this, the class of unknown functions is narrowed down to such an extent, that it is not possible to satisfy the five physical boundary conditions in the classical moment theory of shells and the corresponding system of differentical equations satisfies only four independent boundary conditions.

Thus, both in the moment theory of shells and also in membrane theory, the physical boundary conditions can only be satisfied partially. This contradiction is a serious shortcoming of the classical constructions. Trying to remove the defect, many investigators attempted to find new approaches and develop a more perfect theory. Many important studies along these lines were made by scientists in various countries

among which we mention, for example, the studies of E. Reissner.[1]

Below we will discuss mainly those results which were obtained along these lines in the last 15-20 years by the author of this survey. A general method was developed which makes it possible to construct various variants of the theory of shells which are classified by the order of the approximation. Following a natural and actually relatively simple approach, it was possible to construct a general theory which is mathematically correct.

First the case of the so-called prismatic shell was studied whose middle surface is a plane (1955). Generally, such shells may have a variable thickness. Later these studies were generalized to the case of an arbitrary flat shell, (1964, 1965). Below, we will briefly characterize the essence of the methods that are used, and we will outline the results that were obtained.

§2. Theory of Thin Flat Shells

2.1. Using the notation and concepts of vector and tensor analysis, the equilibrium equations of a continuous medium and the relations of the theory of elasticity can be written in the following form:

1) the equilibrium equation:

$$\frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}\, \mathfrak{P}^k}{\partial x^k} + \widetilde{\mathfrak{F}} = 0; \tag{2.1}$$

2) the stress force acting on an area with normal l:

$$\mathfrak{P}_{(l)} = \mathfrak{P}^k l_k \qquad (l_k = \mathfrak{R}_k l); \tag{2.2}$$

3) the generalized Hooke law:

$$\mathfrak{P}^k = \lambda \left(\mathfrak{R}^i \frac{\partial \mathfrak{U}}{\partial x^i}\right) \mathfrak{R}^k + \mu (\mathfrak{R}^i \mathfrak{R}^k) \frac{\partial \mathfrak{U}}{\partial x^i} + \mu \left(\mathfrak{R}^k \frac{\partial \mathfrak{U}}{\partial x^i}\right) \mathfrak{R}^i. \tag{2.3}$$

---

1. A rather complete bibliography on this problem can be found in the book by A. E. Green and V. Zerny "Theoretical Elasticity" (1954) and in the collection of translations "Elastic Shells" (Moscow, 1962).

These equations are referred to an arbitrary coordinate system. We will explain the notation used here: g is the discriminant of the metric quadratic form in the space

$$ds^2 = g_{ik}dx^i dx^k, \quad g_{ik} = \mathfrak{R}_i\mathfrak{R}_k. \tag{2.4}$$

$\mathfrak{F}$ is the body force, $\mathfrak{P}^i$ are the contravariant components of the stress force ($\mathfrak{P}^i = p^{ik}\mathfrak{R}_k$, where p^{ik} are the contravariant components of the stress tensor), $\mathfrak{R}_k$ and $\mathfrak{R}^k$ are the basis and conjugate basis vectors of the selected coordinate system $\mathfrak{U}$ is the displacement vector and λ and μ are the Lame elasticity constants.

The problem of integrating the system of equations (2.1) and (2.3) taking into account the boundary conditions is the fundamental problem studied in the theory of elasticity. The basic boundary value problems are problems of finding a solution of the system which satisfies the boundary condition

$$\mathfrak{P}_{(n)} = \mathfrak{f} \quad \text{on F} \tag{2.5}$$

or

$$\mathfrak{U} = \mathfrak{f} \quad \text{on F}, \tag{2.6}$$

where F denotes the boundary of the elastic body under consideration.

In addition, an equally important practical problem is the study of various types of mixed boundary value problems.

2.2. As was already mentioned above, the main goal of the theory of shells is to develop methods which can be used, taking into account the specific properties of shells, to construct approximate solutions for the boundary value problems that were mentioned above. We now pass to the systematic presentation of the method that was proposed for this purpose.

Let S^+ and S^- be the outer surfaces, S the middle surface of the shell and Σ the lateral surface by which the shell is bounded. Taking the scalar product of both members of equation (2.1) with some vector and then integrating over the region D occupied by the shell and applying the integral Ostrogradskiy-Gauss formula, we obtain

$$\iint_{\Sigma} \mathfrak{V}\mathfrak{P}_{(t)}\, d\Sigma + \iint_{S^+} \mathfrak{V}\mathfrak{P}_{(n^+)}\, dS^+ + \iint_{S^-} \mathfrak{V}\mathfrak{P}_{(n^-)}\, dS^- = \qquad (2.7)$$

$$= \iiint_D \mathfrak{V}\mathfrak{F}\, dD + \iiint_D \mathfrak{P}^i \frac{\partial \mathfrak{V}}{\partial x^i}\, dD,$$

where $\mathfrak{P}_{(n^+)}$ and $\mathfrak{P}_{(n^-)}$ are the stress forces acting on S^+ and S^-, respectively. This equality is valid for any continuously differentiable vector field $\mathfrak{V}$, when the $\mathfrak{P}^i$ satisfy the equation (2.1). It can be easily proved that the integral equation (2.7) is equivalent to equation (2.1).

Below we will use a special coordinate system in which the coordinate lines x^3 are a family of normals to the middle surface S. Then the surface S will be the coordinate surface $x^3 = 0$ and the radius vector $\mathfrak{R}$ of a point on the shell will be described by the formula

$$\mathfrak{R} = \mathfrak{r}(x^1, x^2) + x^3 \mathfrak{n}(x^1, x^2), \qquad (2.8)$$

where $\mathfrak{n}$ is the unit normal to S at the point (x^1, x^2) and r is the radius vector at this point.

Now let us take for $\mathfrak{V}$ in equality (2.7) a vector of the form

$$\mathfrak{V}_n(x^1, x^2) P_n\left(\frac{x^3}{h}\right) \quad (n = 0, 1, \ldots), \qquad (2.9)$$

where P_n are Legendre polynomials of degree n, and $\mathfrak{V}_n(x^1, x^2)$ is any continuously differentiable vector-function on S which is different from zero in some subregion S' in the interior of S. Then we obtain from (2.7) the following equalities:

$$\iint_{S} \Big\{ \frac{1}{\sqrt{a}} \frac{\partial \sqrt{a}\, \overset{(n)}{\mathfrak{P}}{}^{\alpha}}{\partial x^{\alpha}} - (2n+1)\big[\overset{(n-1)}{\mathfrak{P}}{}^{3} + h^2 \overset{(n-3)}{\mathfrak{P}}{}^{3} + \ldots\big] +$$

$$+ (2n+1) h \frac{\partial h}{\partial x^{\alpha}}\big[\overset{(n-2)}{\mathfrak{P}}{}^{\alpha} + h^2 \overset{(n-4)}{\mathfrak{P}}{}^{\alpha} + \ldots\big] + \overset{(n)}{\mathfrak{F}} \Big\} \frac{1}{h^n} \mathfrak{V}_n\, dS = 0 \qquad (2.10)$$

$$(n = 0, 1 \ldots;\ \overset{(k)}{\mathfrak{P}}{}^{i} = 0 \ \text{при}\ k < 0),$$

where

$$\overset{(n)}{\mathfrak{P}}{}^{i}(x^1, x^2) = \left(n + \frac{1}{2}\right) h^{n} \int_{-h}^{h} \sqrt{\frac{g}{a}}\, \mathfrak{P}^{i}(x^1, x^2, x^3)\, P_n\left(\frac{x^3}{h}\right) dx^3, \tag{2.11}$$

$$\begin{aligned}\overset{(n)}{\mathfrak{F}} = \left(n + \frac{1}{2}\right) \overline{h}^{n} \Big[\mathfrak{P}_{(n+)} \left(\sqrt{\frac{g}{a}}\right)_{x^3 = h} + (-1)^{n} \mathfrak{P}_{(n-)} \left(\sqrt{\frac{g}{a}}\right)_{x^3 = -h} + \\ + \int_{-h}^{h} \sqrt{\frac{g}{a}}\, \mathfrak{F}(x^1, x^2, x^3)\, P_n\left(\frac{x^3}{h}\right) dx^3 \Big].\end{aligned} \tag{2.12}$$

Here a denotes the discriminant of the first fundamental quadratic form of the middle surface

$$d\sigma^2 = a_{\alpha\beta} dx^{\alpha} dx^{\beta}, \qquad a_{\alpha\beta} = \mathbf{r}_{\alpha} \mathbf{r}_{\beta}, \tag{2.13}$$

i.e.,

$$a = a_{11} a_{22} - a_{12}^2 > 0. \tag{2.14}$$

$$\begin{aligned}\frac{1}{\sqrt{a}} \frac{\partial \sqrt{a}\, \overset{(n)}{\mathfrak{P}}{}^{\alpha}}{\partial x^{\alpha}} - (2n + 1) [\overset{(n-1)}{\mathfrak{P}}{}^{3} + h^2 \overset{(n-3)}{\mathfrak{P}}{}^{3} + \dots] + \\ + (2n + 1) h \frac{\partial h}{\partial x^{\alpha}} [\overset{(n-2)}{\mathfrak{P}}{}^{\alpha} + h^2 \overset{(n-4)}{\mathfrak{P}}{}^{\alpha} + \dots] + \overset{(n)}{\mathfrak{F}} = 0 \\ (n = 0, 1, \dots).\end{aligned} \tag{2.15}$$

follows immediately from equations (2.10).

It should be noted that this system of equations is exactly equivalent to the original equation (2.1). It has the advantage that the unknown functions are functions only of the two independent variables x^1 and x^2. It disadvantage is that it contains an infinite number of equations. Below we will show how this disadvantage can be eliminated.

2.3. We will make two assumptions. The first assumption is a purely mathematical assumption and is based on the well-known Weirstrass theorem which says that any continuous function can be uniformly approximated by polynomials on a closed interval. On the basis of this theorem, it can be assumed that the approximate solution of the problem (the components of the displacement vector $\mathcal{U}$ and the stress tensor) can be found in the form of polynomials of degree N in the variable x^3 $(-h \leq x^3 \leq h)$.

The second assumption imposes a constraint on the class of shells under consideration. We will consider so-called thin flat shells for which we can assume

$$1 - k_1 x^3 \approx 1, \quad 1 - k_2 x^3 \approx 1 \quad (-h \leqslant x^3 \leqslant h), \tag{2.16}$$

where k_1 and k_2 are the principal curvatures of the middle surface S. It should be noted that in the case when the middle surface is a plane, assumption (2.16) is satisfied exactly, since $k_1 = k_2 = 0$. Hence, in the case of prismatic shells, (in particular for a plate) the second assumption is unnecessary and the theory is based only on the first assumption.

Let us take as the coordinate lines x^1 and x^2 the lines of curvature. Then

$$\mathfrak{R}_1 = \mathfrak{r}_1 (1 - k_1 x^3), \quad \mathfrak{R}_2 = \mathfrak{r}_2 (1 - k_2 x^3), \quad \mathfrak{R}_3 = \mathfrak{n}. \tag{2.17}$$

Hence, in view of assumptions (2.16), we can take

$$\mathfrak{R}_1 \approx \mathfrak{r}_1, \quad \mathfrak{R}_2 \approx \mathfrak{r}_2, \quad \mathfrak{R}_3 = \mathfrak{n}, \tag{2.18}$$

$$g_{\alpha\beta} \approx a_{\alpha\beta} = \mathfrak{r}_\alpha \mathfrak{r}_\beta, \quad g_{\alpha 3} = 0, \quad g_{33} = 1 \quad (\alpha, \beta = 1, 2), \tag{2.19}$$

$$g = a\,[1 - 2Hx^3 + K(x^3)^2] \approx a, \tag{2.20}$$

where H and K are the mean and principal curvature of the middle surface. Therefore, formulas (2.11) can be rewritten in the form

$$\overset{(n)}{\mathfrak{P}}{}^{i}(x^1, x^2) \approx \left(n+\frac{1}{2}\right) h^{n} \int_{-h}^{h} \mathfrak{P}^{i} P_n\left(\frac{x^3}{h}\right) dx^3. \tag{2.21}$$

We will now use the formulas which follow from Hooke's law

$$P_{ik} = \lambda\Theta g_{ik} + 2\mu e_{ik}, \tag{2.22}$$

where

$$e_{ik} = \frac{1}{2}\left(\mathfrak{R}_i \frac{\partial \mathfrak{u}}{\partial x^k} + \mathfrak{R}_k \frac{\partial \mathfrak{u}}{\partial x^i}\right), \tag{2.23}$$

$$\Theta = e^i_i = \mathfrak{R}^i \frac{\partial \mathfrak{u}}{\partial x^i} = \operatorname{div} \mathfrak{u}. \tag{2.24}$$

In view of (2.18), we write

$$\left.\begin{aligned}
e_{\alpha\beta} &\approx \frac{1}{2}\left(\mathfrak{r}_\alpha \frac{\partial \mathfrak{u}}{\partial x^\beta} + \mathfrak{r}_\beta \frac{\partial \mathfrak{u}}{\partial x^\alpha}\right),\\
e_{\alpha 3} &\approx \frac{1}{2}\left(\mathfrak{r}_\alpha \frac{\partial \mathfrak{u}}{\partial x^3} + \mathfrak{n} \frac{\partial \mathfrak{u}}{\partial x^\alpha}\right),\\
e_{33} &= \mathfrak{n} \frac{\partial \mathfrak{u}}{\partial x^3} = \frac{\partial u_3}{\partial x^3} \qquad (u_3 = \mathfrak{n}\mathfrak{u}),\\
\Theta &\approx \mathfrak{r}^\alpha \frac{\partial \mathfrak{u}}{\partial x^\alpha} + \frac{\partial u_3}{\partial x^3}.
\end{aligned}\right\} \tag{2.25}$$

Using the formulas

$$\overset{(n)}{P}_{ik} = \left(n+\frac{1}{2}\right) h^{n} \int_{-h}^{h} P_{ik} P_n\left(\frac{x^3}{h}\right) dx^3 \tag{2.26}$$

and representing the displacement vector in an infinite series

$$\mathfrak{u}(x^1, x^2, x^3) = \sum_{k=0}^{\infty} h^k \overset{(k)}{\mathfrak{u}}(x^1, x^2) P_k\left(\frac{x^3}{h}\right), \tag{2.27}$$

where

$$\overset{(k)}{\mathfrak{u}}(x^1, x^2) = \left(k + \frac{1}{2}\right) \frac{1}{h^{k+1}} \int_{-h}^{h} \mathfrak{u}(x^1, x^2, x^3)\, P_k\left(\frac{x^3}{h}\right) dx^3, \tag{2.28}$$

we obtain after some calculations

$$\left.\begin{array}{l} \overset{(n)}{P}_{ik} = \lambda \overset{(n)}{\Theta} g_{ik} + 2\mu \overset{(n)}{e}_{ik}, \quad \overset{(n)}{\Theta} = \overset{(n)}{e}{}^{i}_{i} \\ (g_{\alpha\beta} = a_{\alpha\beta}, \ g_{\alpha 3} = 0, \ g_{33} = 1), \end{array}\right\} \tag{2.29}$$

where

$$\left.\begin{array}{l} 2\overset{(n)}{e}_{\alpha\beta} = h^{2n+1}\left[\nabla_\alpha \overset{(n)}{u}_\beta + \nabla_\beta \overset{(n)}{u}_\alpha - 2b_{\alpha\beta}\overset{(n)}{u}_3 - \overset{(n)}{u}{}'_\alpha \dfrac{\partial \ln h}{\partial x^\beta} - \overset{(n)}{u}{}'_\beta \dfrac{\partial \ln h}{\partial x^\alpha}\right], \\ 2\overset{(n)}{e}_{\alpha 3} = h^{2n+1}\left[\nabla_\alpha \overset{(n)}{u}_3 + b_{\alpha\beta}\overset{(n)}{u}{}^\beta + \overset{(n)}{u}{}''_\alpha - \overset{(n)}{u}{}'_3 \dfrac{\partial \ln h}{\partial x^\alpha}\right], \\ \overset{(n)}{e}_{33} = h^{2n+1}\overset{(n)}{u}{}''_3, \end{array}\right\} \tag{2.30}$$

∇_α is the symbol for the covariant derivative with respect to the metric of the middle surface S, $\overset{(n)}{u}_1$, $\overset{(n)}{u}_2$, $\overset{(n)}{u}_3$, are the components of the vector $\mathfrak{U}^{(n)}$, i.e.,

$$\overset{(n)}{\mathfrak{u}}(x^1, x^2) = \overset{(n)}{u}_\alpha \mathfrak{r}^\alpha + \overset{(n)}{u}_3 \mathfrak{n}. \tag{2.31}$$

Further,

$$\left.\begin{array}{l} \overset{(n)}{u}{}'_i = (2n+1)\displaystyle\sum_{k=0}^{\infty} h^{2k}\overset{(n+2k)}{u_i}, \\ \overset{(n)}{u}{}''_i = (2n+1)\displaystyle\sum_{k=0}^{\infty} h^{2k}\overset{(n+2k+1)}{u_i}. \end{array}\right\} \tag{2.32}$$

Now, in accordance with the first assumption, we represent the components of the displacement vector as polynomials of degree N:

$$\mathfrak{u}(x^1, x^2, x^3) = \sum_{k=0}^{N} h^k \overset{(k)}{\mathfrak{u}}(x^1, x^2)\, P_k\left(\frac{x^3}{h}\right). \tag{2.33}$$

Hence

$$\overset{(k)}{u}_i = 0, \quad \overset{(k)}{e}_{ik} = 0 \quad \text{for} \quad k > N, \tag{2.34}$$

and the sums (2.32) will have only a finite number of terms.

We now represent the $\overset{(n)}{\mathfrak{U}}{}^i$ by the formulas

$$\overset{(n)}{\mathfrak{P}}{}^i = \overset{(n)}{P}{}^{i\alpha}\mathfrak{r}_\alpha + \overset{(n)}{P}{}^{i3}\mathfrak{n}. \tag{2.35}$$

Then, using the well-known Gauss and Weingarten formulas for the derivatives, equations (2.15) can be rewritten in the form

$$\left.\begin{aligned}
&\nabla_\alpha \overset{(n)}{P}{}^{\alpha\beta} - b_\alpha^\beta \overset{(n)}{P}{}^{\alpha 3} - (2n+1)\big(\overset{(n-1)}{P}{}^{\beta 3} + h^2 \overset{(n-3)}{P}{}^{\beta 3} - \ldots\big) + \\
&+ (2n+1)\, h \frac{\partial h}{\partial x^\alpha}\big(\overset{(n-2)}{P}{}^{\alpha\beta} - h^2 \overset{(n-4)}{P}{}^{\alpha\beta} + \ldots\big) - \overset{(n)}{F}{}^{\beta} = 0, \\
&\nabla_\alpha \overset{(n)}{P}{}^{\alpha 3} + b_{\alpha\beta} \overset{(n)}{P}{}^{\alpha\beta} - (2n+1)\big(\overset{(n-1)}{P}{}^{33} + h^2 \overset{(n-3)}{P}{}^{33} - \ldots\big) + \\
&+ (2n+1)\, h \frac{\partial h}{\partial x^\alpha}\big(\overset{(n-2)}{P}{}^{\alpha 3} - h^2 \overset{(n-4)}{P}{}^{\alpha 3} + \ldots\big) - \overset{(n)}{F}{}^{3} = 0 \\
&(\beta = 1, 2;\ n = 0, 1, \ldots, N; \quad \overset{(k)}{P}{}^{ij} = 0 \ \text{при}^{a}\ k < 0).
\end{aligned}\right\} \tag{2.36}$$

Key: a. for

Introducing the expressions

$$\overset{(n)}{p}{}^{ik} = \lambda \overset{(n)}{\Theta} g^{ik} + 2\mu \overset{(n)}{e}{}^{ik} \quad (g^{\alpha\beta} = a^{\alpha\beta},\ g^{\alpha 3} = 0,\ g^{33} = 1), \tag{2.37}$$

where

$$\left.\begin{aligned} 2\overset{(n)}{e}{}^{\alpha\beta} &= h^{2n+1}\left[\nabla^{\alpha}\overset{(n)}{u}{}^{\beta} + \nabla^{\beta}\overset{(n)}{u}{}^{\alpha} - 2b^{\alpha\beta}\overset{(n)}{u}{}_{3} - \overset{(n)}{u}{}'^{\alpha}\nabla^{\beta}\ln h - \overset{(n)}{u}{}'^{\beta}\nabla^{\alpha}\ln h\right], \\ 2\overset{(n)}{e}{}^{\alpha 3} &= h^{2n-1}\left[\nabla^{\alpha}\overset{(n)}{u}{}_{3} + b^{\alpha\beta}\overset{(n)}{u}{}_{\beta} + \overset{(n)}{u}{}''^{\alpha} - \overset{(n)}{u}{}'_{3}\nabla^{\alpha}\ln h\right], \\ \overset{(n)}{e}{}_{33} &= h^{2n-1}\overset{(n)}{u}{}''_{3}, \end{aligned}\right\} \tag{2.38}$$

we obtain

$$\left.\begin{aligned} &\mu\nabla_{\alpha}\left(h^{2n-1}\nabla^{\alpha}\overset{(n)}{u}{}^{\beta}\right) + \mu\nabla_{\alpha}\left(h^{2n+1}\nabla^{\beta}\overset{(n)}{u}{}^{\alpha}\right) + \lambda\nabla^{\beta}\left(h^{2n+1}\nabla_{\alpha}\overset{(n)}{u}{}^{\alpha}\right) + \\ &\qquad\qquad + \overset{(n)}{M}{}^{\beta} + \overset{(n)}{F}{}^{\beta} = 0 \quad (\beta = 1, 2), \\ &\mu\nabla_{\alpha}\left(h^{2n+1}\nabla^{\alpha}\overset{(n)}{u}{}_{3}\right) + \overset{(n)}{M}{}^{3} + \overset{(n)}{F}{}^{3} = 0 \quad (n = 0, 1, \ldots, N), \end{aligned}\right\} \tag{2.39}$$

where $\overset{(n)}{M}{}^{j}$ are homogeneous linear differential expressions containing the unknown functions $\overset{(n)}{u}{}_{i}$ and their first-order partial derivatives with respect to the variables x^1 and x^2. A second order system of partial differential equations of the elliptic type was obtained in this manner. This system contains 3N + 3 equations and the same number of unknown functions $\overset{(n)}{u}{}_{i}$ of two independent variables. Its order is 6N + 6.

Now in formulas (2.32) $\overset{(n)}{u}{}'_{i}$ and $\overset{(n)}{u}{}''_{i}$ are finite sums since $\overset{(n)}{u}{}_{i} = 0$, when $n > N$.

It should be noted that we retained only the first N + 1 equations from the infinite system of equations (2.15) and that we ignored the remaining equations.

2.4. When the stress forces $\mathfrak{P}_{(l)}$ are given on the lateral surface of the shell, the values of $\overset{(n)}{\mathfrak{P}}_{(l)}$ can be calculated along the contour L of the shell using the formulas

$$\overset{(n)}{\mathfrak{P}}_{(l)} = \left(n + \frac{1}{2}\right) h^n \int_{-h}^{h} \mathfrak{P}_{(l)} P_n\left(\frac{x^3}{h}\right) dx^3 \quad (n = 0, 1, \ldots). \tag{2.40}$$

Hence, boundary conditions of the form

$$\overset{(n)}{\mathfrak{P}}_{(l)} = \overset{(n)}{\mathfrak{P}}{}^{\alpha} l_\alpha = \mathfrak{f}_n \quad (n = 0, 1, \ldots, N). \tag{2.41}$$

can be added to the system of equations (2.36) and (2.37).

When the displacement field is given on the lateral surface, the contour values of $\overset{(n)}{\mathfrak{U}}$ can be calculated in exactly the same manner. Hence, boundary conditions of the form

$$\overset{(n)}{\mathfrak{U}} = \overset{(n)}{u}{}^{\alpha} \mathfrak{r}_\alpha + \overset{(n)}{u}{}_3 \mathfrak{n} = \mathfrak{f}_n \quad (n = 0, 1, \ldots, N). \tag{2.42}$$

can be added to the system of equations (2.39). By considering the projections on the coordinate axes, it is easily seen that the boundary conditions (2.41) and (2.42) contain 3N + 3 equalities, respectively.

The following identity (it is assumed that $\overset{(n)}{F}{}^{i} = 0$) plays an important role in the proof of the uniqueness theorem for the solution of these boundary value problems

$$\sum_{k=0}^{N} \frac{1}{2k+1} \int_{L} h^{-2k-1} \overset{(k)}{\mathfrak{U}} \overset{(k)}{\mathfrak{P}}_{(l)} ds = \sum_{k=0}^{N} \frac{1}{2k+1} \iint_{S} h^{-2k-1} \overset{(k)}{p}{}^{ij} \overset{(k)}{e}{}_{ij} dS. \tag{2.43}$$

Using this equality, uniqueness theorems are easily proved for the basic boundary value problems (2.41) and (2.42) and also for a number of other problems of the mixed type (I. N. Vekua, 1965). Existence theorems can be proved with the aid of integral equation methods (ibid.). The uniqueness and existence theorems show that the theory that was constructed is internally consistent. This property is a necessary attribute of any correctly constructed mathematical theory for a physical problem. However, the applied significance of these theorems must also not be underestimated. The quest for practical techniques for the solution of boundary value problems is facilitated on the basis of these theorems. In a number of cases, the method of integral equations can also be used as a practical method for constructing an approximate solution for the problem.

In practice the integration of the system of equations (2.36) and (2.37) is, of course, a difficult problem. The degree of difficulty obviously increases considerably as N increases.

However, in the case of thin shells we can restrict ourselves to approximations of order N = 0 and N = 1. Therefore, we will consider them in greater detail. For a plate and spherical shell of constant thickness, the systems of equations obtained above can be integrated in explicit form. Approximations of order N = 0 correspond to the case when the patterns of the stressed and deformed state are completely independent of the coordinate x^3, i.e., are the same along surfaces which are parallel to the middle surface. This case of the stressed equilibrium of a shell is in fact torqueless. But unlike in classical torqueless theory, in the N = 0 order approximation, we obtain a concrete system of differential equations which is compatible with all physical boundary conditions (in the given case with three conditions). It must be emphasized that here we have an elliptic system of equations which is equivalent to one sixth order elliptic equation, and that in classical torqueless theory the problem reduces to a second order elliptic equation.

For N = 0 the system of equations (2.39) takes on the form

$$\left.\begin{aligned} &\mu \nabla_\alpha (h\nabla^\alpha u^\beta) + \mu \nabla_\alpha (h\nabla^\beta u^\alpha) + \lambda \nabla^\beta (h\nabla_\alpha u^\alpha) - \mu h b_\alpha^\beta b_\gamma^\alpha u^\gamma - \\ &\quad - 2\lambda\nabla^\beta (hHu) - 2\mu\nabla^\alpha (hb_\alpha^\beta u) - \mu h b_\alpha^\beta \nabla^\alpha u + F^\beta = 0 \quad (\beta = 1, 2), \\ &\mu \nabla_\alpha (h\nabla^\alpha u) - 4h\,[(\lambda + 2\mu)\, H^2 - \mu K]\, u + \\ &\quad + 2\lambda H\nabla_\alpha u^\alpha + 2\mu b^{\alpha\beta}\nabla_\alpha u_\beta + \mu\nabla_\alpha (hb^{\alpha\beta} u_\beta) + F = 0. \end{aligned}\right| \qquad (2.44)$$

In the case of a plate of constant thickness (h = const) the system of equations (2.44) simplifies considerably and takes on the form (in a cartesian coordinate system)

$$\left.\begin{aligned} \mu\Delta u_1+(\lambda+\mu)\frac{\partial}{\partial x}\left(\frac{\partial u_1}{\partial x}+\frac{\partial u_2}{\partial y}\right)+\frac{1}{h}F_1=0,\\ \mu\Delta u_2+(\lambda+\mu)\frac{\partial}{\partial y}\left(\frac{\partial u_1}{\partial x}+\frac{\partial u_2}{\partial y}\right)+\frac{1}{h}F_2=0,\\ \mu\Delta u+\frac{1}{h}F=0. \end{aligned}\right| \qquad (2.45)$$

The general solution of the system of equations is described by the formulas

$$\left.\begin{aligned} 2\mu(u_1+iu_2)=\varkappa\varphi(z)-z\overline{\varphi'(z)}-\overline{\psi(z)}+\\ +\frac{1}{8\pi(1-\sigma)h}\iint_D\left[\frac{\bar{\zeta}-\bar{z}}{\zeta-z}(F_1-iF_2)-2(3-4\sigma)(F_1+iF_2)\ln|\zeta-z|\right]d\xi\,d\eta,\\ 2\mu u=\chi(z)+\overline{\chi(z)}-\frac{1}{\pi h}\iint_D F\ln|\zeta-z|\,d\xi\,d\eta, \end{aligned}\right| \qquad (2.46)$$

where φ, ψ and χ are arbitrary analytic functions of the variable $z = x + iy$

$$\varkappa=\frac{\lambda+3\mu}{\lambda+\mu}=3-4\sigma. \qquad (2.47)$$

The integral terms in formulas (2.46) describe the particular solution of the nonhomogeneous system (2.45), and the terms outside the integrals, the general solution of the corresponding homogeneous system of equations. The presence in formulas (2.46) of three arbitrary analytic functions φ, ψ and χ shows that the three boundary conditions can be satisfied. In addition, formulas (2.46) make it possible to construct an infinite set of complete systems of particular solutions of system (2.45), with the aid of which various boundary value problems can be solved in regions with a special form (circle, circular ring, etc.). These particular systems of solutions can also be used to approximate the solutions in regions of any form.

It can be seen from (2.46) that the boundary value problems for the plate reduce to the solution of a plane problem and the Poisson equation. The well-known methods and results of N. I. Muskhelishvili can be used for their solution.

When the homogeneous system of equations ($F_1 = F_2 = 0$) is considered, the averaged components of the stress tensor $p_{\alpha\beta}$ can be described by the formulas

$$p_{11} = \frac{\partial^2 \varphi}{\partial y^2}, \quad p_{12} = p_{21} = -\frac{\partial^2 \varphi}{\partial x \, \partial y}, \quad p_{22} = \frac{\partial^2 \varphi}{\partial x^2}, \tag{2.48}$$

where the function φ satisfies the equation (I. N. Vekua, 1955, 1965)

$$h^{-1}\Delta\Delta\varphi + 2\frac{\partial h^{-1}}{\partial x}\frac{\partial \Delta\varphi}{\partial x} + 2\frac{\partial h^{-1}}{\partial y}\frac{\partial \Delta\varphi}{\partial y} - \frac{\sigma}{1-\sigma}(\Delta h^{-1})\Delta\varphi +$$
$$+ \frac{1}{1-\sigma}\left(\frac{\partial^2 h^{-1}}{\partial x^2}\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 h^{-1}}{\partial y^2}\frac{\partial^2 \varphi}{\partial y^2} + 2\frac{\partial^2 h^{-1}}{\partial x \, \partial y}\frac{\partial^2 \varphi}{\partial x \, \partial y}\right) = 0. \tag{2.49}$$

For h = const (plate) we obtain the biharmonic equation

$$\Delta\Delta\varphi = 0, \tag{2.50}$$

whose general solution, as is well known, is expressed in explicit form.

The study of a plate of variable thickness of the form

$$h = h_0 e^{ax+by} \quad (h_0, \ a, \ b = \text{const}). \tag{2.51}$$

(I. N. Vekua, 1965) is also of interest. Here equation (2.49) takes on the form

$$\Delta\Delta\varphi - 2a\frac{\partial \Delta\varphi}{\partial x} - 2b\frac{\partial \Delta\varphi}{\partial y} - \frac{\sigma}{1-\sigma}(a^2 + b^2)\Delta\varphi +$$
$$+ \frac{1}{1-\sigma}\left(a^2\frac{\partial^2 \varphi}{\partial x^2} + b^2\frac{\partial^2 \varphi}{\partial y^2} + 2ab\frac{\partial^2 \varphi}{\partial x \, \partial y}\right) = 0. \tag{2.52}$$

The general solution for this equation was constructed by A. R. Khvoles.

For a spherical shell of radius R the system of equations in the displacement components takes on the form

$$\left.\begin{aligned} &\mu\nabla_\alpha(h\nabla^\alpha u^\beta)+\mu\nabla_\alpha(h\nabla^\beta u^\alpha)+\lambda\nabla^\beta(h\nabla_\alpha u^\alpha)-\frac{\mu h}{R^2}u^\beta+\\ &\quad+\frac{2(\lambda+\mu)}{R}\nabla^\beta(hu)+\frac{\mu h}{R}\nabla^\beta u+F^\beta=0 \quad (\beta=1,2),\\ &\mu\nabla_\alpha(h\nabla^\alpha u)-\frac{4(\lambda+\mu)h}{R^2}u-\frac{2(\lambda+\mu)}{R}\nabla_\alpha u^\alpha-\frac{\mu}{R}\nabla_\alpha(hu^\alpha)+F=0. \end{aligned}\right\} \tag{2.53}$$

We will consider the case of a homogeneous system of equations ($F^1 = F^2 = F = 0$) and constant thickness (h = const). Then the general solution of the system has the form

$$u=w_1+w_2, \quad u_1+iu_2=\frac{\partial U}{\partial\bar z}, \tag{2.54}$$

where

$$U=2R(w_1-w_2+iw_3). \tag{2.55}$$

Here w_1, w_2, and w_3 are arbitrary solutions of the equations

$$(\nabla^2+k_i^2)w=0 \quad (i=1,2,3), \tag{2.56}$$

where

$$k_1^2=\frac{2}{R^2}, \quad k_2^2=-\frac{1}{(1-\sigma)R^2}, \quad k_3^2=\frac{1}{R^2}, \tag{2.57}$$

and ∇^2 is the Laplace operator on a sphere of radius R. In an isothermic coordinate system it has the form

$$\nabla^2=\frac{(1+x^2-y^2)^2}{4R^2}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\equiv\frac{(1+z\bar z)}{R^2}\frac{\partial^2}{\partial z\,\partial\bar z}, \tag{2.58}$$

$$z=\operatorname{tg}\frac{\theta}{2}e^{i\varphi}. \tag{2.59}$$

The general solution of an equation of the form

$$(\nabla^2 + k^2)\, w = 0 \quad (k^2 = \mathrm{const}) \tag{2.60}$$

is expressed by the formula (I. N. Vekua, 1948)

$$w = a_0 P_n(\cos\theta) + \int_0^1 [\Phi(zt) + \Phi_*(\bar{z}t)]\, P_n[t + (1-t)\cos\theta]\, dt, \tag{2.61}$$

where $n(n + 1) = k^2$, a_0 is an arbitrary constant and Φ and Φ_* are arbitrary analytic functions of z. When w is a real function, a_0 is a real constant and $\Phi_* = \bar{\Phi}$. In the case under consideration w_1, w_2 and w_3 are arbitrary real functions. Therefore, the general solution of the system of equations (2.53) for h = const is expressed in terms of three arbitrary analytic functions. The three boundary value problems are satisfied by an appropriate selection of these functions.

A particular solution for h = const can be constructed in explicit form also for the nonhomogeneous system of equations (2.53) (I. N. Vekua, 1965).

2.5. We will now survey the results pertaining to the case of the approximation of order N = 1 (I. N. Vekua, 1965).

The components of the stress tensor and displacement vector in this case can be written in the form

$$\left.\begin{aligned} P^{\alpha\beta} &= \frac{1}{2h} T^{\alpha\beta} + \frac{3}{2}\frac{x^3}{h^3} S^{\alpha\beta}, \\ P^{\alpha 3} &= \frac{1}{2h} T^{\alpha} + \frac{3}{2}\frac{x^3}{h^3} S^{\alpha}, \\ P^{33} &= T, \end{aligned}\right\} \tag{2.62}$$

$$U^{\alpha} = \frac{1}{2} u^{\alpha} + \frac{3}{2} x^3 v^{\alpha}, \quad U^3 = \frac{1}{2} u + \frac{3}{2} x^3 v, \tag{2.63}$$

where

$$
\left.\begin{aligned}
T^{\alpha\beta} &= \lambda h\varepsilon a^{\alpha\beta} + 2\mu h\varepsilon^{\alpha\beta},\\
T^{\alpha} &= 2\mu h\varepsilon^{\alpha}, \quad T = \lambda h\varepsilon + 6\mu h v,\\
S^{\alpha\beta} &= \lambda h^{3}\eta a^{\alpha\beta} + 2\mu h^{3}\eta^{\alpha\beta},\\
S^{\alpha} &= 2\mu h^{3}\eta^{\alpha},
\end{aligned}\right\} \tag{2.64}
$$

$$
\left.\begin{aligned}
\varepsilon^{\alpha\beta} &= \frac{1}{2}(\nabla^{\alpha}u^{\beta} + \nabla^{\beta}u^{\alpha}) - b^{\alpha\beta}u,\\
\varepsilon^{\alpha} &= \frac{1}{2}(\nabla^{\alpha}u - b^{\alpha}_{\beta}u^{\beta}) + \frac{3}{2}v^{\alpha},\\
\eta^{\alpha\beta} &= \frac{1}{2}(\nabla^{\alpha}v^{\beta} + \nabla^{\beta}v^{\alpha}) - b^{\alpha\beta}v,\\
\eta^{\alpha} &= \frac{1}{2}(\nabla^{\alpha}v + b^{\alpha}_{\beta}v^{\beta}),\\
\varepsilon &= \varepsilon^{\alpha}_{\alpha} + 3v = \nabla_{\alpha}u^{\alpha} - 2Hu + 3v,\\
\eta &= \eta^{\alpha}_{\alpha} = \nabla_{\alpha}v^{\alpha} - 2Hv.
\end{aligned}\right\} \tag{2.65}
$$

The equilibrium equations have the form

$$
\left.\begin{aligned}
&\nabla_{\alpha}T^{\alpha\beta} - b^{\beta}_{\alpha}T^{\alpha} + X^{\beta} = 0 \quad (\beta = 1, 2),\\
&\nabla_{\alpha}T^{\alpha} + b_{\alpha\beta}T^{\alpha\beta} + X = 0,\\
&\nabla_{\alpha}S^{\alpha\beta} - b^{\beta}_{\alpha}S^{\alpha} + Y^{\beta} = 0 \quad (\beta = 1, 2),\\
&\nabla_{\alpha}S^{\alpha} + b_{\alpha\beta}S^{\alpha\beta} + Y = 0,
\end{aligned}\right\} \tag{2.66}
$$

where X^{β}, X, Y^{β}, Y are components of the vectors

$$
\left.\begin{aligned}
\mathfrak{X} &= \int_{-h}^{h}\mathfrak{F}\,dx^{3} + \mathfrak{P}_{(n+)} + \mathfrak{P}_{(n-)} = X^{\beta}\mathfrak{r}_{\beta} + X\mathfrak{n},\\
\mathfrak{Y} &= \int_{-h}^{h}\mathfrak{F}x^{3}\,dx^{3} + h\mathfrak{P}_{(n+)} - h\mathfrak{P}_{(n-)} = Y^{\beta}\mathfrak{r}_{\beta} + Y\mathfrak{n}.
\end{aligned}\right\} \tag{2.67}
$$

We will now consider the case of the homogeneous system of equations ($X^{\beta} = X = Y^{\beta} = Y = 0$). Then the system of equations (2.66) can be integrated in explicit form for a plate and a spherical shell of constant thickness (I. N. Vekua, 1965). In both cases the displacement vector can be described by the formula

$$\mathfrak{U} = \operatorname{grad} P + \mathfrak{n} \times \operatorname{grad} \widetilde{P} + w\mathfrak{n}, \tag{2.68}$$

where grad is taken with respect to the middle surface and P, $\widetilde{P}$ and w are scalar functions expressed in terms of the formulas given below.

1. In the case of plate of constant thickness

$$P = \frac{1}{4}\left[-\sigma h^2\chi - f_2 - \overline{f}_2 + \frac{1-\sigma}{1+\sigma}(\overline{z}f_1 + z\overline{f}_1) + 3x^3(\overline{z}\Phi_2 + z\overline{\Phi}_2 + \Phi_3 + \overline{\Phi}_3)\right], \tag{2.69}$$

$$\widetilde{P} = \frac{1}{4}\left[\frac{2i}{1+\sigma}(z\overline{f}_1 - \overline{z}f_1) + 3x^3\omega\right], \tag{2.70}$$

$$w = \frac{1}{2}(u + 3x^3 v), \tag{2.71}$$

where

$$u = \Phi_1 + \overline{\Phi}_1 - \frac{3}{2}(z\overline{\Phi}_2 + \overline{z}\Phi_2), \tag{2.72}$$

$$v = \chi - \frac{2\sigma}{3(1+\sigma)}(f_1' + \overline{f}_1'), \tag{2.73}$$

f_1, f_2, Φ_1 and Φ_2 are arbitrary analytic functions of z, and χ and φ are arbitrary solutions of the equations

$$\Delta\chi - \frac{6}{(1-\sigma)h^2}\chi = 0, \qquad \Delta\omega - \frac{3}{h^2}\omega = 0. \tag{2.74}$$

Finally

$$\Phi_3 = -\frac{2}{3}\Phi_1 + \frac{8(1-\sigma)h^2}{9}\Phi_2''. \tag{2.75}$$

The general solution of an equation of the form

$$\Delta u - \lambda^2 u = 0, \tag{2.76}$$

where λ^2 is a real constant is described by the formula (I. N. Vekua, 1948)

$$u = \mathrm{Re}\left[f(z) - \int_0^z f(t) \frac{\partial}{\partial t} I_0 \left(\lambda \sqrt{\bar{z}(z-t)}\right) dt\right], \tag{2.77}$$

where f is an arbitrary analytic function of z = x + iy.

Thus the general integral of the system of equilibrium equations for the plate in the case of the N = 1 order approximation contains six arbitrary analytic functions. The six boundary conditions can be satisfied by an appropriate selection of these functions.

2. In the case of a spherical shell of constant thickness, the functions P, $\tilde{P}$ and w have the form

$$P = \frac{1}{4} R^2 (C_1 w_1 + C_2 w_2 + C_3 w_3 + \bar{C}_3 \bar{w}_3) + \frac{3}{4} x^3 R (B_2 w_2 + B_3 w_3 + \bar{B}_3 \bar{w}_3), \tag{2.78}$$

$$\tilde{P} = \frac{1}{4} R^2 (w_4 + w_5) + \frac{3}{4} x^3 R (B_4 w_4 + B_5 w_5), \tag{2.79}$$

$$w = \frac{1}{2} R (A_1 w_1 + A_2 w_2 + A_3 w_3 + \bar{A}_3 \bar{w}_3) + \frac{3}{4} x^3 (w_2 + w_3 + \bar{w}_3), \tag{2.80}$$

where A_i, B_i, C_i are defined constants depending on the elasticity Poisson ratio σ and the ratio h/R. With regard to w_i, these are arbitrary solutions of equations of the form

$$(\nabla^2 + k_i^2) w_i = 0 \quad (i = 1, \ldots, 5), \tag{2.81}$$

where ∇^2 is the Laplace operator on the unit sphere and k_i^2 are constants which have been determined. With the exception of k_3^2, all remaining k_i^2 are real constants. Therefore w_i $(i \neq 3)$ are real functions and w_3 is a complex single-valued function. Thus, formula (2.68) contains, in final form, six arbitrary

analytic functions. Hence, it can satisfy the six independent boundary conditions. For example, in the case of boundary value problem (2.41) the five boundary conditions from classical moment theory are used (the normal and tangential forces are given along the boundary, the shearing force, the torque and bending moments) and an additional new boundary condition which is not considered in classical theory, namely a condition of the form

$$S_{(l)} = S^{\alpha} l_{\alpha} = f_{6}. \tag{2.82}$$

We will clarify the physical meaning of the last condition. According to formulas (2.62), the stress forces

$$\sigma_{(l)} = \frac{1}{2h} T^{\alpha\beta} \mathrm{r}_{\alpha} l_{\beta} + \frac{3}{2} \frac{x^3}{h^3} S^{\alpha\beta} \mathrm{r}_{\alpha} l_{\beta} + \left(\frac{1}{2h} T^{\alpha} l_{\alpha} + \frac{3}{2} \frac{x^3}{h^3} S^{\alpha} l_{\alpha} \right) \mathrm{n}. \tag{2.83}$$

are acting on the transverse area Σ_1 with the normal 1. From the above it can be easily seen that the forces

$$\left.\begin{aligned} &\frac{1}{2} T^{\alpha\beta} \mathrm{r}_{\beta} l_{\alpha} + \frac{3}{4} \frac{1}{h} S^{\alpha\beta} \mathrm{r}_{\alpha} l_{\beta} + \left(\frac{1}{2} T^{\alpha} l_{\alpha} + \frac{3}{4} \frac{1}{h} S^{\alpha} l_{\alpha} \right) \mathrm{n}, \\ &\frac{1}{2} T^{\alpha\beta} \mathrm{r}_{\alpha} l_{\beta} - \frac{3}{4} \frac{1}{h} S^{\alpha\beta} \mathrm{r}_{\alpha} l_{\beta} + \left(\frac{1}{2} T^{\alpha} l_{\alpha} - \frac{3}{4} \frac{1}{h} S^{\alpha} l_{\alpha} \right) \mathrm{n}. \end{aligned}\right\} \tag{2.84}$$

are acting on each half of the area Σ_1 which is symmetric with respect to the middle surface. Thus, the total force

$$T^{\alpha\beta} \mathrm{r}_{\alpha} l_{\beta} + T^{\alpha} l_{\alpha} \mathrm{n} \tag{2.85}$$

is acting on the area Σ_1, and the force couple

$$\left(\frac{3}{4h} S^{\alpha\beta} l_{\alpha} \mathrm{r}_{\beta} + \frac{3}{4h} S^{\alpha} l_{\alpha} \mathrm{n},\ -\frac{3}{4h} S^{\alpha\beta} \mathrm{r}_{\alpha} l_{\beta} - \frac{3}{4h} S^{\alpha} l_{\alpha} \mathrm{n} \right). \tag{2.86}$$

The last couple is a sum of three couples

$$\left.\begin{array}{c}\frac{3}{4h}(S^{\alpha\beta}l_\alpha l_\beta, \ -S^{\alpha\beta}l_\alpha l_\beta)\,\mathfrak{l}, \\ \frac{3}{4h}(S^{\alpha\beta}l_\alpha s_\beta, \ -S^{\alpha}l_\alpha s_\beta)\,\mathfrak{l}, \\ \frac{3}{4h}(S^{\alpha}l_\alpha, \ -S^{\alpha}l_\alpha)\,\mathfrak{n}.\end{array}\right\} \qquad (2.87)$$

The first two couples (with an accuracy up to a constant factor) coincide with the bending and torsional couples of classical moment theory, and the last is a couple of forces oriented in opposite directions and parallel to the normal of the middle surface which we will agree to call the transverse couple. Its moment is zero. If we adopt the hypotheses of the rigidity of the transverse areas, clearly the mechanical effect of the action of such a couple reduces to zero. In fact, the transverse areas undergo a deformation. Therefore, there is no a priori justification for ignoring the transverse couple.

The N = 1 order approximation differs from classical moment theory only by the fact that an additional transverse couple of forces is introduced into the discussion, which is usually ignored in classical formulations and considered as a higher order infinitely small magnitude. It is true that in magnitude the force $S^\alpha l_\alpha$ may be negligible. However, the consideration of the transverse couple of forces is important theoretically. By using it, it is possible to construct a noncontradictory variant of the theory of shells which is compatible with the corresponding boundary conditions. This theory is a modification of classical moment theory, but its advantage is that it is free of the internal contradictions which occur in classical formulations. It should also be noted that the more complex mathematical apparatus does not lead to new essential difficulties. This can be seen on the examples of a plate and a spherical shell.

§3. Torqueless Theory of Shells

A characteristic feature of the torqueless (or membrane) theory of shells is that it leads to a statically determinate problem. In the final analysis this problem reduces to a first order system of partial differential equations in two unknown variables. The type of system of equations is determined by the sign of the Gaussian curvature K of the middle surface of the shell. When K > 0, we have a system of the

elliptic type and when $K < 0$ or $K = 0$, a system of the hyperbolic or parabolic type, respectively.

3.1. The membrane equilibrium state is a special case of the general equilibrium state. The condition that it occur is described by the equality

$$\iint_S \mathfrak{X}\mathfrak{U}\,dS + \int_L \mathfrak{T}\mathfrak{U}\,dS = 0, \tag{3.1}$$

where $\mathfrak{X}$ is the reduced loading surface of the shell,

$$\mathfrak{X} = \int_{-h}^{h} \mathfrak{F}\,dx^3 + \mathfrak{P}_{(n+)} + \mathfrak{P}_{(n-)}, \tag{3.2}$$

$\mathfrak{T}$ is the load on the contour and $\mathfrak{U}$ is the displacement vector when the bending of the middle surface is infinitesimal, i.e., $\mathfrak{U}$ satisfies the equation

$$d\bar{\mathfrak{U}}\,d\mathfrak{r} = 0. \tag{3.3}$$

Equation (3.1) must be satisfied for any continuously differentiable vector field $\mathfrak{U}$, satisfying equation (3.3). Relation (3.1) relates organically the torqueless theory of shells to the theory of the infinitesimal bending of surfaces. This relation is very useful. The joint study of the problems makes it possible to study them more fully and in greater depth. The mechanical meaning of equation (3.1) is easily understood. It says that a necessary and sufficient condition that the membrane stressed equilibrium state be realized is that the work of the external load on the displacement corresponding to infinitesimal bends of the middle surface of the shell be zero.

Clearly, the wider the class of vector fields $\mathfrak{U}$, satisfying equation (3.3), the more constraining the condition (3.1).

In the case of a closed convex shell, this equality becomes the well-known condition for the static equilibrium of an absolutely rigid body. It is known (I. N. Vekua, 1965) that such a shell is rigid, i.e., the displacement field has the form

$$\mathfrak{U} = \mathfrak{r} \times \Omega + \mathfrak{C} \quad (\Omega, \mathfrak{C} - \text{const}). \tag{3.4}$$

In other cases equation (3.3) has an infinite set of linearly independent solutions. Therefore, condition (3.1) will only be satisfied for a particular class of loads $(\mathfrak{X}, \mathfrak{Z})$ on the shell. In spite of this, the membrane theory of shells is widely used in engineering calculations. The point is that the shells in engineering structures are sufficiently rigid in the majority of cases. Therefore, if two external loads $(\mathfrak{X}, \mathfrak{Z})$ and $(\bar{\mathfrak{X}}, \bar{\mathfrak{Z}})$ are close, the statemant can be made that the corresponding stress fields will also be close. (Below we will define exactly the concept of the closeness of two loads.) If one load $(\mathfrak{X}, \mathfrak{Z})$ satisfies the torquelessness condition (3.1), it can be assumed that the shell is practically torqueless and close to the load $(\bar{\mathfrak{X}}, \bar{\mathfrak{Z}})$. This fact makes it possible to apply torqueless theory to a very large class of engineering problems.

If, due to practically insignificant variations in the internal load, it is possible to obtain another load which satisfies the torqueless condition, we will say that the given load admits membrane regulation. At first sight, the general problem of membrane regulation has a relatively undefined character. However, as we will see below, it can be formulated completely mathematically.

Below, we will focus our attention exclusively on convex shells ($K > 0$). Then the problem reduces as we already mentioned above to an integration of a first order system of partial differential equations of the elliptic type.

The results presented below were obtained by the author in connection with the development of a general theory of complex-valued functions satisfying the so-called generalized system of Cauchy-Rieman equations (I. N. Vekua, 1952, 1959)

$$\left.\begin{aligned} \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + au + bv = 0, \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + cu + dv = 0. \end{aligned}\right\} \tag{3.5}$$

The theory of functions of the form w = u + iv has a great deal in common with the classical theory of analytic functions of a complex variable z = x + iy. Therefore, it is called the theory of generalized analytic functions.

The membrane equilibrium equations have the form

$$\left.\begin{aligned} &\nabla_\alpha T^{\alpha\beta} + X^\beta = 0 \quad (\beta = 1, 2), \\ &b_{\alpha\beta} T^{\alpha\beta} + X = 0. \end{aligned}\right\} \tag{3.6}$$

It is known that on a surface with a positive Gaussian curvature a conjugate isometric net of coordinate lines can be introduced relative to which the coefficients of the second fundamental quadratic form have the form

$$b_{11} = b_{22}, \quad b_{12} = 0. \tag{3.7}$$

Then the orientation of the normal n to the middle surface can be selected in such a way that the equalities $b_{11} = b_{22} = \sqrt{aK}$, hold

where a is the discriminant of the corresponding metric form of the middle surface. The third equation in (3.6) gives

$$T^{22} = -T^{11} - \frac{X}{\sqrt{aK}}. \tag{3.8}$$

Now, introducing into the discussion the complex stress function

$$w' = aK^{1/4}(T^{11} - iT^{12}) + \frac{1}{2}\sqrt{\frac{a}{\sqrt{K}}}\,X, \tag{3.9}$$

the first two equations in (3.6) can be written as a single complex equation

$$\frac{\partial w'}{\partial \bar{z}} - \overline{B}\overline{w'} = F', \tag{3.10}$$

where

$$B = \frac{1}{4}\left(\Gamma^1_{22} - \Gamma^1_{11} + 2\Gamma^2_{11}\right) + \frac{i}{4}\left(\Gamma^2_{22} - \Gamma^2_{11} - 2\Gamma^1_{12}\right), \tag{3.11}$$

$$F' = \frac{1}{2}\sqrt{a}\,K^{3/4}\frac{\partial}{\partial z}\left(\frac{X}{K}\right) - \frac{1}{2}aK^{1/4}\left(X^1 - iX^2\right), \tag{3.12}$$

and $\Gamma^{\lambda}_{\alpha\beta}$ are the Christoffel symbols of the second kind,

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right). \tag{3.13}$$

Thus, having found the solution of equation (3.10), it is possible to determine from formulas (3.9) and (3.8) the corresponding field of stresses.

Thus, the problem of determining the torqueless equilibrium stressed state field of a convex shell reduces to the integration of a nonhomogeneous generalized Cauchy-Rieman equation (3.10). Hence, the problem is a topic in the theory of generalized analytic functions to which an extensive literature is devoted at the present time.

It should also be pointed out that equation (3.3) can be writeen in complex form. It is equivalent to the homogeneous complex equation

$$\frac{\partial w}{\partial \bar{z}} + B\bar{w} = 0, \tag{3.14}$$

which is the conjugate of (3.10). Here w is a complex valued function given by the formula

$$w = \frac{u_1 + iu_2}{\sqrt{a}\sqrt{K}} \qquad (u_\alpha = \mathfrak{U}\mathfrak{r}_\alpha). \tag{3.15}$$

The vector field $\mathcal{U}$, satisfying equation (3.3), is described by the formula

$$\mathfrak{U} = \operatorname{Re}\left\{-2K^{-1/4}w(z)\frac{\partial \mathfrak{n}}{\partial z} + \frac{1}{K\sqrt{a}}\frac{\partial}{\partial z}\left[\sqrt{a}\,K^{3/4}w(z)\right]\mathfrak{n}\right\}, \tag{3.16}$$

where $\mathfrak{n}$ is the unit normal to the middle surface of the shell.

For two arbitrary solutions of equations (3.10) and (3.14), the identity

$$\operatorname{Re}\left(\iint_D F'w\,dx\,dy - \frac{1}{2i}\int_\Gamma w'w\,dz\right) = 0, \tag{3.17}$$

is satisfied where D is the region on the plane z onto which the middle surface S is mapped and Γ is the boundary of the region D. It can be proved that equalities (3.1) and (3.17) are equivalent.

It follows from (3.17) that if w and w' satisfy the homogeneous equations

$$\frac{\partial w}{\partial \bar z} + B\bar w = 0, \qquad \frac{\partial w'}{\partial \bar z} - \bar B\bar w' = 0, \tag{3.18}$$

the identity

$$\operatorname{Re}\left(\frac{1}{2i}\int_\Gamma ww'\,dz\right) = 0. \tag{3.19}$$

is satisfied. The force acting on the element of area Σ_1 whose width is ds with the normal l, is described by the formula

$$\mathfrak{T}_{(l)}\,ds = \left\{\frac{2K^{3/4}}{k_s}\operatorname{Re}\left[w'\left(\frac{dz}{ds}\right)^2\right] + X\right\}\mathfrak{P}_{(l)}\,ds + \frac{K^{1/4}}{k_s}\operatorname{Im}\left[w'\left(\frac{dz}{ds}\right)^2\right]d\mathfrak{r}, \tag{3.20}$$

where

$$\mathfrak{P}_{(l)} = \frac{1}{2K}\frac{d\mathfrak{n}}{ds}\times\mathfrak{n} = -\frac{1}{2K}(k_s\mathfrak{l} + \tau_s\mathfrak{t}). \tag{3.21}$$

Here k_s and τ_s are the normal curvature and the geodesic rotation of the middle surface in the direction $\mathfrak{f}$, perpendicular to $\mathfrak{l}$, and the normal is oriented in the concave direction of the surface $\mathfrak{n} = \mathfrak{l} \times \mathfrak{j}$.

From (3.20) we obtain easily

$$w' \left(\frac{dz}{ds}\right)^2 = -K^{1/4}\left[\left(\sqrt{K} + i\tau_s\right) T_{(ll)} - ik_s T_{(ls)} + \frac{k_s}{2K} X\right], \qquad (3.22)$$

where $T_{(ll)}$ and $T_{(ls)}$ are the normal and tangential stresses acting on Σ_1. A number of important corrollaries can be obtained from this equality.

It can be seen immediately from equality (3.22) that $w'dz^2$ is an invariant, i.e., a quantity which is independent of the coordinate system selected on the middle surface.

Equality (3.22) also shows that its right member cannot take on any given (complex) values along the boundary of the shell. The point is that the function w' satisfying the generalized nonhomogeneous Cauchy-Rieman equation (3.10), has the property that generally it satisfies on the contour either the given real or imaginary part. Therefore, the values w' cannot be given arbitrarily on the contour of the region (more precisely the value must not be given arbitrarily even on any arbitrarily small arc of the boundary). From this it follows that for a given load $(\mathfrak{X}, \mathfrak{Z})$, a necessary and sufficient condition for the torqueless stressed equilibrium of a shell is that the normal and tangential forces $T_{(ll)}$ and $T_{(ls)}$ and also the normalized surface load X satisfy the equality (3.22), i.e., the expression

$$-K^{-1/4}\left[\left(\sqrt{K} + i\tau_s\right) T_{(ll)} - ik_s T_{(ls)} + \frac{k_s}{2K} X\right]\left(\frac{ds}{dz}\right)^2 \qquad (3.23)$$

must take on the boundary values of a solution of equation (3.10).

This requirement can always be satisfied by selecting appropriately the boundary values of $T_{(ll)}$, $T_{(ls)}$ and X.

If a load $(\bar{\mathfrak{X}}, \bar{\mathfrak{T}})$, is obtained which is close to the given external load $(\mathfrak{X}, \mathfrak{T})$, this means that the given load admits boundary membrane regulation. A particularly important case is the case when membrane regulation is achieved by selecting appropriately the normalized normal load X along the edge of the shell. This problem has been studied thoroughly in the case when the normalized surface load has a potential, i.e.,

$$\mathfrak{X}_0 = K \operatorname{grad}_* V + KV, \qquad (3.24)$$

where V is a scalar, and $\operatorname{grad}_* V$ denotes the gradient of the scalar V on the spherical image of the middle surface. For this load, it is characteristic that the right member of equation (3.10) vanishes (F' = 0), i.e.,

$$X^1 = \frac{K}{2\sqrt{aK}} \frac{\partial}{\partial x}\left(\frac{X}{K}\right), \quad X^2 = \frac{K}{2\sqrt{aK}} \frac{\partial}{\partial y}\left(\frac{X}{K}\right). \qquad (3.25)$$

Thus, the problem reduces to the following generalized Rieman-Hilbert problem: to find a solution w' of equation (3.10) satisfying on L the boundary condition

$$\operatorname{Im}\left[w'\left(\frac{dz}{ds}\right)^2\right] = -K^{-1/4}[\tau_s T_{(l)} - k_s T_{(s)}]. \qquad (3.26)$$

When this problem has a solution, the boundary value of the unknown potential V is determined from the formula

$$V = -\frac{X}{K} - \frac{2}{k_s}\operatorname{Re}\left[w'\left(\frac{dz}{ds}\right)^2\right] - \frac{2K^{1/4}}{k_s} T_{(l)}. \qquad (3.27)$$

When the boundary value problem (3.26) has a solution, the problem of the membrane regulation of the load $(\mathfrak{X}, \mathfrak{T})$ in the manner indicated has an infinite set of solutions. Formula (3.27) defines the values of the unknown potential only on the boundary of the shell. In the interior of the region, the potential V can be continued arbitrarily, and it

is sufficient to ensure that the continuation of the function has piecewise continuous first order derivatives. In addition, clearly, it is important that the load whose form has changed, differ little from the original load. This can be achieved by taking into account that the function V can be continued in such a way that it is identically zero in the interior of an arbitrarily narrow boundary strip, which can be achieved when max $|V|$ in the region is less than or equal to $|V|$ on the boundary. It can be seen from formula (3.20) that the additional stress field corresponding to the additional potential load $\mathfrak{X}_0$, will be described by the formula

$$\mathfrak{T}^0_{(l)} = KV\mathfrak{N}_{(l)}. \tag{3.28}$$

Hence, in the exterior of the boundary strip this field is identically zero.

Thus if membrane regulation is possible through the loading of the shell by forces of type (3.24), this can be achieved so that the real stressed state pattern is only distorted inside a narror boundary strip. (Of course, we are assuming that the shell has a sufficiently high rigidity.) This effect is usually called the boundary effect.

What must still be clarified is the question whether membrane regulation is always possible by means of adding a potential load. This depends on whether the Rieman-Hilbert problem (3.26) has a solution. The last question has been studied thoroughly by now. Certain results pertaining to this question are available (I. N. Vekua, 1959).

A study of problem (3.26) shows that a convex shell with two and a larger number of holes ($m \geq 2$) always admits membrane regulation by means of additional loading by forces of the type (3.24). If the shell is bounded by one ($m = 0$) or two ($m = 1$) closed simple smooth contours, it generally does not admit this kind of membrane regulation. It is only possible in exceptional cases.

For simply and doubly connected shells, it is necessary to vary not only the normal surface load X but also the normal and tangential forces on the boundary.

It can be seen from (3.2) that the normal surface load X depends on the thickness h. Therefore, condition (3.27) can be satisfied in a number of cases by selecting appropriately the thickness $h(x, y)$ of the shell along its edge. In other words, in such cases the membrane regulation of the shell can be achieved by making the shell thicker and thiner along its boundary. It is well known that this method is often used in practice.

If the body force $\mathfrak{F}$ reduces to gravity, we will have

$$X = Q\nu + X_+ + X_-, \tag{3.29}$$

where $Q = 2\rho\gamma h$, ν is the projection of the unit normal onto the vertical direction at the corresponding boundary point, Q is the weight of an element of volume of the shell calculated per unit area. In a number of cases, membrane regulation can be ensured by an appropriate selection of the weight Q of the shell. The weight can be varied both by an appropriate selection of the thickness h and also by varying the density ρ of the material of the shell along the edge.

3.2. In the study of the membrane regulation problem, it is useful to consider a Hilbert space H, whose elements are the loads $e = (\mathfrak{X}, \mathfrak{T})$. Defining the scalar product by the formula

$$(e_1, e_2) = \iint\limits_S \mathfrak{X}_1\mathfrak{X}_2\, dS + \int\limits_L \mathfrak{T}_1\mathfrak{T}_2\, ds, \tag{3.30}$$

we introduce the following norm and distance:

$$\| e \| = [(e, e)]^{1/2}, \quad d(e_1, e_2) = \| e_1 - e_2 \|. \tag{3.31}$$

Let H_0 be a subspace of the elements $(\mathfrak{X}, \mathfrak{T})$, satisfying condition (3.1). Then, to each load, $e = (\mathfrak{X}, \mathfrak{T})$ we make correspond the nonnegative number

$$d(e) = \min_{e_0 \in H_0} \| e - e_0 \|. \tag{3.32}$$

Since, H_0 is a closed subspace of the space H, an element e_* exists in H_0 such that

$$d(e) = \| e - e_* \|. \tag{3.33}$$

Clearly, the condition for the regular membrane state is the equality d(e) = 0.

The number d(e) can be considered as a measure of the degree of deviation of the load $e = (\mathfrak{X}, \mathfrak{T})$ from the set of membrane loads. If the number d(e) is sufficiently small, the load $(\mathfrak{X}, \mathfrak{T})$ can be considered as a membrane load for all practical purposes.

Let H_+ be a subspace whose elements are the couples $g = (\mathfrak{U}, \mathfrak{U}^{(s)})$, where $\mathfrak{U}$ is the displacement vector during infinitesimal bending of the middle surface and $\mathfrak{U}^{(s)}$ is the tangential component of $\mathfrak{U}$ along the boundary. Let $\mathfrak{U}_j$ be a complete system of particular solutions of equation (3.3) satisfying the conditions

$$(g_i, g_j) = \delta_{ij}, \quad g_j = (\mathfrak{U}_j, \mathfrak{U}_j^{(s)}) \tag{3.34}$$

(complete systems of particular solutions $\mathfrak{U}_j$ of equation (3.3) can be constructed according to formula (3.16), taking the complete system of particular solutions of equation (3.14) as the w_j). Then, it is easily proved that

$$d(e) = \sum_{j=0}^{\infty} c_j^2, \tag{3.35}$$

where c_j are the Fourier coefficients of the load $(\mathfrak{X}, \mathfrak{T})$, i.e.,

$$c_j = \iint_S \mathfrak{X}\mathfrak{U}_j \, dS + \int_L \mathfrak{T}\mathfrak{U}_j^{(s)} \, ds. \tag{3.36}$$

From (3.35) it follows that $c_n \to 0$ when $n \to \infty$.

Thus, the torquelessness condition (d = 0) is equivalent to the requirement

$$c_j = 0 \quad (j = 1, 2, \ldots). \tag{3.37}$$

However, in practice it is difficult to verify whether an infinite number of equalities (3.37) is satisfied, especially in those cases when the loads are given not in terms of analytical expressions but in tabular or graphical form. But in practical problems, it is not necessary that all equalities (3.37) be satisfied exactly, and it suffices to ensure that the number d(e) is small, i.e., that some finite number of the first equalities be satisfied.

$$c_j = 0 \quad (j = 1, 2, \ldots, n). \tag{3.38}$$

Since $c_n \to 0$ when $n \to \infty$, for the given $\epsilon \to 0$ an $n = n(\epsilon)$ can always be found such that

$$d(\epsilon) = \sum_{i=n}^{\infty} c_i^2 < \epsilon. \tag{3.39}$$

Thus, to ensure in practice the membrane equilibrium state of the shell, it is sufficient if a finite number n of equalities of the type (3.38) is satisfied. Clearly, always $n \geq 6$, since equalities (3.38) must necessarily contain six static equilibrium conditions for the shell as an absolutely rigid body.

It is important to note that it is not necessary to require in equalities (3.38) that the vectors $\mathcal{U}_j$ be orthogonal. When conditions (3.38) are satisfied for an orthogonal set of vectors $\mathcal{U}_j$, they will also be satisfied for any linear combination of these vectors. Therefore, it is not necessary to use a complex and difficult orthogonalization process in practice.

We will describe an additional practical way of ensuring membrane regulation. This is loading the shell by point surface concentrated forces (I. N. Vekua, 1960).

3.3. The practical construction of a torqueless field of stresses is related to the solution of the generalized Cauchy-Rieman equation (3.10). This problem is simply solved in the case when B = 0. The nonhomogeneous Cauchy-Rieman equation is valid.

$$\frac{\partial w'}{\partial \bar{z}} = F' \tag{3.40}$$

and its general solution is described by the formula

$$w' = f(z) - \frac{1}{\pi} \iint\limits_D \frac{F'(\xi, \eta)\, d\xi\, d\eta}{\zeta - z}, \tag{3.41}$$

where f is an arbitrary analytic function of z = x + iy. The case B = 0 occurs for second order algebraic surfaces. Therefore, it includes a very large class of problems that are important in engineering applications. Using the well-known methods for the solution of boundary value problems for analytic functions of a single complex variable, a number of applied problems can be studied. In individual cases (for example, for circular regions) their solutions can be expressed in explicit form.

In particular, the case of a closed convex shell loaded by forces of the form (3.24) should be considered. Such a load and the corresponding stress field is determined in explicit form:

$$\mathfrak{T}_{(t)} = X\mathfrak{P}_{(t)} = -\frac{1}{2}\frac{X}{K}(k_s\mathfrak{t} + \tau_s\mathfrak{s}). \tag{3.42}$$

The last case occurs, for example, when the normalized normal surface load is proportional to the principal curvature at the corresponding point of the middle surface:

$$X = cK \qquad (c \text{ — const}). \tag{3.43}$$

Then, the potential V = const and the surface load $\mathfrak{X}_0$ has the form

$$\mathfrak{X}_0 = cK\mathfrak{n}. \tag{3.44}$$

and the corresponding stress field is described by the formula

$$\mathfrak{T}_{(t)} = cK\mathfrak{P}_{(t)} = -\frac{1}{2}c(k_s\mathfrak{t} + \tau_s\mathfrak{s}). \tag{3.45}$$

Let us assume that the force $\mathfrak{T}_{(1)}$ described by formula (3.42) has the direction $\mathfrak{l}$. Then $\tau_s \equiv 0$, i.e., the middle surface is a sphere. Thus, when the closed convex shell is loaded by forces of type (3.24) the corresponding force $\mathfrak{T}_{(1)}$ at each point of the middle surface has the direction $\mathfrak{l}$ if and only if the middle surface is a sphere. In this case formula (3.42) has the form

$$\mathfrak{T}_{(l)} = -\frac{1}{2} RX\mathfrak{l}, \tag{3.46}$$

where R is the radius of the sphere.

3.4. We conclude this survey by the following remarks, which, in a number of cases, can facilitate considerably the practical solution of concrete problems. Equations (3.10) and (3.14) are invariant with respect to a projective transformation of the space (I. N. Vekua, 1959). Therefore, it is easy to obtain the transformation formulas for the displacement and force fields during the transition from the given shell to another shell whose middle surfaces are projectively equivalent. Using these projective properties, it is possible, having solved the problem for the given shell, to construct solutions of the corresponding problems for projectively equivalent shells. In view of this, for example, many problems for ellipsoidal shells can be reduced to problems for a spherical shell.

DYNAMICS OF DEFORMABLE SOLIDS

N. V. Zvolinskiy, M. I. Reytman, G. S. Shapiro

In the first stage of development, the dynamics of deformable solids studied elastic bodies.

The equations of motion of elastic bodies were already derived in the beginning of the last century. First they were used for the solution of one-dimensional problems in the dynamic tension-compression and torsion of rods, the flexure of beams and vibrations of circular cylinders and spheres. Only at the beginning of our century were these equations applied to the solution of seismic problems.

Seismology requires the study of laws for the propagation of waves from the earthquake focus to the earth's surface and of those changes which the waves undergo during reflection and refraction on separation boundaries. The maximum information about the mechanism of the focus must be obtained from observations of movements on the earth's surface, in particular, the energy liberated during an earthquake must be estimated. The study of the structure of the earth's core (or its surface layer) on the basis of observations of the propagation of wave perturbations is very important. These problems are usually solved on the basis of representing the soil as an elastic body.

After World War II, as a result of the scientific technological revolution, the dynamics of deformable rigid bodies changed drastically. This applies primarily to the theory of loads of short duration acting on the body. The effective use of an impulsive load (using an explosive material, an electromagnetic or electrohydraulic effect, etc.) brought about a genuine revolution in such technological processes as riveting, molding, welding, hardening and cutting of metallic billets. The use of explosive technology in useful excavations and oil drilling, in the excavation of trenches and depressions in soils in seismological research in the building of dams in the soil, the reinforcement of soils and the drilling of holes is just as important.[1] The evaluation of the destruction during an impulsive load (splitting-off, the effect of earthquakes and explosions on structures, etc.) became important.

In the engineering applications that were mentioned above, usually the plastic or viscous properties of the materials are of fundamental importance. These properties also turned out to be important in the description of the behavior of a number of new materials (in particular polymers) whose characteristics are highly sensitive to changes in the temperature and the deformation rates. Heterogeneous and reinforced media, such as soils, fiberglass plastics, reinforced concrete, etc. display even more complex properties. It is not surprising that recently the center of gravity of the investigations in the branch of mechanics under consideration shifted to the dynamics of nonelastic media. This survey does not claim completeness. It discusses mainly those branches of the dynamics of deformable solids which reflect predominantly the scientific interests of the authors. The authors use studies which they published earlier or in which they participated (Kh. A. Rakhmatulin and G. S. Shapiro, 1955, N. V. Zvolinskiy, 1965, N. V. Zvolinskiy, B. M. Malyshev and G. S. Shapiro, 1966, M. I. Reytman and G. S. Shapiro, 1968).

§1. Dynamics of Elastic Bodies

The successes in the dynamics of elastic bodies in the Soviet Union were, based to a certain extent on the achievements of scientists in prerevolutionary Russia. The first studies in general integration methods for the equations of

1. References in the literature on engineering applications of impulsive loads can be found, for example, in the bibliography in the survey by N. V. Zvolinskiy, B. M. Malyshoev and G. S. Shapiro (1966).

dynamic elasticity theory were already made in 1831 by M. V. Ostrogradskiy, who constructed (at the same time as S. Poisson) the solutions for the equations of motion with arbitrary initial conditions. Integrating solutions of the simple harmonic type, M. V. Ostrogradskiy obtained a solution corresponding to the propagation of two types of waves in an infinite elastic medium: expansion waves and distortional waves. When waves of the first type are considered, compressions, tensions and displacements occur in the medium, but there are no rotations. Waves of the second type cause displacements and rotations without volume expansion.

Relatively little attention was given to the dynamics of an elastic body in prerevolutionary Russia. In the beginning of this century, A. N. Krylov studied the propagation of elastic waves in cylindes and rods in connection with problems of the stressed state in barrels in artillery guns and projectiles during fire. S. P. Timoshenko developed the theory taking into account both local and general deformations during the impact of a ball on a beam. A. N. Dinnik studied dynamic stresses in elevator ropes.

In the first years after the Revolution, the studies of Soviet scientists were primarily devoted to the solution of special important, one-dimensional problems. Ye. L. Nikolay (1919) studied the movement of a string of variable length. His study was the starting point for many subsequent studies in this field that were applied to the calculation of elevator ropes in mines. A detailed survey of these studies is available in the book by G. N. Savin and O. A. Goroshko (1962). N. M. Belyayev (1925) laid the foundation for the development of the dynamic stability theory of motion of elastic systems by solving the stability problem for a rectilinear prismatic rod with hinged supports compressed by a longitudinal harmonic force that varied over time. The results of the subsequent studies in this field are summarized in the monograph by V. V. Bolotin (1956).

A new stage in the development of the plane and three-dimensional theory of the propagation of elastic waves began in the 30's which was connected with the achievements of mathematicians at the Leningrad University working in the Seismological Institute (now the Earth Physics Institute) at the USSR Academy of Sciences. The Mathematical School in this Institute whose outstanding representatives in the past were P. L. Chebyshev, A. M. Lyapunov, A. N. Korkin and V. A. Steklov, achieved outstanding results in the solution of mathematical problems related to theoretical problems in natural sciences and engineering. The outstanding representatives of the mathematical school of the Leningrad State University and their followers, V. I. Smirnov, S. L. Sobolev, V. G. Gogoladze, S. G. Mikhlin,

Ye. A. Naryshkina, D. I. Sherman and others were engaged in work in the Seismological Institute, and published their studies in the "Trudy" (Publications) of this Institute. After World War II, studies in the propagation of elastic waves were continued at Leningrad State University under the leadership of G. I. Petrashen and in the Earth Physics Institute under the leadership of N. V. Zvolinskiy and V. I. Keylis-Borok It should be mentioned that in the postwar years, the development of studies in dynamic problems for elastic media was very broad. This is connected with the intense development of a number of engineering and economic branches (seismological research, earthquake-proof construction, etc.). The last two decades are characterized not only by a large number of publications but also by a great variety of trends in the study of elastic waves.

Below we will list the individual trends and supplement them by a brief characterization and references to the main published studies. The list includes the entire 50 year period. At the same time the references do not exhaust the entire literature in each direction. This disadvantage is compensated by a full list of studies by Soviet authors which make up the final (fourth) volume of this edition.

1. Results of a General Character, Methods for the Solution of Equations in the Theory of Elasticity.[1] The oscillations of a linear elastic homogeneous medium are described by the vector equation

$$(\lambda+2\mu)\,\mathrm{grad}\,\mathrm{div}\,u-\mu\,\mathrm{rot}\,\mathrm{rot}\,u=\rho\frac{\partial^2 u}{\partial t^2} \qquad (1.1)$$

or by the corresponding system of scalar equations. Here, u is the displacement vector, t is time, and λ and μ are the Lamé constants. This system of equations has real characteristics and therefore can be classified as a system of the hyperbolic

1. The basic results in this field were obtained by V. I. Smirnov and S. L. Sobolev (1932), S. L. Sobolev (1934, 1937), V. I. Smirnov (1936), N. P. Yerugin (1944), D. I. Sherman (1946, 1949), S. G. Mikhlin (1947), V. D. Kupradze (1950, 1953), A. G. Sveshnikov (1953), G. I. Petrashen, A. S. Alekseyev and B. Ya. Gel'chinskiy (1959), G. A. Skuridin (1959), B. A. Bondarenko (1960), B. M. Naymark (1960), A. S. Alekseyev (1962), V. M. Babich (1962, 1967), K. I. Ogurtsov, L. S. Pakhomenko and A. I. Sutyagina (1962), I. P. Tsay (1962), G. I. Petrashen (1964, 1966), A. S. Blagoveshchenskiy (1966), L. Ya. Aynola (1967).

type. The boundary value problems also include initial conditions, and generally, boundary conditions. Thus, problems in dynamic elasticity theory are either Cauchy problems or mixed boundary value problems.

One second-order vector differential equation can be replaced by a system of two wave equations for the scalar and vector potentials.

An important class of particular solutions is the class of functional-invariant solutions, i.e., solutions f(x, y, z, t) of the wave equations which generate solutions F(f(x, y, z, t)) for any (twice differentiable) function F. These solutions were found and studied originally for the two-dimensional problem (S. L. Sobolev, 1934), and then generalized to the three-dimensional case (N. P. Yerugin, 1944). Applications to concrete problems were obtained for two-dimensional problems. It is essential that important singular solutions of the concentrated forces type are described by functional-invariant solutions. Functional-invariant solutions are particularly suitable for the description of similar two-dimensional problems.

Another series of plane and axisymmetric problems in dynamic elasticity was brought all the way to an analytical solution and analysis using the method of integral transformations (the method of incomplete separation of variables). These studies are the work of G. I. Petrashen and his students. While it has certain advantages in the analysis of the solution and the study of the physical consequences, the method of integral transformations is more complex to justify rigorously mathematically. In applied problems, incidentally, such a justification is usually not required.

The reciprocity principle which states that there is a certain symmetry between the external forces and the observed results of the deformation of the elastic body is known in statics and it can be described by the known Betti formula. It was extended to dynamic elastic phenomena by V. M. Babich (1962).

The stationary oscillations of an elastic medium are described by an elliptic system of differential equations. These can be reduced to integral equations (V. D. Kupradze, 1953), which to some extent, are similar to the integral equations in potential theory, but are more complex (due to the presence of eigenvalues, the frequencies of the natural oscillations of bounded volumes). In the case of external problems, radiation conditions must be formulated at infinity which will ensure the uniqueness of the solution (A. G. Sveshnikov, 1953).

The few attempts to formulate inverse problems in the dynamic theory of elasticity, in which conclusions are drawn about the properties of the inhomogeneous halfspace on the basis of the known properties of the oscillation source and the movement of the boundary of the elastic inhomogeneous halfspace should be noted.

2. Concrete Problems with Simplest Geometry.[1] Problems with simple boundary surfaces were studied: halfspace, half-plane, layer, sphere, cylinder. Their selection is determined, on one hand, by the fact that the boundary surfaces must be considered as coordinate surfaces, and on the other hand that in these simple situations, real practically important problems can be idealized. A particularly great deal of attention was given to the elastic halfspace (axisymmetric and plane problems). These were used to model certain seismic problems. The first study on the propagation of waves in an elastic half-space was published already in 1904 by G. Lamb. Subsequently it was again solved by S. L. Sobolev, who used the method of functional-invariant solutions and by G. I. Petrashen and his students using the method of integral transformations. Such repeated solutions are justified by the fact that the new solution method provides the investigator with new possibilities of analyzing the solution. Wave fields in the halfspace were studied in sufficient detail. In addition to the traditional source (a normal concentrated force on the surface) other problems were also studied: the expansion center, and tangential forces on the boundary surface.

Next, a moving load on the boundary of the halfspace was studied. This idealized the displacements of the atmospheric pressure centers on the surface of the earth or the displacements of a propagating shock wave from an explosion

1. The basic results in this field were obtained by V. D. Kupradze and S. L. Sobolev (1930), Ye. A. Naryshkina (1933, 1934), V. I. Smirnov (1937), G. I. Petrashen (1945, 1946, 1949, 1950), D. I. Sherman (1946), Kh. L. Smolitskiy (1947), G. I. Petrashen, G. I. Marchuk and K. I. Ogurtsov (1950), K. I. Ogurtsov and G. I. Petrashen (1951), L. N. Sretenskiy (1952, 1955, 1956), V. A. Sveklo (1954), Ye. I. Shemyakin and V. A. Faynshmidt (1954), G. S. Markhasev (1955), K. I. Ogurtsov (1956, 1960, 1966), B.Ya. Gel'chinskiy (1958), A. S. Stavrovskiy (1959), A. N. Margot'ev (1960), Ya. S. Uflyand (1961), Ye. I. Shemyakin (1961), K. I. Ogurtsov, L. S. Pakhomenko and A. I. Sutyagina (1962), D. N. Klimova and K. I. Ogurtsov (1966), Zh. M. Imenitova and K. I. Ogurtsov (1967), L. A. Molotkov (1967, 1968).

The geophysical applications also include studies on the effect of gravitational vibrations in water basins on seismic vibrations (microseisms). Studies on the movement of the elastic halfspace when the initial and boundary conditions are arbitrary are of general theoretical significance.

3. Reflection and Refraction on One Boundary.[1] The study of reflection and refraction on one isolated plane boundary dividing two media is the main link in the calculation of multiple reflections in a medium consisting of layers. At the same time the study of a single reflection-refraction act describes important qualitative properties of the phenomenon.

The most typical and frequent case is the case of the complete contact of two media with different elasticity constants on both sides of the separation boundary when there are no relative displacements of the media on the boundary.

V. G. Gogoladze (1947) has shown that the reflection and refraction coefficients are meaningful for a plane incident wave of arbitrary "shape" (in the absence of dispersion) and he found these coefficients. The dependence of the coefficients on the parameters of the media and the angle of incidence is complex. This is due to a change in the regimes of the phenomenon during the transition through the critical angles and the formation of a total internal reflection. To facilitate the calculations in applications, extensive tables of the reflection and refraction coefficients have been compiled. Generally, the coincident surface waves confined within the separation boundaries are related to the separation boundaries.

In addition to the full contact case that was described other boundary conditions were studied on the contact surface. Cases of non-rigid contact or conditions when the three-dimensional waves are not reflected were also studied. Studies dealing with "weak" separation boundaries, i.e., on which the elasticity moduli are continuous, but the derivatives of the moduli are discontinuous, are also available. Incidentally, this situation already applies to inhomogeneous media. If a nonplanar wave is incident to a plane boundary, generally front waves are also formed. These were the subject of many studies. Front waves, together with other types of waves

1. The basic results in this field were obtained by S. G. Mikhlin (1941), V. G. Gogoladze (1945, 1947), M. A. Isakovich (1956), T. I. Oblogina (1956), N. V. Zvolinskiy (1957, 1958), B. Ya. Gel'chinskiy (1958), L. M. Flitman (1958), G. S. Pod"yapol'skiy (1959, 1963), K. I. Ogurtsov (1960), V. V. Tyutekin (1962), V. A. Sveklo (1962).

are also important in seismic research.

Finally, we mention the study of a reflection on a curvilinear boundary. This case refers essentially to the diffraction phenomenon.

4. Media Consisting of Layers.[1] The main object of study here was a system of plane parallel homogeneous layers with different elastic properties. Such a structure models the thickness of the earth's core. Studies along these lines were primarily in the field of general research and engineering seismology.

Two approaches can be used in the calculation of the seismic field of a medium consisting of layers. One approach is based on solving the problem by the method of integral transformations. It is not connected (at least in the first stage) with the isolation of individual waves. The second approach is based (when the original formulation allows it) on the study of successive reflections and refractions. The latter is connected with difficulties in taking into account a large number of waves which continuously increase with each reflection-refraction act. Incidentally, in a number of cases modern electronic computing technology makes it possible to cope successfully with this problem.

When a wave propagating parallel to the boundaries in a layer medium is studied, depending on the elastic properties of the layer, it can be described in some layers by equations of the elliptic type (wave of the surface type) and in other layers by equations of the hyperbolic type (interference waves).

1. The basic results in this field were obtained by I. N. Vekua (1937), P. K. Ishkov (1941, 1956), D. I. Sherman (1945), V. G. Gogoladze (1947), N. V. Zvolinskiy (1947), M. A. Naymark (1948), V. I. Keylis-Borok (1952, 1954, 1956), G. I. Petrashen (1952, 1956, 1957), I. M. Khaykovich (1954), G. G. Pogonyaylo and I. N. Uspenskiy (1959), G. S. Pod"yapol'skiy (1959), T. Ya. Barinova (1961), K. I. Ogurtsov (1961, 1962), L. B. Levitin, G. A. Skuridin and K. P. Stanyukovich (1963) and Z. A. Yanson (1965).

A general study of interference and surface waves can be found in the monograph by V. I. Keylis-Borok (1961). The frequencies of the interference oscillations of the layers are the internal characteristics of the layer as an element of the layer structure. Studies were made in which liquid layers were also considered. Such formulations can be used on one hand to evaluate the differences between the phenomena occurring in solid and liquid (compressible) media and on the other hand they can be applied to the propagation of waves in the solid base at the bottom and in the water volume itself.

The global geophysical study of the wave globe is given in the work of Z. A. Yanson, in which a spherical layer medium is considered.

The well thought out and carefully written monograph of L. M. Brekhovskiy (1957), on waves in layer media (including also predominantly acoustical and electromagnetic waves) should be mentioned.

5. Media Consisting of Thin Layers.[1] An elastic layer can be considered "thin" if the time the wave takes to cover the path over its thickness is much smaller than the characteristic time of the process as a whole (for example, the characteristic time of the external action). During this condition, the wave phenomena in the "transverse direction to the layer" can be considered totally (in a certain sense asymptotically). This point of view gave rise to a large series of studies which derived approximate solutions for the transverse oscillations of plates (also rods and shells). Numerous improvement and corrections for the usual approximate ("engineering") theory of oscillations of plates were proposed. The basic idea is to separate the low frequency oscillation mode. When a thin layer is surrounded by media with smaller propagation velocities, a so-called interference leading wave is formed. Each reflection in the layer generates the front of a leading wave. These waves are superimposed on one another with a small phase shift and form the total low frequency wave (L. A. Molotkov and P. V. Krauklis, 1963). The formation of screening waves is related

1. The main results in this field were obtained by M. A. Naymark (1947, 1948), G. I. Petrashen and V. A. Yenal'skiy (1956), G. I. Petrashen and L. A. Molotkov (1958, 1964), Yu. A. Voronin (1959), T. B. Yanovskaya (1959), L. A. Molotkov (1961), G. S. Pod"yapol'skiy (1961), P. V. Krayklis (1962), P. V. Krauklis and L. A. Molotkov (1962, 1963), G. I. Petrashen (1966), L. A. Molotkov and D. K. Ozerov (1967).

to a thin layer with a higher propagation velocity. They also occur as a result of interference waves reflected inside the layer, which form the smooth low frequency transmitted wave which has the shape of a "smoothed" refracted wave. The smoothing is greater, the thicker the layer. Therefore, the screening wave is only well observed in thin layers (G. I. Petrashen' (1954, A. S. Alekseyev and V. M. Babich, 1954, Yu. A. Voronin, 1959).

6. Asymptotic Rays.[1] The front of a propagating wave is a discontinuity surface for derivatives of a certain order of the displacements. In view of this, in the vicinity of the front, the change in the displacement field in the direction of the normal to the front is much more intense than the same change along the front. This makes it possible to consider the neighborhood of each point of the front as a locally-plane wave. The asymptotic method for the study of the neighborhood of fronts (for the stationary observer, the neighborhoods where some wave appears for the first time) is based on this idea. This method has been known for a long time in acoustics and optics. It was extended to the theory of elasticity for the first time in the study of M. L. Levin and S. M. Rytov (1956). Subsequently, it was developed further and used as a means for the approximate solution of reflection and refraction problems. The field in the neighborhood of the front can be described with various degrees of accuracy. In applied problems, usually the first approximation is used but cases exist when it is theoretically inadequate (G. S. Pod"yapol'skiy, 1959). On one hand, the ray approach is very general, for example, it can be applied without particular difficulties to inhomogeneous media. On the other hand, exceptional situations exist when it breaks down or when it must be essentially reformulated, for example, in the neighborhood of the initial points of the leading waves (and generally at points where the fronts intersect), in a caustic neighborhood and other neighborhoods (V. M. Babich, 1961, Yu. L. Gasaryan, 1961, B. T. Yanovskaya, 1964).

1. The basic results in this field were obtained by A. A. Gvozdev (1952), N. V. Zvolinskiy and G. A. Skuridin (1956), M. L. Levin and S. M. Rytov (1956), A. F. Filippov (1957), V. M. Babich and A. S. Alekseyev (1958), A. S. Alekseyev and B. Ya. Gel'chinskiy (1959, 1961), G. S. Pod"yapol'skiy (1959), A. S. Alekseyev, V. M. Babich and B. Ya. Gel'chinskiy (1961), V. M. Babich (1961), T. B. Yanovskaya (1964).

7. Surface Waves.[1] These waves are characterized by the fact that they propagate "along" the boundary surface. They are particularly important in seismology and have been mainly studied from the point of view of this science. Being confined to the boundary surface, they are damped slower than three-dimensional waves and they carry the energy over greater distances. The structure of the medium in which they were formed and through which they passed leaves its trace in the form of the dispersion law. Attempts are being made to obtain information about the structure of the medium (the earth's core) on the basis of observations of this law.

The surface waves may be related both to the free boundary and to the separation boundary of the media (V. D. Kupradze and S. L. Sobolev, 1930, V. I. Keylis-Borok, 1960).

In addition to media with a plane-layer structure, continuous-inhomogeneous media were also studied in connection with surface waves (V. M. Babich and I. A. Molotkov, 1966), as well as regions with non-planar boundaries. In the last case, high frequency waves, which were rapidly damped as the distance from the boundary increased, were studied. The limiting case of this kind is the spreading of the discontinuity of the derivative of the stresses along the boundary surface (I. G. Petrovskiy, 1945). For curvilinear boundaries of the simplest types (sphere, cylinder) exact particular solutions for the problem of the surface waves can be obtained. In addition to the typical surface waves that were mentioned here, wave motions which have the character of surface waves which were formed under more complex conditions were also detected and studied (L. P. Zaytsev, 1960, G. S. Pod"yapol'skiy and Yu. I. Vasil'ev, 1960).

---

1. The basic results in this field were obtained by V. D. Kupradze and S. L. Sobolev (1930), Ye. A. Naryshkina (1934, 1936, 1940), I. G. Petrovskiy (1945), Ya. A. Mindlin (1946), G. I. Petrashen (1946), V. G. Gogoladze (1948), I. A. Viktorov (1958), L. P. Zaytsev (1959, 1960), Yu. I. Vasil'ev and G. S. Pod"yapol'skiy (1960), V. I. Keylis-Borok (1960), V. M. Babich (1961), V. M. Babich and N. Ya. Rusakova (1962), A. G. Alenitsin (1963, 1964), Ya. A. Mindlin (1963), T. Ya. Barinova (1964), R. V. Gol'dshteyn (1965), V. Yu. Zavadskiy (1965), V. M. Babich and I. A. Molotkov (1966), L. M. Brekhovskiy (1966, 1967), V. M. Babich and T. S. Kravtsova (1967), I. A. Molotkov and I. V. Mukhina (1967).

8. Inhomogeneous Media.[1] This is the name of elastic media in which the Lame coefficients λ, μ and the density ρ are functions of the coordinates. When λ, μ and ρ are continuous functions and the derivatives of these functions are discontinuous on certain surfaces, such surfaces are usually called "weak" boundaries. Some information about the studies of continuous media was given above in connection with asymptotic rays and surface waves. The equations of motion for inhomogeneous elastic media which retain the same highest order derivative have additional terms with first order derivatives of the displacement vector. For these equations fundamental solutions have been constructed (V. M. Babich, 1961). Primarily media that were inhomogeneous with respect to one of the coordinates were considered (this choice is dictated both by the requirements of simplicity and by the geophysical applications). Generally in an inhomogeneous medium the motion cannot be decomposed into the sum of longitudinal and transverse waves. However, this can be done when certain conditions (differential conditions) are satisfied which the functions λ, μ and ρ must obey (V. Yu. Zavadskiy, 1964).

An inhomogeneous medium as well as a homogeneous medium has two types of front waves, longitudinal and transverse waves. Each propagates at its own local rate, the longitudinal and transverse rate, respectively. The rotor of the vector $\underset{\sim}{u}$ has a discontinuity at the longitudinal front and the divergence on the transverse front (A. A. Gvozdev, 1959).

In an inhomogeneous medium the rays are curvilinear. This leads to new effects which are far from simple. Thus, in inhomogeneous media, geometric shadow zones are formed which the perturbations can only penetrate by way of diffraction. Near the separation boundary, refracted waves can be formed as a result of the multiple reflection of curvilinear rays.

1. The basic results in this field were obtained by S. L. Sobolev (1930), A. A. Gvozdev (1959), N. V. Tsepelev (1959), B. S. Chekin (1959, 1964), V. M. Babich (1961), Yu. A. Gazaryan (1961), V. Yu. Zavadskiy (1964, 1965), I. A. Chaban (1964, 1965), A. G. Alenitsyn (1966, 1967).

We mention a study of the Lamb problem for an inhomogeneous medium (A. G. Alenitsyn, 1966).

9. Anisotropic Media.[1] Materials with elastic anisotropy are frequently encountered in nature and in engineering. Besides the genuine (molecular) anisotropy, "structural" anisotropy caused, for example, by the microlayer structure of the material is encountered.

The great variety of different types of anisotropic bodies (the number of constants varies from 3 to 21) makes their study more complex. Predominantly special types of media with a small number of elastic constants have been studied. The wave processes are described by a hyperbolic system of equations with constant coefficients (for a homogeneous medium). Three types of waves rather than two exist for anisotropic bodies.

Among the results of a more general character, the construction of singular (fundamental) solutions for special types of anisotropy and the functional-invariant solutions that were generalized to these cases and the solution of the Lamb problem should be mentioned (B. A. Sveklo, 1961).

A sufficiently large number of studies deals with the reflection and refraction of waves on a plane boundary (which coincides with one plane of anisotropy) and also with surface waves of the Rayleigh type (I. O. Osipov, 1961). The diffraction problem on the halfline has also been solved.

1. The basic results in this field were obtained by V. G. Gogoladze (1935), I. M. Lifschitz and L. M. Rosenzweig (1946), V. A. Sveklo (1949, 1961), I. M. Lifschitz and G. D. Parkhomovskiy (1952), I. O. Osipov (1961-1963), I. N. Uspenskiy and K. I. Ogurtsov (1962).

10. <u>Diffraction</u>.[1] By diffraction in the broad sense are usually meant wave phenomena which cannot be described with the aid of ray concepts or plane waves. Typical diffraction problems are the interaction of waves with different obstacles. The analytical difficulties in diffraction problems in the theory of elasticity are connected with the presence of two types of waves (longitudinal and transverse waves) which are intertwined in the boundary conditions.

Until now very few exact solutions are available. For obstacles with a simple geometry (a sphere, cylinder, ellipsoid) the solution can be constructed by the method of separation of variables (V. D. Kupradze, 1935). However, such infinite series solution has a formal character, it is difficult to justify and very difficult to analyze physically (perhaps this situation can be improved by modern computing technology.

Diffraction on a semiinfinite section (with free or supported edges admits a convenient application of functional-invariant solutions and leads to a solution in closed form (M. M. Fridman, 1949, A. F. Filippov, 1956).

The diffraction problems in acoustics are much simpler. At the same time they are closely allied to analogous problems for a solid elastic medium, and therefore the study of the diffraction of sound waves is important in the theory of elasticity. Sophisticated analytical studies which studied the nonstationary diffraction front separating the geometric shadow region behind a convex body from the perturbed medium were carried out (V. S. Buldyrev and I. A. Molotkov, 1958, V. S. Buldyrev, 1959).

---

1. The main results in this field were obtained by V. D. Kupradze (1935), S. L. Sobolev (1935), D. I. Sherman (1945), G. I. Petrashen (1946), M. M. Fridman (1949, 1959), A. M. Kuskov (1950), G. I. Petrashen, N. S. Smirnova and B. Ya. Gel'chinskiy (1953), V. V. Tyutekin (1956), A. F. Filippov (1956, 1959, 1964), G. A. Skurdin (1957), V. S. Buldyrev and I. A. Molotkov (1958, 1961), G. I. Petrashen, B. G. Nikolayev and D. N. Kouzov (1958), V. A. Sveklo (1958), V. A. Sveklo and V. A. Syukiyayn (1958), V. S. Buldyrev (1959), G. D. Malyuzhinets (1959), V. V. Tetekin (1959, 1960), Yu. L. Gazaryan (1961), A. S. Golubev (1961), I. A. Molotkov (1961), P. I. Tsoy (1961), A. V. Borovikov (1962), I. G. Filippov (1963), I. A. Chaban (1963), I. M. Yavorskaya (1964-1967), B. V. Kostrov (1965), V. Yu. Zavadskiy (1966).

Approximate methods have also been applied to diffraction problems. The approximate method can be constructed on the short wave or long wave asymptote (compared to the characteristic dimension of the obstacle). The approximate method proposed by G. D. Malyuzhinets (1959) is worthy of attention.

11. Contact Problems, Waves Caused by Sudden Cracks.[1] In wave processes of this kind, diffraction participates in an essential way. Therefore, generally speaking, they could be combined with the previous section. Problems of waves caused by an instantaneous violation of continuity are forecast by seismology. Contemporary concepts about the mechanism of the earthquake focus require the solution of the following problem: in an originally stressed medium a crack (slit) is formed instantaneously and the stresses are removed from the edges of the slit. The wave field that is formed in the process must be determined. For a crack of finite length, such a problem was first solved in a two-dimensional formulation by L. M. Flitman (1963). Subsequently, this formulation was generalized to the case of a crack formed on the separation boundary of two different elastic media and to axisymmetric cracks. In these formulations, the dimension of the crack that is formed or the law according to which it spreads are considered to be given in advance. This means that fracture conditions and the fracture process are not considered. This second aspect, the study of the crack as a result of fracture, requires an analysis which is outside the scope of elasticity theory and will not be touched on here.[2]

1. The basic results in this field were obtained by L. M. Flitman (1958, 1959, 162, 1963), V. A. Sveklo (1959, 1962), N. M. Borodachev (1960, 1962, 1964, 1966), B. V. Kostrov (1964, 1966), L. P. Zaytsev and L. M. Flitman (1965), R. V. Gol'dshteyn (1966), L. O. Sigalov (1966), I. V. Simonov and L. M. Flitman (1966), O. Ya. Shekhter (1966), A. N. Kovshov and I. V. Simonov (1967).

2. See the survey "Mechanics of Fracture" by V. Z. Parton and G. P. Cherepanov (pp 424-574) at the end of this volume (editors)

The mixed problems which we here have in mind can be of two kinds. Primarily they are dynamic problems dealing with the action of a die on an elastic body. In the simplest formulations, by the body is meant an elastic halfspace and the die is considered either as an infinite strip (plane problem) or a circular region in the plane (L. M. Flitman, 1959, N. M. Borodachev, 1960). Problems of this type were solved analytically, but for completeness the calculations of successive types of diffractions on the edges of the die, or the behavior on the long wave asymptote was calculated. It was assumed that there were no tangential stresses at the base of the die (free slippage).

A kind of reverse of this type of problems is the following formulation: a rigid massive body is placed in an elastic medium or it lies (continuously) on its surface. The incident wave is given and it is required to determine the motion of the body. This formulation was stimulated by problems in engineering seismology. Problems were solved when the massive body is a strip on the surface of a half space (L. M. Flitman, 1962), or a strip soldered in a plastic medium (A. M. Kovshov and I. V. Simonov, 1967). In these problems which clearly belong to the diffraction field, the attention is shifted from the diffraction field (which in certain cases must not be completely known) to the law of motion of the massive body).

§2. General Problems in the Dynamics of Nonelastic Bodies

2.1. Introductory Historical Survey

The dynamics of nonelastic bodies is a comparatively young branch in the dynamics of deformable media which came into being shortly before and during World War II. Many fundamental results in it were obtained by Soviet scientists. The dynamics of nonelastic bodies followed a somewhat different path than the dynamics of elastic bodies. The first results in the dynamics of elastic bodies refer to the nature of perturbations (the expansion waves and distortional waves) propagating in an unbounded medium. Only several decades ago, concrete problems dealing with the propagation of longitudinal waves in rods were studied. On the other hand, in the theory of the propagation of elastoplastic waves, the propagation of waves in rods was studied first and only after this the problem of the propagation of the perturbations in an unbounded medium.

In the beginning of §1 it was noted that the main achievement in the dynamics of elastic bodies are linked to the Mathematical School of the Leningrad University and the Seismological Institute at the USSR Academy of Sciences. To some extent a similar statement can be made about the first achievements in the dynamics of plastic and viscoelastic bodies and their connection with the Mechanics-Mathematics School at the Moscow University and the Institute of Mechanics at the USSR Academy of Sciences. The founders of the dynamics of viscoplastic and plastic media are the contemporary representatives of the school that was mentioned, A. A. Il'yushin and Kh. A. Rakhmatulin. Their studies were continued at the Mechanics Institute (V. V. Sokolovskiy, G. S. Shapiro, et al.) and at Moscow State University (V. S. Lenskiy, P. M. Ogibalov, et al.). The main results in this field are published in the Publications of the Mechanics Institute, the journal "Prikladnaya Matematika i Mekhanika" (Applied Mathematics and Mechanics) and "Inzhenernyy sbornik" (Engineering Collection) and also in "Uchenyye zapiski" (Learned Notes) and "Vestnik" (Herald) of the Moscow University.

The Structural Mechanics School of A. A. Gvozdev, I. M. Rabinovich and A. R. Rzhanitsyn at the Moscow Construction Engineering Institute (MCEI) which is intimately connected with the Central Scientific Research Industrial Construction Institute (CSRICI) (now the Central Scientific Research Building and Construction Institute, CSRBCI), played an important role in the development of the dynamics of nonelastic media.

A. A. Gvozdev was the founder of the theory of limiting equilibrium which uses a simplified plastic model of the body without taking into account elastic deformations and hardening (the so-called rigid-plastic model). This model was applied on a wide scale in the statistical theory of plasticity. It was also used for the first time in the solution of dynamic problems by A. A. Gvozdev (1942). Ten years later this method was perfected in the USA by E. Lee, P. Simondson, V. Praeger and G. Hopkins and it is successfully used to this very day both in the USSR and abroad.

Simplified methods for the solution of dynamic problems in the theory of plasticity in which the structures are considered as systems with one degree of freedom were developed by I. M. Rabinovich (1948). This engineering direction was subsequently developed on a wide scale.

The very promising studies of variational methods for the solution of dynamic problems for nonelastic media that were started by A. R. Rzhanitsyn (1959) were continued at Moscow State University by V. P. Tauszh (1962) and at the Central Scientific Research Institute for Building and Construction by M. 1. Reitman (1964). The representatives of the same school at the Moscow Construction Engineering Institute

(G. A. Geniyev, 1959, 1961, M. I. Estrin, 1958, 1961, 1962) were among the first who studied the propagation of discontinuous waves in a two-dimensional and three-dimensional plastic medium.

The studies of the scientists of the given school were mainly published in the collections of the Central Scientific Research Industrial Construction Institute (The Central Scientific Research Building and Construction Institute) published by "Stroyizdat" (State Publishing House of Construction Literature).

The sphere of investigations in the dynamics of nonelastic media expanded considerably in the last two decades. In Moscow, in addition to Moscow State University and the Institute for Problems in Mechanics at the USSR Academy of Sciences (organized in the Institute of Mechanics) problems in the dynamics of nonelastic media are now studied in many academic and departmental institutes and in institutions of higher learning. Studies on these problems also go on outside Moscow, in Alma-Ata, Baku, Voronezh, Gor'kiy, Kiev, Kishinev, Leningrad, Minsk, Novosibirsk, Riga, Tartu, Tashkent, Tbilisi (Tiflis) etc. All-Union symposia on the propagation of elastoplastic waves in continuous media are being held regularly (Moscow, 1962, Baku, 1963, Tashkent, 1966, Kishinev, 1968, Alma-Ata, 1971).

The behavior of materials under a dynamic load often differs considerably from their static behavior. This shows that an adequate description of the dynamic behavior of materials may require the use of defining equations which depend on time. Thus, in addition to the difficulties of a predominantly mathematical character, which arise in the solution of problems in the dynamics of elastic bodies, in the dynamics of nonelastic bodies, new difficulties are added, which are connected with the selection of the appropriate model of the material, i.e., with the selection of the defining equations.

Under static conditions, one of the simplest characteristics of the material is the load-elongation curve. Under a dynamic load the determination of the load-elongation curve becomes a nontrivial problem. As a result of the inertial forces that are formed (which must be taken into account at deformation rates exceeding 10 per 1/sec) the stress and strain fields in the samples are inhomogeneous. Since the stresses and strains at the same point of the sample cannot be determined simultaneously in practice, the form of the defining equation cannot be determined directly from the data of such tests. Usually the form of the defining equation is specified in advance with an accuracy up to a certain number of free parameters and then the corresponding wave problem

is solved and the unknown parameters are determined from the experimental data. This shows the fundamental importance of the simplest dynamic problem of the tension of a rod under different assumptions about the properties of its material.

Clearly attempts to determine universal defining equations which can be used in any range in which the stresses, strains and their derivatives vary with respect to time, temperature, hydrostatic pressure, etc., will lead to almost hopeless difficulties. Therefore, in practice an attempt is made to use idealized models describing the behavior of the materials in limited ranges which correspond to the conditions of the problem under consideration.

A complete solution of the problem of the selection of an appropriate model of the material even in this simplified form is far from complete; however, examples of useful special solutions are available. Thus, at superhigh pressures (on the order of the elasticity modulus) which develop during superfast collisions, the model of an ideal liquid is used successfully (M. A. Lavrent'ev, 1949). For materials of the polymer type for which the incomplete elasticity effects are essential, sometimes the model of an elastoplastic body is used (see, for example, A. Yu. Ishlinskiy, 1940). With regard to materials such as metals under the action of moderately high stresses on the order of the yield limit (to which this survey is predominantly devoted), these can be studied using two approaches. The first approach is based on the assumption that beyond the elasticity limits the material makes the transition to the viscoelastic state and its defining equation depends on time. The studies of A. A. Il'yushin (1940, 1941) in which the defining equations used were the viscoplastic flow equations not taking into account the elastic deformations pioneered this trend. In these studies the solution of a number of theoretical problems was obtained (the impact of a solid body on a cylindrical sample, the deformation of a smooth cylinder under the action of internal pressure), and in it the author also described the first air operated hammer designed by the author with which deformation rates on the order 10^4 1/sec were attained (the viscosity coefficients of certain metals were obtained with the aid of this hammer). Shortly afterwards the students of A. A. Il'yushin solved problems dealing with the rotation of a cylinder in an elastoplastic medium (P. M. Ogibalov, 1941) and the impact of a cylinder along a viscoplastic plate (F. A. Bakhshiyan, 1948. The publication of this study was postponed for five years.). From the mathematical standpoint, the equations of the dynamics of a uniaxial viscoplastic body belong to the class of equations of the parabolic type.

World War II interrupted work along these lines. However, after the war, the work began to develop in our country and abroad on an ever-increasing scale. A new trend in these studies was established by the work of V. V. Sokolovskiy (1948) in which the well-known elasto-visco-plastic model of a material was used in the analysis of the propagation of longitudinal waves in a rod (the model proposed by K. Howenemzer and V.Praeger). At deformation rates which are zero, the equations of this model become the equations of ideal plasticity, and at infinite deformation rates, the equations of elasticity theory. A modified model taking into account the deformation hardening of the material was proposed in 1951 in the USA by L. Malver. The equations of uniaxial motion based on this model are of the hyperbolic type.

The second approach which is due to Kh. A. Rakhmatulin assumes that under a dynamic load the material beyond the elasticity limit makes the transition to the plastic state. This point of view is justified by the fact that the deformation curves of many materials, especially metals, show a weak dependence on the deformation rates. For example, in tempered steels, these curves coincide almost exactly under static and dynamic loads. On the other hand for a number of problems the deformation rates vary only by two to three orders of magnitude, which may almost not be reflected at all in the relation between the stresses and strains. Thus, during a dynamic load, it is often possible, at least in first approximation, to use the deformation laws of elasto-plastic media, even though the parameters of these laws may differ from the static parameters.

The study of Kh. A. Rakhmatulin (1945) on the propagation of longitudinal waves in a semiinfinite rod started the investigations which studied the elasto-plastic rays. Taking as the basis the stress-strain diagram with different loading and unloading laws, Kh. A. Rakhmatulin detected the existence of the so-called unloading wave which separates the "space-time" plane into loading and unloading regions. One year later G. Taylor in England and T. Karman in the USA published less complete studies of this problem (without taking into account unloading).

The problem of determining the unloading wave occupies a key position in the one-dimensional theory of the propagation of elasto-plastic waves. An analysis has shown that this problem does not reduce to the classical Goursat, Cauchy problems or to a mixed problem in the theory of hyperbolic equations. A special solution method was developed for it (G. S. Shapiro, 1946), which was subsequently further developed (V. L. Biderman, 1952). Specific cases of the propagation of fractures

were also studied (Kh. A. Rakhmatulin and G. S. Shapiro, 1948) and in the case of a longitudinal impact of a rod along a rigid barrier the possibility of the existence of stationary ruptures was detected (V. S. Lenskiy, 1949). The construction of similar solutions was analyzed by G. I. Barenblatt (1952). An original approach to the problem of the propagation of elasto-plastic waves was proposed by K. P. Stanyukovich (1955).

The solution of the next more complex problem after the problem of longitudinal waves, the problem of the propagation of longitudinal-transverse waves in wires is also due to Kh. A. Rakhmatulin (1946) (see §4). His studies served as the source for a long series of studies in this field.

The theory of the propagation of longitudinal waves was soon generalized to the case of spherical symmetry (L. V. Al'tshuler, 1946, F. A. Bakhshiyn, 1948, Ya. B. Lunts, 1949).

Some methods for the application of electronic computers to problems in the dynamics of plastic media were systematically studied by V. K. Kabulov (the results of his studies are summarized in his 1966 monograph).

The following part of this survey does not clarify exhaustively all aspects of the dynamics of nonelastic media. It is devoted to dynamic plasticity and visco-plasticity problems. It does not touch at all on problems dealing with the modeling of inhomogeneous and anisotropic media, visco-elastic media, fracture phenomena, superhigh pressure effects (on the order of the elasticity modulus) and also penetration effects. Experimental studies are almost not mentioned at all.

In this general introduction, the main attention was given to a survey of the initial development stages in each trend. An idea about their further development is given below. Extensive material on the topics in the survey can also be obtained from the monographs of I. I. Gol'denblat and N. A. Nikolayenko (1961), Kh. A. Rakhmatulin and Yu. A. Dem'yanov (1961), I. L. Dikovich (1962), L. P. Orlenko (1964), N. N. Popov and B. S. Rastorguyev (1964), Yu. Ya. Voloshenko-Klimovitskiy (1965), N. N. Popov and B. S. Rastorguyev (1966), and also from the papers read at the 2nd and 3rd All-Union Symposia on the Propagation of Elasto-Plastic Waves in Continuous Media (1966, 1969).

2.2. Propagation of Strong and Weak Distortional Waves

Beyond the elasticity limits the total deformation ϵ_{ij} of elasto-plastic bodies is represented as the sum of the elastic ϵ^{e}_{ij} and of plastic ϵ^{p}_{ij} parts. When the deformations are small, the relation between the elastic deformations and the stresses is determined by Hooke's law $\epsilon^{e}_{ij} = H_{ijkl}\sigma_{kl}$. It is assumed that the three-dimensional deformation is elastic, i.e., $\epsilon^{p}_{ii} = 0$. Then the ϵ^{p}_{ij} will be the components of the deviator of the deformations.

In deformation plasticity theory, it is assumed that the relations between the principal stresses depend only on the relations between the principal strains. The simplest variant of the isotropic relations between the plastic deformations and stresses has the form $\epsilon^{p}_{ij} = F(I_2)s_{ij}$, where $I_2 = 1/2s_{ij}s_{ij}$ (the s_{ij} are components of the stress deviator). The difference between nonlinear elastic and plastic deformations manifests itself only during unloading. For $dI_2 = s_{ij}ds_{ij} > 0$ loading occurs, for $dI_2 < 0$ unloading, according to Hooke's law and for $dI_2 = 0$, neutral loading. When $I_2 < k^2$, where k^2 is the plasticity constant, an elastic state occurs.

A number of studies made in the USSR have shown that the laws for the spreading of weak and strong fractures in nonlinear elastic and elasto-plastic media differs considerably from the classical case of the spreading of fractures in a linear elastic medium.

In the case of a three-dimensional nonlinear elastic medium, three types of waves are formed (V. M. Babich, 1954). If the displacement vector is continuous together with its first derivatives, and its second derivatives have discontinuities on some nonstationary discontinuity surface, the maximum and minimum propagation velocities of the waves depend on the direction. Thus, the stress field creates a unique kind of anisotropy, the fastest and slowest waves are neither longitudinal nor transverse waves. Waves moving with an intermediate velocity have the character of transverse waves. The direction of the distortional vector for these waves depends on the stress field; however, the rate at which they propagate is independent of the direction.

An analysis of the propagation of waves in a two-dimensional compressible plastic medium (G. A. Geniyev, 1959, 1961) has shown that the propagation velocities of lines of first order discontinuities differ from the local velocity of sound. They only coincide during the propagation of the first order in the direction of the principal normal stresses. The propagation velocity of first order discontinuity lines in the directions coinciding with the normals to the principal stesss component areas is zero. Any first order discontinuity line is a characteristic. In the case of stationary movement, real characteristics may exist even during subsonic velocities. The orientation of the characteristics depends both on the direction and magnitude of the modulus of the velocity vector and the orientation of the principal axes of the stresses.

In particular the propagation of strong and weak distortional waves has been studied in the plane deformed state of an ideal plastic medium assuming a linear relation between the first invariants of the stress tensors and the deformation rates (M. I. Estrin, 1961). The propagation of weak distortional waves in the plane stressed state has also been studied (M. I. Estrin, 1962, A. D. Chernyshev, 1966). The propagation of strong distortional waves in a medium with a nonlinear rigid[1] characteristic during loading characterized by linear unloading has also been studied (G. I. Bykovtsev, 1961).

A considerable simplification in the analysis of the propagation of distortional waves is introduced by the concept of piecewise linear yield surfaces and the associated yield law. First, such an analysis for weak distortional waves was made on the assumption that the stress point lies on the edge of a prism on the Tresk yield surface.

In the Prandtl-Reiss incremental plasticity theory (flow theory) it is assumed that the increment in the plastic deformations $d\varepsilon_{ij}^{p}$ is determined by the values of the stresses which are proportional to the increments in these stresses. When the loading surface coincides with the plastic potential surface, the increment in the plastic deformation will be orthogonal to the loading surface, and the simplest relation between the increments in the stresses and strains will have the form

1. A decrease or increase in the increment of the stresses for a given increment in the strains by comparison with the initial linear behavior characterizes, respectively, the "soft" and "hard" behavior of the material.

$$d\varepsilon^p_{ij} = G \frac{\partial f}{\partial \sigma_{ij}} df, \quad df = \frac{\partial f}{\partial \sigma_{kl}} d\sigma_{kl} > 0, \tag{2.1}$$

where f is the loading function, G is a scalar function depending on the stresses, strains and the loading history.

The propagation of weak distortional waves in isotropic ideal elasto-plastic media was studied most extensively. The most detailed analysis can be carried out for media satisfying piecewise-linear yield conditions (G. I. Bykovtsev, D. D. Ivlev and T. N. Martynova, 1966). For a stressed state corresponding to some edge of the yield prism, it was detected that three waves can propagate in the body in any direction, and that the three propagation velocities of the waves are real and independent of the characteristics of the edge. The maximum propagation velocity of the waves for the Tresk yield condition is $[(\lambda + 2\mu)/\rho]^{1/2}$ and it is attained when the wavefront coincides with the surface of the tangential stresses. For an arbitrary orientation of the normal of the wave surface relative to the principal axes, the wave is accompanied by a change both in the dilational and shearing strain. For a stressed state corresponding to an edge of the yield prism one wave propagates as an elastic wave with the velocity $(\mu/\rho)^{1/2}$, and it does not cause changes in the plastic deformations. The velocities of the other two weak distortional waves depend both on the direction of the normal to the discontinuity surface relative to the principal axes of the stress tensor and also on the form of the yield conditions.

In the case of an incompressible elasto-plastic material, two propagation velocities of the distortional waves exist. Both have a shearing character, and one of them does not cause changes in the plastic deformations.

Certain results of a more special character were obtained earlier for the plane deformed (M. I. Estrin, 1961) and plane stressed (M. I. Estrin, 1962, A. D. Chernyshev, 1966) states. The propagation of strong distortional waves in a medium with a nonlinear hard characteristic during loading characterized by linear unloading was also studied (G. I. Bykovtsev, 1961).

For an elasto-visco-plastic medium with defining equations in the form (D. D. Ivlev, 1959)

$$
\left.\begin{aligned}
&\varepsilon_{ij} = \varepsilon^{e}_{ij} + \varepsilon^{p}_{ij}, \quad \sigma_{ij} = \lambda\varepsilon^{e}_{kk}\delta_{ij} + 2G\varepsilon^{e}_{ij}, \\
&(s_{ij} - \eta\dot{\varepsilon}^{p}_{ij})(s_{ij} - \eta\dot{\varepsilon}^{p}_{ij}) = \sigma_T^2, \quad \dot{\varepsilon}^{p}_{ij} = \psi(s_{ij} - \eta\dot{\varepsilon}^{p}_{ij}),
\end{aligned}\right\} \qquad (2.2)
$$

where η is the viscosity coefficient, ψ is a positive multiplier and $\psi > 0$ for $s_{ij}s_{ij} \geq \sigma_T^2$, $\psi = 0$ for $s_{ij}s_{ij} < \sigma_T^2$ an analysis of the propagation of weak and strong distortional waves (G. I. Bykovtsev and N. D. Verveyko, 1966) has shown that the plastic velocity components of the deformations cannot be discontinuous: $[\dot{\varepsilon}^{p}_{ij}] = 0$. Acceleration waves in such a medium propagate at the velocity of elastic waves, which is $(G/\rho)^{1/2}$. The damping of the waves in the material under consideration is faster than in an elastic material. For developing and convex wave surfaces, the intensity of the waves with equal volume tends uniformly to zero when a plastic deformation occurs on both sides of the wave surface.

An interesting analysis of the propagation of distortional waves was made for a special dynamically hardening medium whose current state was determined by certain parameters characterizing the properties acquired by the material under a dynamic load (V. A. Skripkin, 1962).

2.3. Extremum Principles

The basic difficulty in formulating variational principles for nonelastic media is that such media are infinite dimensional mechanical systems with nonholonomic, nonideal relations for which the Lagrange principle does not hold. The exception is the movement of a hard ideal plastic body whose shape does not change with time (M. I. Reytman, 1965). It is natural that the extremum principle was first proposed for such a case. (A. R. Rzhanitsyn, 1959). Later a principle was proposed which was free of the limitation that was mentioned (V. P. Tamuzh, 1962). It requires that the functional I attain a minimum

$$
J = \frac{1}{2}\int\limits_V \rho\ddot{u}_i\ddot{u}_i\,dV - \int\limits_V P_i\ddot{u}_i\,dV - \int\limits_{S_T} T_i\ddot{u}_i\,dS + \int\limits_V \sigma_{ij}\ddot{\varepsilon}_{ij}\,dV = \int\limits_V F\,dV, \qquad (2.3)
$$

where ρ is the density, the $\ddot{u}_i$ the accelerations, the P_i the body forces, the T_i the surface forces, σ_{ij} the components of the stress tensor, ε_{ij} the components of the strain tensor, V the volume of the body, S_T the part of the surface of the body on which the external forces are given. The kinematically possible accelerations of the strains ε_{ij} are related to the components of the acceleration vector by the formulas

$$\ddot{\varepsilon}_{ij}=\frac{1}{2}(\ddot{u}_{i,j}+\ddot{u}_{j,i}). \tag{2.4}$$

The principle remains valid when the accelerations have strong discontinuities. In this case for a medium without discontinuities, a constraint on the discontinuity surfaces must be added to the Gaussian constraint

$$\int \sigma_{ij}\nu_j[\ddot{u}_i]\,dl, \tag{2.5}$$

where ν_j are the components of the normal to the discontinuity surface l. It is easily seen that the Ostrogradskiy-Euler equations for the functional I are the equilibrium equations for a continuous medium. Subsequently, it was shown (M. I. Reytman, 1965) that this principle can be generalized to media with very general properties. The properties of the media are included in the last term in the functional I. For the Weirstrass-Erdman conditions for I, conditions on the first order discontinuity surfaces of the accelerations can be obtained

$$[\sigma_{ij}]\cos(\nu x_j)=0,\quad [F]=\left[\frac{\partial F}{\partial \ddot{u}_i}\right][\ddot{u}_{i,j}]. \tag{2.6}$$

It should be mentioned that the general variational principle with the aid of which the invariant equations of motion which determine the equations (model) and various additional conditions (boundary conditions, initial conditions on the discontinuity surface, etc.) are found was formulated by L. I. Sedov (1965). This principle was used to study discontinuities in a solid medium by M. V. Lur'e (1966).

§3. Propagation of Waves in Nonelastic Media

3.1. Elastoplastic Bodies

The first two parts of this section deal with plane waves. Plane waves are divided into two classes: plane stress waves and plane strain waves. The first occur in rods and are characterized by the three-dimensional deformed state and the one-dimensional stressed state (more precisely nearly one-dimensional). The second waves are formed in plates and are characterized by the three-dimensional stressed state and the one-dimensional deformed state (see, for example, G. S. Shapiro, 1952).

The deformation rates for elasto-plastic bodies do not enter the defining law in explicit form. The dynamic character of the load is taken into account by using a different relation $\sigma = \sigma(\varepsilon)$ between the stresses and strain than in the static case.

The propagation of longitudinal loading waves is described by the nonlinear hyperbolic equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}. \tag{3.1}$$

where u is the longitudinal displacement, $a^2 = (1/\rho)(d\sigma/d\varepsilon)$, and ρ is the density of the material. Under a monotonic continuous load on the boundary of a semiinfinite medium for $d^2\sigma/d\varepsilon^2 < 0$, shock waves are not formed in the body and for $d^2\sigma/d\varepsilon^2 > 0$, shock waves occur (Kh. A. Rakhmatulin, 1945, G. S. Shapiro, 1946). The general case of similar solutions of equation (3.1) was investigated, which included the possible change of sign of $d^2\sigma/d\varepsilon^2$ and the occurrence of shock waves (G. I. Barenblatt, 1953, 1957). The propagation of similar perturbations under the action of a constant load applied to the end of a semiinfinite rod was also considered (G. Ya. Galin, 1958).

Beyond the elasticity limits, the relation $\sigma = \sigma(\varepsilon)$ for elasto-plastic media has a different form during loading and unloading. The problem of the propagation of elasto-plastic waves in a semiinfinite medium for $d^2\sigma/d\varepsilon^2 < 0$ on the assumption that unloading takes place in accordance with the linear

elastic law was analyzed for the first time by Kh. A. Rakhmatulin (1945). Let x be the longitudinal coordinate and t be time. In the case of a semiinfinite medium, the region (x, t) is divided into two parts. In one part loading and in the other part unloading occurs. The difficulty of solving the corresponding system of two hyperbolic equations is related to the fact that the boundary between the zones that were mentioned, which is called the unloading wave, is not known in advance. Various solution methods were proposed for the case when the unloading wave is a weak distortional wave, namely the method of power series (Kh. A. Rakhmatulin, 1945) the method of characteristics (G. S. Shapiro, 1946, V. L. Biderman, 1952) the graphical method (S. D. Mochalov, 1952) and other methods.

Existence and uniqueness problems for the unloading wave were studied by A. M. Skobeyev (1962), who has also shown that as $t \to \infty$ the propagation velocity of the unloading wave tends asymptotically to the propagation velocity of elastic waves.

The case of an impact load during which the unloading wave is a strong distortional wave, was also studied in great detail (Kh. A. Rakhmatulin and G. S. Shapiro, 1948, V. S. Lenskiy, 1949, N. F. Lebedev, 1952). This case is important since it is encountered in problems dealing with longitudinal collisions of rods beyond the elasticity limit (V. G. Cheban, 1952, R. I. Nadeyeva, 1953). Taking into account simultaneously the local crimping and the propagation of the waves is of interest in such problems (S. A. Zegzhda, 1965). It was possible to detect the existence of a dimensionless parameter which determined the process (including the collision time and the increase in the contact force, the maximum value of the contact force and the recovery coefficient). In addition to this for a semiinfinite rod and a rod of finite length, using the condition for the equality of the potential energy of the deformation, it was possible to linearize the relation between the contact force and the local crumpling.

Progress in the study of the propagation of plane elastoplastic waves was reflected both in the perfected analytical methods and in the application of electronic computers.

Considerable simplifications in the analytical procedure were achieved by means of useful approximations of the relation between the stresses and strains or by transformations of the original system of equations. Thus, if we write the equations of motion in the form (G. A. Dombrovskiy and G. V. Litvinov, 1966)

$$\frac{\partial h}{\partial v} = a(v)\frac{\partial t}{\partial u}, \quad \frac{\partial h}{\partial u} = a(v)\frac{\partial t}{\partial v}, \tag{3.2}$$

where

$$a^2 = \frac{1}{\rho_0}\frac{d\sigma}{d\varepsilon}, \quad v = \int a\, d\varepsilon, \tag{3.3}$$

and h is the Lagrangian coordinate, t is time, u is the velocity, ρ_0 is the initial density and take a(v) in one of the following forms:[1]

$$a(v) = n^2 \operatorname{tg}^2 (mv), \quad a(v) = n^2 \operatorname{th}^2 (mv), \quad a(v) = n^2 \operatorname{cth}^2 (mv) \tag{3.4}$$

Key: a. tan
b. tanh
c. coth

(here m and n are arbitrary constants), we can obtain from the formulas

$$\varepsilon(v) = \int \frac{dv}{a(v)}, \quad \sigma(v) = \rho_0 \int a(v)\, dv \tag{3.5}$$

three families of relations between the stresses and strains $\sigma = \sigma(\varepsilon, m, n, C_1, C_2)$, each of which depends on the four parameters m, n, C_1 and C_2 (C_1 and C_2 are arbitrary integration constants in (3.5)). If the given relation $\sigma = \sigma(\varepsilon)$ belongs to one of the three classes, the problem has an exact solution. In the contrary case, simple approximate solutions can be obtained. Examples of using such a transformation are given by G. A. Dombrovskiy and G. V. Litvinov (1966) and G. V. Litvinov (1965)

1. These approximations have been proposed earlier by G. A. Dombrovskiy (1963) for the equations of gas dynamics.

Taking the relation between the stresses and strains in the form of segments of two lines (a straight line and a parabola) made it possible to study the decay phenomenon of an arbitrary fracture during the interaction of the unloading wave and a shock wave on the example of the problem of a rod of finite length to one end of which a constant load was applied for a certain period and then an instantaneous compression force (A. I. Buravtsev and N. A. Yesenina, 1966). Using the relation

$$\sigma = \sigma_* - \frac{A}{(\varepsilon + b)^k}, \tag{3.6}$$

where A, b, σ_* and k are the material constants, the problem of a rod of finite length, one of whose ends moves with a constant velocity was solved (A. I. Buravtsev, 1965).

A number of interesting solutions was obtained with the aid of piecewise-linear, bilinear (L. R. Stavnitser, 1964) and trilinear (A. P. Sinitsyn, 1964) approximations of the $\sigma - \epsilon$ diagram. The last case made it possible to study the propagation of waves in a hardening elasto-plastic layer. With regard to the use of electronic computers, the advantage of using computer technology in the case of the method of characteristics was demonstrated by N. A. Nesterenko (1964). The problem of the damping of a one-dimensional wave during exponential damping of the pressure at the end of the end of a rod was also studied (L. P. Orlenko and G. F. Yefremova, 1965).

The particular features of the propagation of elasto-plastic waves in rods with a variable yield point which are important in the study of multiple impacts on the rod were studied by Kh. A. Rakhmatulin (1946).

From a theoretical and practical point of view, the problem of the reflection and refraction of a plane plastic wave in the presence of a boundary surface is important. It is not surprising that it attracted a great deal of attention among investigators. However, the studies dealing with this problem (G. M. Lyakhov and N. I. Polyakova, 1962, N. V. Zvolinskiy and G. V. Rykov, 1963, 1965, G. M. Lykhov, R. A. Osadkhenko and N. I. Polyakov, 1965, G. M. Lyakhov, 1966, Z. V. Narozhnaya,

1966, N. V. Zvolinskiy, 1967) either contained solutions based on the simplest approximation of the compression law, or did not take into consideration the boundary surface, or led to a complex analytical description from which it was difficult to draw any conclusions.

The assumption of rigid unloading (N. V. Zvolinskiy, 1967) made it possible to study certain general characteristics of the reflection problem and also investigate the character of the phenomena on the basis of numerical solutions. It became evident that the usual a priori assumption that unloading occurs in the region of the reflected wave is erroneous, although usually the errors resulting from this error are small. In particular, when the compression law is linear, the unloading hypothesis is justified. A study of the effect of the boundary surface with the given stresses on it on the propagation of the reflected wave has shown that the reflected wave begins to "feel" the external load immediately after the reflection begins. Initially this effect is small, but gradually, as it increases, it becomes very significant and finally it leads to the destruction of the shock wave which cannot reach the surface, except in the case of a stationary wave. It turned out that this fact, which was noticed in special cases (see, for example, Z. V. Narozhnaya, 1965), has a general character.

In certain materials, for example, annealed low carbon steel, the so-called "lagged yield effect" was detected. It became evident that when a constant pressure exceeding the static yield point was suddenly applied, the plastic deformation does not occur immediately, but after a certain time. To every particular value σ of the stress corresponds it own lagged yield time t_1. When the applied load increases with time, usually the formula

$$\int_0^{t_1} \left(\frac{\sigma}{\sigma_*}\right)^{\alpha} dt = C, \tag{3.7}$$

is used for t_1 where σ_*, α and C are constants of the material. Formula (3.7) which agrees well with the experimental data was proposed

on the basis of theoretical concepts by George Campbell, who used as the yield criterion a critical value of the density of the liberated dislocations where the quantity t_1 must not exceed 1 sec. The effect of temperature on the yield lag was studied by Yu. Ya. Voloshenko-Klimovitskiy (1962, 1965). A theory for the propagation of longitudinal elasto-plastic waves in rods, taking into account the yield lag effect, was proposed by Yu. N. Rabotnov (1967).

Effects connected with the propagation of plane waves during thermal shock in an elastic medium were studied by V. I. Danilovskaya (1952). An analogous problem for an elasto-plastic material with linear hardening was studied by Yu. P. Suvorov (1964), who studied a thermal shock at the end of a semiinfinite rod with a linear law for the temperature increase over time (the heat conductivity coefficient was assumed to be proportional to the temperature and the mechanical characteristics of the material to be independent of the temperature). For such a law, the nonlinear heat conductivity equation has a simple solution which simplifies considerably the equation for the propagation of elasto-plastic waves. It became evident that when the propagation velocity was equal to the velocity with which the elastic or plastic perturbations propagate, strong dislocational waves are formed.

One-dimensional problems dealing with the propagation of waves in the complex stressed state are of considerable interest. Kh. A. Rakhmatulin (1952) formulated this problem and obtained a solution for it on the basis of deformation plasticity theory for the case of a torsional-compressional shock. Later an analogous problem was studied for a shearing-compression shock (Kh. A. Rakhmatulin and V. S. Antsiferov, 1964). Subsequently, this problem attracted a great deal of attention abroad. A detailed study of a shearing-compression shock on the basis of the Prandtl-Reiss theory was made by A. M. Skobeyev (1965).

3.2. Visco-elastic-plastic Bodies

In spite of the fact that an elasto-plastic model reflects correctly the dynamic behavior of metals in many cases, the studies on the propagation of nonlinear waves in solids that were made in the last two decades are characterized by a critical approach to the theory of elasto-plastic waves and an attempt to improve it. Certain experimental facts have been detected which cannot be explained on the basis of the model of an elasto-plastic body. This pertains primarily to observations of the propagation of additional load impulses (waves) in stressed rods beyond the elasticity limits. The theory of the propagation of elasto-plastic waves

predicts that the rate at which an additional load impulse propagates on a deformed rod is determined by the inclination of the dynamic pattern during the given deformation. However, experiments (see, for example, M. V. Malyshev, 1961) have shown that in metallic rods the front of the additional impulsive load propagates for any preliminary deformations with the velocity of elastic waves. There is reason to assume that the effect connected with the plastic deformation rates depends on the quantity $\sigma - f(\varepsilon)$ which represents the overshoot of the instantaneous stress over the stress corresponding to the same deformation during the static test. Therefore, usually for a model of an elasto-visco-plastic medium the following deformation law is used

$$\left.\begin{array}{ll} E\dot{\varepsilon} = \dot{\sigma} & \overset{a}{\text{при}}\ \sigma \leqslant \sigma_s, \\ E\dot{\varepsilon} = \dot{\sigma} + \Phi(\sigma - f(\varepsilon)) & \overset{a}{\text{при}}\ \sigma > \sigma_s, \end{array}\right\} \qquad (3.8)$$

Key: a. for

where σ_s is the static yield point, or even the more general law

$$\left.\begin{array}{ll} E\dot{\varepsilon} = \dot{\sigma} & \overset{a}{\text{при}}\ \sigma \leqslant \sigma_s, \\ E\dot{\varepsilon} = \dot{\sigma} + g(\sigma, \varepsilon) & \overset{a}{\text{при}}\ \sigma > \sigma_s, \end{array}\right\} \qquad (3.9)$$

Key: a. for

which agrees with dislocation theory. When the deformation rates are high, the model of a visco-elasto-plastic medium behaves elastically with the E modulus, which is why it explains the propagation of the additional load waves at elastic wave rates.

When the equation of motion are added to these defining equations

$$\frac{\partial\sigma}{\partial x} = \rho\frac{\partial v}{\partial t} \qquad (3.10)$$

and the continuity equations

$$\frac{\partial \varepsilon}{\partial t} = \frac{\partial v}{\partial x} \qquad (3.11)$$

a system of equations of the hyperbolic type is obtained which has three families of characteristics: $dx \pm c\,dt = 0$, $dx = 0$, where $c = (E/\rho)^{1/2}$. Here

$$
\begin{array}{lll}
\text{a вдоль } dx = & c\,dt & d\sigma - \rho c\,dv = -g(\sigma,\ \varepsilon)\,dt, \\
\text{a вдоль } dx = & -c\,dt & d\sigma + \rho c\,dv = -g(\sigma,\ \varepsilon)\,dt, \\
\text{a вдоль } dx = 0 & & E\,d\varepsilon - d\sigma = g(\sigma,\ \varepsilon)\,dt.
\end{array}
$$

Key: a. along

When the differential relations are replaced along the characteristics by finite difference equations, the problems can be solved numerically. The first solutions in this domain were obtained in this way by V. V. Sokolovskiy (1948) for the case $f(\varepsilon) = \sigma_s$, i.e., in the case of a material without deformation hardening.

From the mathematical point of view, the most thorough study of the system of equations (3.8), (3.10), (3.11) was made by V. N. Kukudzhanov (1965, 1967). In his first study he obtained a solution for the problem of the propagation of an elastic unloading wave and in the second the system of equations was analyzed using an asymptotic method on which the computations on the electronic computer were based.

The propagation of longitudinal waves in a semiinfinite rod consisting of two parts with different yield points was studied on the basis of the assumption $f(\varepsilon) = \sigma_s$ (V. N. Kukudzhnov and L. V. Nikitin, 1966). Various cases were considered. In particular, if in the zone adjacent to the end to which the shock is applied, the yield point is higher than in the farther zone, and the magnitude of the shock is selected so that the yield point is exceeded only in the second zone, the first part of the rod remains elastic. By introducing new variables, both equations reduce to a form to which the Laplace transform is easily applied.

Certain generalizations of the deformation law (3.8) were studied. In particular, an analysis of the propagation of the load waves for a medium with a nonlinear viscosity of the type

$$\frac{\partial \varepsilon}{\partial t} = \frac{1}{E(\varepsilon)} \frac{\partial \sigma}{\partial t} + \Phi(\sigma/f(\varepsilon)) \tag{3.12}$$

has shown (A. M. Skobeyev, 1967) that in such a medium the perturbations can propagate with a velocity which is different from the elastic velocity (the motion is assumed to be nearly similar).

The yield lag phenomenon, taking into account viscous effects was studied by V. S. Lenskiy and L. N. Foina (1959) and V. A. Kotlyarevskiy (1962).

When a linear visco-plastic model was used (which ignored elastic deformations), the velocities and stresses in the region where the plastic deformations are formed must satisfy the heat conductivity equation. A number of well-know solutions from the theory of heat conductivity can be directly applied to problems dealing with the propagation of perturbation in visco-plastic bodies. For example, the problem of a shock with a constant velocity on a semiinfinite visco-plastic rod is equivalent to the problem of the sudden heating of a semiinfinite rod, at the end of which the temperature suddenly increases and remains constant (V. V. Sokolovskiy, 1949). In the case of a visco-plastic body with hard unloading, the analogous problem reduces to the Stefan problem in the theory of heat conductivity (G. S. Shapiro, 1966).

Great interest was shown in the problem of the impact of a visco-plastic rod of finite length on a hard obstacle. Its solution has shown (G. I. Barenblatt and A. Yu. Ishlinskiy, 1962) that during the impact, the rod is divided into two parts. In one part, the part adjacent to the end in which the impact occurs, visco-plastic flow occurs and the other part moves as a solid. The position of the moving boundary is determined during the solution of the problem. The validity of this scheme was proved by A. M. Skobeyev (1966).

First the system of basic equations for the problem was solved approximately using averaging methods that are used in the theory of the boundary layer. Subsequently, the same problem was solved using a discrete technique (A. Yu. Ishlinskiy and G. P. Sleptsova, 1969). The rod was replaced by a system of

concentrated masses connected by visco-plastic rods. The solution of the heat conductivity equation with a moving boundary reduces to the solution of ordinary differential equations.

The propagation of the perturbations during the impact of a solid on a semiinfinite visco-plastic rod taking into account the linear deformation hardening was investigated by I. N. Zverev (1950). Later the problem was generalized to the case of an elasto-visco-plastic material (G. L. Komissarova and S. A. Lezhov, 1965).

The dynamic stability of rods beyond elasticity limits is of considerable interest. This problem, taking into account the yield lag effects and viscosity was considered by A. K. Pertsev and A. Ya. Rukolayne (1965).

3.3. Spherical, Cylindrical and Multidimensional Waves

When elasto-plastic waves are formed in a semiinfinite rod, the plastic deformations propagate to infinity (it is easily shown that the unloading wave never catches up with the front of the elastic wave). In the case of spherical and cylindrical waves the plastic deformations propagate only over a finite distance.

The problem of the propagation of a spherical or loading wave was first formulated by L. V. Al'tshuler (1946). The solution for a loading wave which is valid until the strong distortional wave separating the elastic and plastic deformation regions spreads, was obtained by F. A. Bakhshiyan (1948). A complete study of the problem of the propagation of loading and unloading waves, including the instant at which the strong distortional wave spreads, was carried out by Ya. B. Lunts (1949).

A study of the propagation of cylindrical shearing waves has shown (Kh. A. Rakhmatulin, 1948) that in the case of linear hardening of the material a drop in the velocities and deformations on the front of the elastic waves is inversely proportional to the square root of the distance from the center of symmetry. The problem of the stresses in a cylindrical pipe made from an ideal plastic incompressible material to which a load is applied suddenly is relatively easy to analyze. It reduces to the integration of an ordinary first-order nonlinear differential equation (Ye. Kh. Agababyan, 1953). In the case of a compressible material, with the same compression modulus both in the region of elastic and plastic deformations, the problem is solved by the method of characteristics (Ye. Kh. Agababyan, 1955). The presence of special types of waves propagating from the internal surface of the cylinder

at the same rate, which are subsequently separated, was detected.

Analogous problems in the propagation of perturbations during spherical and cylindrical symmetry taking into account the viscous effect were also obtained. The numerical solutions based on the elasto-visco-plastic model were found for cylindrical shearing waves (V. V. Sokolovskiy, 1948), for spherical compression waves (V. N. Kukudzhanov, 1959) and for cylindrical pressure waves (L. V. Nikitin, 1959).

Problems on the deformation of a cylinder under the action of internal pressure were solved on the basis of the visco-plastic model (A. A. Il'yushin, 1940), as well as problems in the propagation of cylindrical shearing waves (P. M. Ogibalov, 1941, F. A. Bakhshiyan, 1948).

A number of explosion problems under spherical symmetry conditions oriented toward the dynamics of soils[1] were solved.

Successes in the solution of higher dimensional dynamic problems on the basis of the plastic model of bodies were only achieved in the last decade. In particular, certain methods from gas dynamics were used in the process. It is known that when the flow around a thin body has a supersonic velocity, the medium moves mainly along surfaces which are perpendicular to the direction of flight, which considerably simplifies the analysis. This was used in the solution of problems in the propagation of waves in a halfspace on the boundary of which normal pressures are acting. Here the characteristic direction of motion which coincides with the direction in which the pressure acts can be isolated. An approximate solution for an elasto-plastic halfspace under the action of normal pressure on the part of the boundary was obtained on the basis of this concept by Kh. A. Rakhmatulin (1959).

It was possible to reduce the system of equations for the plane deformed movement of a compressible ideal plastic medium to the wave equation for the motion of a barotropic gas (G. A. Geniyev, 1962). Using the method of natural coordinates used in gas dynamics, it was possible to construct approximate techniques for the solution of the equations of the plane deformed motion of a rigid-plastic and elasto-plastic medium (O. D. Grigor'ev, 1962).

1. The survey by S. S. Grigoryan and V. A. Ioselevich (pp [these pages (203-225) are missing from the Russian text]) is devoted to the mechanics of soils. (editors).

Similar movements of an incompressible ideal plastic body under plane deformation conditions were studied by M. I. Estrin in plane stressed state conditions (1958) and in the case of a compressible medium obeying the Tresk yield condition (1962). The problem of a stepped load with a constant velocity was studied by A. M. Skobeyev (1965)

§4. Dynamics of Nonelastic Structures

By structures we will mean bodies in which one or two dimensions exceed considerably the third dimension. These include rods, pliable wires, membranes, beams, frames, plates and shells. A survey of the studies on the propagation of waves in rods is given in Section 3.

4.1. Pliable Wires and Membranes

The difficulties of constructing an elasto-plastic theory of transverse shock along wires are connected with the necessity of taking into account the double nonlinearity of the problem (the deviations of the wire from the original shape, the nonlinear form of the relation between the stresses and strains) as well as the conditions for the contact of the wire with the body to which the shock is applied.

In the theory developed in several studies by Kh. A. Rakhmatulin (1945, 1947, 1952), problems of the propagation of longitudinal and transverse waves in wires were separated. In the first study the solution of the problem of a shock along a pliable wire of infinite length was obtained when the body making the impact moves with a constant velocity. Analytically the problem reduces to the solution of two differential equations in the two displacement components. In particular, the practical important case was considered when the load-elongation curve of the wire can be represented by a broken line consisting of two segments (bilinear law). In addition the normal shock by a body of finite mass with infinitesimally small dimensions was considered. The strain formed as a result of the shock immediately after the collision reduces the velocity of the body. At the same time a Rieman wave propagates simultaneously to the right and left from the collision point. The subsequent solution depends on the postulated relation between the velocities of these waves. The first solutions dealt with infinite wires. I. N. Zverev (1950) considered the impact of a wire on a wedge. He introduced in his study an impulsive force acting on particles of the wire on the boundary of the adjacent and free movement regions. In problems dealing with wires of finite length the pattern becomes more complex due to reflections from the ends. In these cases numerical methods turned out to be most effective. The subsequent improvements took into account real loading conditions.

Thus Kh. A. Rakhmatulin studied the impact of a body of given form (for example, a blunt wedge) on a pliable wire.

We note that the relative analytical simplicity of the problem of a wire made it possible to use it in indirect methods to determine the mechanical characteristics of materials.

A. A. Ryabis (1966) developed this trend. In his formulation, an infinitely long pliable rectilinear wire impinges on a blunted body and breaks down into two regions, the free movement region of the wire and the region where the wire is adjacent to the body. The equations of motion and the continuity equations are derived for the boundary of these regions, which is called a strong distortional wave. In both cases it is assumed that the movement satisfies the wave equations. The problem is closed by adding geometric conditions. The relation $\sigma(\varepsilon)$ is selected on the basis of the linear hardening scheme. The deformation of the wire and the adjacent region until the instant when the transverse wave penetrates the rectilinear part of the wire was found. A. A. Ryabis introduced a frictional force caused by the impulsive force in the region adjacent to the strong distortional wave. It was shown that the frictional force is substantially greater than the tangential stress formed on the strong distortional wave. Due to this, it was possible to introduce the additional condition that the tangential velocity of the particles in the region adjacent to the strong distortional wave be zero. This made it possible to determine the deformation of the particles that were mentioned. The deformation was positive, i.e., elongation occurs on the strong distortional wave, unlike in the preceding results that were mentioned above, in which compression occurred and an unloading wave had to be introduced in the adjacent region. The conditions for the applicability of the above scheme were also analyzed.

New numerical and analytical methods made it possible to expand the class of problems on wires that were studied.

A. L. Pavlenko, B. M. Pavlov and G. S. Roslyakov (1965, 1966) studied numerically the movement of nonlinear elastic wires.

N. N. Popov and B. S. Rastorguyev (1966), using a trigonometric series expansion along the length obtained an approximate solution of the problem of the movement of a wire made from an elasto-plastic material with a bilinear hardening law, as well as for a material whose deformation law has the form

$$\sigma = \sum_{k=1}^{n} E_k \varepsilon^k, \qquad (4.1)$$

under the action of a uniformly distributed load. They also studied the same problem for an elasto-visco-plastic material. The movement of an elasto-visco-plastic wire in the plastic stage is described by a second order nonlinear differential equation which was solved by the authors using the numerical Adams-Störmer method. P. A. Rakhmanov (1959) added to the conditions of the problem for the shock along the wire the effect of the resistance of the medium to its motion.

The analogy in the mechanical behavior of pliable wires and thin membranes made it possible, soon after the shock phenomenon along a pliable wire was investigated, to analyze the analogous problem of a pliable membrane. The first approximate solution, with the condition that the circular stresses in the circular membrane were ignored, was obtained by D. M. Grigoryan (1949). In the series of studies which followed, this assumption was removed. Thus, M. P. Galin (1949) studied the impact at a single point of a body moving with a constant velocity on a circular membrane. Later, the impact of an axisymmetric body along a membrane was studied (U. Bektursunov 1966). In the last case it was assumed that the radial and transverse motion are not related, and that the solution of the problem can be obtained by integrating separately two different equations for the propagation of the waves.

A more complex problem was studied by S. M. Belonosov, A. L. Pavlenko, B. M. Pavlov and G. S. Roslyakov (1966), who studied the impact of an absolutely rigid cylinder on a membrane. The initial jump in the velocity is transferred along the membrane in the form of two waves: the transverse and longitudinal wave.

4.2. Beams

A systematic discussion of the theory of dynamic loads on beams is available in the monographs by Kh. A. Rakhmatulin and Yu. A. Dem'yanov (1961), I. L. Dikovich (1962), I. I. Gol'denblat and N. A. Nikolayenko (1961).

Obtaining sufficiently accurate solutions for the dynamic loading of elasto-plastic beams is beset by serious difficulties, which can only be overcome in individual cases in the loading and support of beams. The study of I. L. Dikovich (1962) describes a solution for the movement of a freely supported beam under the action of a suddenly applied

uniform load which is constant over time and does not exceed in magnitude the limiting static load. At a particular instant a plastic joint is formed in the middle of the beam, after which the movement of the two halves of the beam is considered, which are analyzed to obtain the expression for the displacement which remains valid until the angular deformation in the plastic joint changes sign. For the hardening I. L. Dikovich proposed approximate methods, for example, the Bubnov-Galerkin method. One term of the approximating series was retained, which is often done in nonlinear problems. It was necessary to introduce the assumption that the plastic joints are stationary, which, as is well known, is no longer justified as the intensity of the sudden load increases and may lead to serious errors. The use of electronic computers in the calculation of beams has great promise. Thus, V. K. Kabulov (1963) used a system of unequal concentrated masses suspended to an imponderable elasto-plastic element for the representation of the flexural oscillations of a cantilever beam.

The mechanical behavior of beams made from reinforced materials has specific features. Reinforced concrete beams have a number of special features (N. N. Popov and B. S. Rastorguyev, 1964). This is due to the fact that the work of reinforced concrete elements breaks down into four stages: 1) from the loading instant until a crack appears in the expanded concrete zone, 2) from the end of the first stage until the beginning of reinforcement yield, 3) from the end of the second stage until the fracture of the compressed concrete zone, 4) loss of loading bearing capacity by the structure. In nonreinforced structures, the third stage does not occur and the brittle fracture of the concrete occurs immediately after the end of the second stage.

The character of reinforced concrete beams is connected primarily with the intensity of the load which determines the origin of the work stages of the material that were mentioned. All stages except the first, require that plastic deformations be taken into account, and in the second and third stages, a damping oscillatory process may occur. In the case when the load bearing capacity is lost the results of rigid-plastic analysis can be used, taking as the limiting plastic moment the corresponding limiting value for reinforced concrete sections. The problem of the movement of a beam undergoing brittle fracture is studied in an analogous manner. The relation between the angle of rotation and the moment used has the form of the bilinear weakening law. Since, according to this curve, the resistance drops as the bends increase and finally becomes zero, it is possible to find for each type of load a value of the bend for which the structure will break down when this value is exceeded.

N. N. Popov and B. S. Rastorguyev also considered the movement of a reinforced concrete beam in the first three stages taking into account the linear dependence of the reinforcement yield point on time. The breakdown scheme for the beam adopted, was as before, a mechanism with one joint in the center.

An analysis of the results that were obtained for a concrete beam showed considerable differences from the calculations obtained on the basis of the Prandt scheme. It became evident that as the intensity of the impulse increases, the dynamic yield point tends to a constant value equal to 1.75 M_0 (M_0 is the static limiting plastic moment).

As was already mentioned, the considerable mathematical difficulties which arise during the solution of elasto-plastic problems and also the fact that under intense loads the elastic work stages of the beam can be ignored, create conditions for the application of rigid-plastic analysis. Variational principles can be applied successfully (see, for example, the solution obtained by A. R. Rzhanits (1959) describing the movement of a beam on two supports, in which the result obtained by the variational method coincides with the exact result).

At the same time the application of rigid-plastic analysis makes it possible to take into account certain additional factors which cannot be accounted for by elasto-plastic analysis. Among these we will include the effect of the external medium on the movement of the beam. The movement of rigid-plastic beams in a resisting medium was first studied by G. S. Shapiro (1962). As he further developed this study, A. A. Amandsov (1965) considered the movement of a rigid-plastic beam in a resisting medium under the action of a concentrated force in which the velocity of the movement of one section was given at an arbitrary instant of time. It was assumed that the resistance of the medium depended on the rate at which the beam was displaced. For a special given displacement function and a fixed cross section of the beam the problem was solved in quadratures.

A number of solutions for problems dealing with the movement of rigid ideal plastic beams is available in the book of I. L. Dikovich (1962). In particular, it includes solutions of problems dealing with the movement of infinite beams when one section is displaced at a constant rate and a concentrated force is acting in some section, the movement of a beam of finite length which is not supported under the action of a concentrated load, and the movement of a freely supported beam under the action of a load distributed along a parabola.

4.3. Arcs and Frames

V. P. Tamuzh (1962) studied the movement of a circular rigid-plastic arc under the action of a concentrated load applied at the center. It was assumed that the movement of the arc, like a static deformation, takes place while three plastic joints are formed. Next, the author used, to determine the two independent parameters characterizing the deformation mechanism, the same variational principle developed by him, as a result of which the problem reduced to the solution of two transcendental equations. To validate the correctness of the solutions that were obtained, it is also necessary to verify that the yield point is not exceeded in the rigid parts of the arc. The pattern of motion that is obtained is generally satisfactorily confirmed by the experiment. The study that was mentioned is also interesting since it is a first example in which quadratic programming is used in the dynamics of a nonelastic body. If the arc is broken up into n equal parts, according to (2.3), the problem reduces to finding the minimum of a quadratic function subject to linear constraints, i.e., to a problem in quadratic programming. The author proposed that the Wolfe method be used for the solution of this problem.

An analogous approach from the standpoint of the kinematic mechanism was applied to circular reinforced concrete arcs by N. N. Popov and B. S. Rastorguyev (1966), who found an expression for the bends in elasto-plastic arcs under the action of a symmetric and nonsymmetric load.

In all the studies that were mentioned dealing with the movement of arcs, the effects of the normal and transverse forces on the load bearing capacity was ignored. Judging by the effects of these factors that were investigated for rectilinear beams, they may have a considerable effect on the deformation pattern.

Starting in the 60's, many studies appeared which dealt with the description of the dynamic behavior of multirod systems in which plastic joints are formed.

The studies of G. V. Ivanov, Yu. V. Nemirovskiy, Yu. N. Rabotnov (1963) studied the dynamics of cross beams covering a rectangular span located at equal distances from one another. Depending on the relation between the spans, the distances between the beams and the limiting plastic moments, two cases can occur: 1) the cross beams remain stationary during the entire movement and each principal beam behaves like a continuous beam on s supports (s is the number of cross beams);

2) after the movement of the beams in the principal direction began, the load bearing capacity of the cross beams is exhausted For each case the equations of motion were set up for the principal and cross beams. The form of the equations of motion and the number of joints depends on whether the number of beams with the same direction is even or odd. Thus, when the number of cross beams is even, the problem reduces to the solution of a system of linear differential equations.

The calculation of frames for dynamic effect is carried out mainly in connection with their checking for seismic loads. This extremely complex and topical problem is currently the center of attention of scientists, and here the plastic deformations must be taken into account. The requirement that the deformation resulting from the seismic action in the body of the equipment remain elastic leads to an excessive use of materials. The mathematical difficulties connected with the calculation of frames in the elasto-plastic work stage, as well as in three-dimensional structures, are usually overcome by reducing the number of degrees of freedom of the system and by concentrating the mass at one or several points. Most frequently the frame is reduced to a system with one degree of freedom, a cantilever with the mass concentrated at the end. A systemmatic discussion of this approach and its generalization to systems with two degrees of freedom is available in the monograph by I. I. Gol'denblat and N. I. Nikolayenko (1961). The authors consider the movement of a system with one degree of freedom when the material of the load bearing element is determined by the Prandtl diagram under the action of an instantaneous rectangular impulse. The work of frames under seismic loads is characterized by a complete fracture of the elements at points at which the largest bending moments are acting. For this reason not plastic but ideal joints are formed at these points. From the mathematical point of view the solution of such problems does not present additional difficulties compared to elastic calculations, although the results differ considerably. This difference is also due to the fact that seismic loads acting on the structure depend on the magnitude of the reaction of the structure and the latter decreases considerably when plastic deformations are taken into account and individual links are disconnected from work.

The same monograph contains a presentation of problems on the movement of systems in which the load bearing element is strengthened under the action of an instantaneous rectangular, sine wave and exponentially decreasing pulse. As a generalization a system with two degrees of freedom is studied, in which the material of the load bearing elements obeys the Prandtl scheme (the unloading is parallel to the direct loading). Free oscillations of the system that was described are studied under the condition that at the initial instant it receives a given velocity. Such an approach can be used in the first approximation to determine the residual deformations.

It should be noted that the determination of the seismic loads during the calculation of structures is highly arbitrary. A more accurate approach is to take into account the accelograms of real earthquakes, which in the general case, is done using the theory of random processes. However, a deterministic approach to the problem can also be used as an approximation, when the input actions operate with the mathematical expectations of the accelerations of the base of the structure. At about the same time, in the early 60's it became evident that the possibilities of an analytical approach to the problem of the dynamic calculation of nonelastic frames have been exhausted for all practical purposes, and that it was necessary to make a transition to numerical methods based on the use of electronic computers. The study by A. S. Tyan (1964) considers in stages the movement of a system with some degree of freedom. The use of electronic computers made it possible to state the law for the changes in the accelerations in nonanalytic form.

E. Ye. Khachiyan (1966) studied in a similar manner the oscillations of elasto-plastic frames with an arbitrary number of degrees of freedom The accelograms of real earthquakes were taken as the actions. The numerical example was calculated for a four-story frame whose material followed the Prandtl diagram. The system was integrated numerically using an electronic computer. As a result of this, the residual deformations obtained by the author are not more than 10-15% of the amplitudes and their role in the oscillatory process is not great. This conclusion, obtained for one particular example, does not yet justify generalizations.

We draw attention to the fact that the solution of this and similar problems whose immediate goal is to describe the behavior of frames actually reduces to the calculation of cantilever beams, due to the approximations that are introduced. For this reason here the results obtained for the description of the movement of cantilever beams (see Section 4.2) can be used successfully. This makes it possible to take into account the dispersion along the length of the mass, in particular, to solve the problem of the propagation of flexural waves caused by a seismic shock along a tall building.

Of course, the approximation of a frame by a cantilever does not satisfy all the requirements. Due to this a number of recent studies proposed to consider the dynamic pattern of the movement mechanism which is obtained when plastic joints are formed in the frame. The experience obtained from studying the movement of rigid-plastic beams has shown that here real results can only be obtained when the joints are considered to be stationary.

4.4. Plates and Shells

The specific features of thin-walled three-dimensional structures often allow us to assume that for many forms of loads, all points of the structure operate simultaneously so that the corresponding wave process need not be investigated. However, under such conditions, the problem is very complex and it is necessary to take into account the three-dimensional operation of the material and the kinematics of the motion which is not simple. The studies of Soviet scientists first dealt with visco-plastic and rigid ideal plastic axisymmetric plates.

Visco-plastic plates were studied by F. A. Bakhshiyan (1948). He considered the material as a Bingham material with a linear relationship between the stresses and the deformation rates and considered the case when the impacting mass was much greater than the mass of the part of the plate experiencing the shock, in view of which the changes in the rate can be ignored at the shock instant. Later the bending of a circular plate made from a visco-elastic material was studied by G. M. Gizatulina (1964) in a more precise formulation.

A. M. Kochetkov (1950) considered the shock of an absolutely rigid cylinder on a plate made from an ideal plastic material and obtained a numerical solution. Shortly afterwards the studies extended to other models of the material of the plate. Elasto-plastic plates were studied by M. P. Galin (1958, 1959). He studied transverse oscillations of beams and plates loaded beyond the elasticity limit. It was assumed that the strengthening of the material was linear and that the material was incompressible. The effect of the shearing forces and torsional inertia were ignored. The solution was obtained with the aid of expansions in series.

The study of A. P. Sinitsyn (1965) studied the general conditions for the propagation of thermoelastic-plastic stress waves and calculated elasto-plastic plates of three types (rectangular plate, a plate on an elastic base and a three-layer plate) under the action of external heat flow changing with time. Two specific forms of oscillations were studied for the three-layer plate and a criterion for the optimal ratios of the rigidities of elements of the plates was obtained. The effect of the plastic zones was estimated.

A. D. Bagdasarov (1964) derived a system of differential equations for the description of the oscillations of arbitrary elasto-plastic plates during large deflections. Ya. Aminov (1964) set up the corresponding system for circular plates.

The use of the rigid-plastic material model enabled G. S. Shapiro (1959) to obtain a solution of the problem of a shock on a circular plate.

The state of numerical methods and computing technology until the middle 60's did not make it possible to use the elasto-plastic model in the analysis of the dynamic behavior of plates with the exception of the axisymmetric problem. Due to this a number of solutions were proposed for structures with more complex contours in the plane (in particular, for rectangular plates) which were based on the concept of plastic joint lines, i.e., a generalization of the concept of a plastic joint in the bent beam.

In the calculation of rectangular plates for a transverse load, N. N. Popov and B. S. Rastorguyev (1964) assumed that after the moment in the direction of the smaller span in the middle of the plate attains the limiting value, linear plasticity joints are formed instantaneously, whose outlines correspond to the usual "envelope" scheme which is used in determining the upper load-bearing capacity boundary in static calculations (the angles of inclination of the joints in the corners were taken to be 45°). Such a scheme, of course, is a rough approximation, but nevertheless it is an improvement over ignoring the elastic work of the plate, which is done in rigid-plastic analysis. Thus, in the plastic stage the plate was represented as a system with one degree of freedom. When the equations of motion were set up in the plastic work stage, the equations for the work were used. Clearly, this approach can only be used when the deformation mechanism is given. The equality of the number of movements at the end of the elastic and at the beginning of the plastic stage were used as the initial conditions for the integration of the equations of motion.

It was emphasized by V. P. Tamuzh (1963) that the variational principle (2.3) can be used to make the deformation mechanism of the plate in the plastic stage more precise.

A number of studies were oriented to improve the concept of the external medium in which the movement of the deformed plates occurs.

The problem of elasto-plastic deformations of a plate on a fluid base was studied by L. I. Slepyan (1964).

A shock with a constant velocity on an annular rigid-plastic plate in a medium whose resistance is proportional to the velocity with which the plate moved was studied by A. A. Amandosov (1962).

Attempts were made to introduce into the discussion more complex models for the mechanical behavior of the material of plates. The study of L. V. Nikitin in which he studied the movement of elasto-visco-plastic beams and plates was already published in 1959.

The problem of the dynamic behavior of axisymmetric shells in which the plastic deformations are taken into account is particularly topical in connection with the studies of the effect of an explosion and thermal shock on such structures. Therefore, it became the subject of many studies. We will begin with studies on cylindrical shells.

The problem of dynamic stability for an elasto-plastic shell with initial imperfections was solved by A. K. Pertsev (1964). The author studied the loss of stability of a circular cylindrical shell under the action of external hydrostatic pressure to whose lateral surface a dynamic load was applied. It was assumed that in the plastic zones the stress components remain constant. Next, the stress function was introduced for the deflections and the initial counter The effect of the fluid on the flexural movement of the shell was taken into account using an approximate coefficient. As a result of a number of assumptions that were made, it turned out that the continuity equations can be integrated exactly and the equations of motion, using the Bubnov-Galerkin method. Finally, the author analyzed the behavior of the overloading coefficient which determined the overshoot in the critical dynamic load over the corresponding static load. As the duration of the loading effect increases, the overloading coefficient decreases and when the duration is equal or greater than three periods of the natural oscillations, it is practically equal to one.

An elasto-plastic analysis of reinforced concrete shells under the action of a dynamic load was made by N. N. Popov and B. S. Rastorguyev (1964), who studied an axisymmetric and flat rectangular shell in the plane. The condition for the transition to the plastic work stage used in the analysis of flat shells is the condition that the yield be obtained in the lateral elements of the shell. As usual, the authors ignored the tangential inertial forces. A system of joints at the corners of the shell directed at a 45° to the sides, and joints parallel to the sides so that the middle rectangular part of the shell moved as a rigid whole, was adopted as the fracture mechanism in the plastic work stage. When the work of the internal forces was calculated using the work of the bending moments in the joints, the yields were ignored.

For cylindrical shells, it was possible to study the entire deformation process of the structure analytically by breaking it up into a series of stages.

P. A. Kuzin (1963, 1964) studied the dynamic deformation of a rigid-plastic cylindrical shells with fixed and free supported edges. It was assumed that the load was applied to a section of the shell along a ring.

A complete study of the problem of the movement of a semiinfinite shell with a free edge under the action of an annular concentrated load is given in the study of P. A. Kuzin and G. S. Shapiro (1965).

Several studies deal with the movement of rigid-plastic spherical domes. These include the studies by N. N. Popov and B. S. Rastorguyev (1964), M. I. Reytman (1964) and M. I. Yerkhov (1966).

N. N. Popov and B. S. Rastorguyev (1964) proposed a meridional deformation scheme for the dynamic loading of a rigid-plastic reinforced concrete dome, which is realized when the support of the contour is not sufficiently firm.

In the study by M. I. Reytman (1964), the problem of the dynamic deformation of a rigid-plastic shell whose material obeys the Tresk condition is solved using the variational principle (2.3) and the generalized Ritz method. The deformation mechanism unlike in the studies described above is characterized not by concentrated but by distributed elongation and bending deformations.

We see that in many approximate studies the authors ignored the work of the bending moments. This justifies the application of ideal plastic shells, usually from the torqueless theory of shells, to dynamic load problems.

M. I. Reytman (1964) studied an ideal plastic shell on the assumption that the entire shell was in the yield stage. This makes it possible to find a simple system of equations which is similar to the equations for the plane problem in the theory of plasticity under a static load. M. I. Yerkhov (1966) studied a flat spherical shell under the action of a load acting in a given finite time interval. It was assumed that the material of the shell followed the yield condition that was proposed by the author earlier.

A. A. Amandosov (1962) generalized the problem of the movement of a cylindrical rigid-plastic shell under the action of internal pressure to the case of a resisting medium. The resisting force was assumed to be proportional to the normal displacement rate. The author reached the conclusion that

the effect of the resisting medium is considerable even for "medium" loads.

Problems in the dynamic deformation of shells under explosive and electrohydraulic effects were solved by L. P. Orlenko and are presented in his monograph (1964).

STABILITY OF ELASTIC AND NONELASTIC SYSTEMS

V. V. Bolotin, E. I. Grigolyuk

§1. General Historical Survey

The theory of stability of elastic and nonelastic systems belongs to a branch of mechanics whose development is intimately connected with the development of technology. A great part of problems in the theory of stability originated directly in engineering practice. The increase in the strength of structural materials, the trend to reduce the weight of equipment and machinery, the introduction of rational thin-walled

structures, all this stimulated the development of the theory. The last few decades were characterized by a sharp rise in the rates, accelerations, temperatures and other parameters, the introduction of new materials and new technological processes, outstanding progress in aviation, rocket technology, electrical engineering and technology. As a result of this, new trends in the theory of elastic and non-elastic stability were formed. It can be said without exaggeration that the stability theory of deformable systems will never lose its topical character. The problem of ensuring stability is inseparable from the problem of increasing the strength of structural materials.

The interests of Russian scientists in the theory of stability of elastic and nonelastic systems has traditions that are rooted in the distant past. The origin of these traditions goes back to the classical studies of L. Euler (1744-1757) in longitudinal bending theory. Among these studies from the pre-Revolutionary period, the studies of F. S. Yasinskiy (1892-1895) on elasto-plastic longitudinal bending problems should be mentioned in which linear equations were properly used for the calculation of the critical forces, the studies of I. G. Bubnov (1902, 1904, 1912) on the stability and postcritical deformations of elements of ship structures, the studies of S. P. Timoshenko (1905-1916) on the longitudinal bending of rods and rod systems, the plane bending of rods, the bending of plates and shells, and also the studies of B. G. Galerkin (1909), A. N. Dinnik (1911, 1913) and A. P. Korobov (1911, 1913). The energy method for determining the critical loads formulated by S. P. Timoshenko (1907) had an especially great influence on the subsequent development of the theory as well as the approximate method proposed for the first time by I. G. Bubnov (1911, 1913, 1914), which was later called the Bubnov-Galerkin method.

The last 50-year period in which the theory developed and which is the subject of this survey can be broken down naturally into two periods, the prewar period (1917-1941) and the postwar period (1945-1967). Studies published during the war will be classified arbitrarily as studies in the prewar period. The fundamental trend in the prewar period was the development of the stability theory of elastic rods and rod systems. Shortly after the Revolution, the studies of Ye. L. Nikolai (1918, 1923) on the stability of elastic rings and curvilinear rods were published. The studies of I. Ya. Shtayerman (1929, 1930, 1937) and A. N. Dinnik (1929, 1933, 1935, 1936) belong to the same trend. The studies of A. N. Krylov (1931, 1935), I. M. Rabinovich (1932), N. V. Kornoukhov (1935), A. P. Korobov (1936), N. G. Chentsov (1936),

A. A. Belous (1937), N. K. Snitko (1938), P. F. Papkovich (1939), I. Ya. Shteyerman and A. S. Pikovskiy (1939) deal with the stability of rods and rod systems. The general stability theory of thin-walled rods with an open cross section was developed by V. Z. Vlasov (1938, 1940). Relatively few studies were published on the stability of elastic plates and shells. In particular these include the studies of L. S. Leybenzon (1917), P. F. Papkovich (1920, 1929), I. Ya. Shtayerman (1929), Kh. M. Mushtari (1934, 1938), and V. V. Novozhilov (1941).

At the same time in the prewar period, studies were published, whose new formulations and the results obtained in them were not fully appreciated for several decades and whose merit was only recognized later. These include primarily the article by N. M. Belyayev (1924) in which the problem of the dynamic equilibrium of a rod compressed by a periodic longitudinal force was formulated and solved for the first time. Before the war these studies were continued by N. M. Krylov and N. N. Bogolyubov (1935), V. N. Chelomy (1938, 1939) who also introduced the term "dynamical stability" of elastic systems, and finally by G. Yu. Dzhanelidze and Yu. M. Radtsig (1940) and V. A. Bodner (1940). Ye. L. Nikolai (1928, 1929) began to develop another aspect of the theory of elastic stability, by considering certain problems in the stability of elastic rods under the action of "followup" forces.

The postwar period is characterized by attention to more general and theoretical problems in the theory, by the development of stability theory of elasto-plastic and visco-elastic systems and a theory of dynamic stability. But the main attention was focused on the development of the stability theory of shells. The development of the linear stability theory of elastic shells was completed by V. Z. Vlasov (1944, 1949), Yu. N. Rabotnov (1946), Kh. M. Mushtari and his collaborators (1946-1958) and others. The general nonlinear stability theory of shells was developed by N. A. Alumyae (1949-1956), Kh. M. Mushtari and K. Z. Galimov (1948-1967), I. I. Vorovich (1955-1957), et al. An original method for the solution of nonlinear problems was developed by A. V. Pogorelov (1960-1966). Many problems were studied by N. A. Kil'chevskiy (1942, 1967), V. I. Feodos'ev (1946, 1954), D. Yu. Panov and V. I. Feodos'ev (1948, 1949), S. A. Ambartsumyan and his collaborators (1950, 1955), A. S. Vol'mir and his collaborators (1950-1964), M. A. Koltunov (1952, 1961), E. I. Grigolyuk (1954, 1955), Kh. M. Mushtari and his collaborators (1954-1966), N. A. Alumyae (1955-1958), et al. Problems in the stability of orthotropic and anisotropic plates and shells in layers were also developed. This trend includes the studies of Kh. M. Mushtari (1938, 1961), S. G. Lekhnitskiy (1947), V. I. Korolev (1956, 1965), A. P. Prusakov (1951-1958), E. I. Grigolyuk and his collaborators (1957-1966), A. Ya. Aleksandrov and his collaborators (1959, 1960), S. A. Ambrtsumyan and his collaborators (1961-1966), Yu. M. Tarnopol'skiy and G. A. Teters (1965-1967), et al.

The studies in the dynamic stability of elastic systems were continued. Parametric oscillations were discussed in the studies of I. I. Gol'denblat (1947, 1948), V. V. Bolotin (1951-1956), G. Yu. Dzhanelidze (1953, 1956), V. N. Chelomey (1956), V. A. Yakubovich (1958), et al. The study of V. N. Chelomey (1956) was also applied to an important engineering problem. The stability of elastic systems under the action of forces depending on the deformation ("followup") forces were studied by V. V. Bolotin (1956-1961), G. Yu. Dzhanelidze (1958), et al. The article by M. A. Lavrent'ev and A. Yu. Ishlinskiy (1949) began the study of the phenomena of the loss of stability during an impact load. These phenomena were further studied by A. S. Vol'mir and his collaborators (1959-1966), E. I. Grigolyuk and his collaborators (1963) and others. The stability of plates and shells interacting with a liquid or gas was analyzed by V. V. Bolotin (1956-1961), E. I. Grigolyuk (1956), A. A. Movchan (1956, 1957), R. D. Stepanov (1957-1960), V. V. Bolotin and his collaborators (1959, 1961), S. A. Ambartsumyan and his collaborators (1961), P. M. Ogibalov (1961), E. I. Grigolyuk and his collaborators (1962-1965), G. N. Mikishev (1959), V. V. Bolotin (1958-1961), I. I. Vorovich (1959) and A. S. Vol'mir and his collaborators (1964, 1965) began the application of physical methods to problems in the theory of elastic and nonelastic stability.

Important results in the stability of elasto-plastic systems were obtained by A. A. Il'yushin (1944, 1948), L. A. Tolokonnikov (1949), L. M. Kachanov (1951-1956), Yu. N. Rabotnov (1952), Ya. G. Panovko (1954, 1962), Yu. R. Lepik (1956, 1957), E. I. Grigolyuk (1957, 1958), V. D. Klyushnikov (1957, 1966), L. V. Yershov (1961), et al. The stability of linear visco-elastic systems was studied by A. R. Rzhanitsyn (1946, 1949). The stability of rods, plates and shells in creep conditions was studied by Yu. N. Rabotnov and S. A. Shesterikov (1957, 1959, 1961, 1963), L. M. Kurshin (1961, 1963), E. I. Grigolyuk and Yu. V. Libovtsev (1965, 1966), M. A. Koltunov (1965-1967), et al.

Rapid progress was made in the general theory. V. V. Novozhilov (1948) analyzed the problem in elastic stability theory from the standpoint of nonlinear elasticity theory. These studies were continued by G. Yu. Dzhanelidze (1955), V. V. Bolotin (1956, 1958), et al. The studies of V. V. Bolotin (1961, 1965) are devoted to the derivation of the stability equations for elastic systems from general variational principles. The studies begun by A. Yu. Ishlinskiy (1954) in which the equilibrium stability problem of a rod is solved on the basis of the equations of elasticity theory started another trend. This trend includes, for example, the studies of D. D. Ivlev (1965), A. N. Guz' (1967), et al. A point of view was generally accepted according to which not only the

problems in the stability of movement but also problems in the equilibrium stability of elastic systems must be considered from the point of view of the general stability theory of motion. The problem of the stability of distributed systems was formulated rigorously by extending A. M. Lyapunov's theory to metric functional spaces (V. I. Zubov, 1957, A. A. Movchan, 1959, 1960).

The development of computational methods for the calculation of the stability of rods and rod systems was continued in the postwar period. A. F. Smirnov (1947) proposed an efficient matric computational method. This method was further developed in the subsequent work of A. F. Smirnov and his collaborators (1950, 1957, 1958). A. R. Rzhanits (1948) proposed a method for calculating the stability of composite rods. Ye. P. Popov (1948) studied the post-critical behavior of ductile rods and classified the possible computational schemes. The studies of N. V. Kornoukhov (1949), Ya. L. Nudel'man (1949), N. K. Snitko (1952, 1956), V. G. Chudovskiy (1952), I. K. Snitko (1960), S. A. Rogitskiy (1961), R. R. Matevoskyan (1961), N. I. Bezukhov and O. V. Luzhin (1963), et al., are devoted to the calculation of the stability of rods and rod systems. The stability of thin-walled rods was studied by V. Z. Vlasov (1947, 1959), S. A. Ambartsumyan (1952), I. F. Obraztsov (1953), et al. Applied problems in the calculation of the stability of structures were developed by B. M. Broude (1949), A. V. Hemmerling (1949), A. I. Segal (1949), V. V. Piandzhan (1956), et al.

In the prewar years the books by A. N. Dinnik (1939), I. Ya. Shtayerman and A. A. Pikovskiy (1939) were widely read. The contemporary state of the theory of elasticity of elastic systems is discussed in a series of monographs. These include the books by A. S. Vol'mir (1956, 1963, 1967), V. V. Bolotin (1956, 1961), A. F. Smirnov (1958), Kh. M. Mushtari and K. Z. Galimov (1957), P. M. Ogibalov (1963), et al. Stability calculations occupy an important place in the classical three-volume monograph on the structural mechanics of a ship by P. F. Papkovich (Part 2, 1941) and in the volume edited by S. D. Ponomarev, et al. (1959).

Stability problems are widely represented in the proceedings of symposia and conferences on mechanics, in particular in the Proceedings of All-Union Conferences on Stability Problems in Structural Mechanics and also in the theory of shells and plates.

§2. The Concept of the Stability of Elastic Systems

Stability is a property of motion (in a special case of equilibrium) understood in the broad general scientific sense of the word. Let us consider a mechanical, electrical thermodynamic, biologial, etc. system. We will assume that a motion of this system which occurs for a particular combination of the system parameters and of the surrounding medium is known. We will assume that this motion is not perturbed. Now, we will imagine that the parameters that were mentioned (all of them or some of them) underwent small changes. Then the motion of the system also changes. A very important question is how large these changes will be, i.e., by how much the perturbed motion will differ from the unperturbed motion. When small actions cause small deviations from the unperturbed motions, the perturbed motions will more or less cluster densely around the unperturbed motion. In this case the unperturbed motion is said to be stable. On the other hand, if small actions cause large deviations of the system from unperturbed motion, the motion is said to be unstable. Thus, stability is a property of a system which deviates little from unperturbed motion during small perturbing effects.

The concept of stability is of fundamental importance Both in nature and in human activity only stable phenomena and processes can be used over a lengthy period. Unstable motions can only be observed over short periods. Thus, the concept of stability is intimately related to the concept of realizability.

Stability problems occupy an important place in engineering calculations. The idealized structure designed by the engineer differs from the actual real structure based on this design. This difference is due to the many more or less small deviations from the design, the defects and imperfections. To the engineer it is absolutely essential that in spite of the presence of these deviations, the real structure will function in approximately the same manner as the corresponding idealized structure. If there were no such a guarantee, designing would be meaningless. It is easily seen that the stability concept is used here. The equilibrium or motion of the design structure will be stable if the small imperfections and defects, the small deviations from the calculated scheme cause small deviations from the idealized operating conditions. If the small imperfections cause incommensurately large deviations, the equilibrium (motion) will be unstable. The designer must select the dimensions of the design in such a way that the equilibrium (motion) of the structure remains stable with respect to all possible combinations of loads and with respect to all types of perturbations that may be encountered. In addition, the

structure must have a certain stability margin.

We will point out four elements which must be included in any definition of stability. First, it is finding the unperturbed motion (equilibrium), whose stability is investigated. We cannot speak about the "stability of a system" in general, we can only speak about the stability of a particular motion (equilibrium) of the system. Second, the definition of stability must include a specification of the parameters of motion with respect to which the stability is studied. The motion may be stable with respect to one group of parameters and unstable with respect to another group. The third element in the definition is specifying the class of perturbing actions causing the deviations from unperturbed motion. The fourth element is specifying the time interval during which the unperturbed and perturbed motions must be close.

A rigorous mathematical definition of the stability of motion of elastic systems goes back to the classical definition of stability due to A. M. Lyapunov (1892). Lyapunov's theory was constructed for systems with a finite number of degrees of freedom whose motion is described by ordinary differential equations. The extension of Lyapunov's theory to continuous systems became possible after it was formulated in terms of functional analysis (N. N. Krasovskiy, 1956, V. I. Zubov, 1957, A. A. Movchan, 1959, 1960). This made it possible to generalize many concepts, theorems and methods developed by Lyapunov and his followers for a finite dimensional Euclidian space to a very large class of metric spaces.

We present the basic definitions (V. I. Zubov, 1957 and omit certain mathematical subtleties. For simplicity we will restrict ourselves to the case when the motion is described by one function $u(x, t)$ of the coordinate x and time t. We will consider a set of motions satisfying the boundary conditions, the continuity conditions and the initial condition $u(x, 0) = \varphi(x)$. Let us denote the elements of this set by $U = U(\varphi, t)$ and introduce the metric distance between the elements of the set U and V which we denote by $\rho(U, V)$. Suppose that to the unperturbed motion U_0 corresponds the initial condition $u(x, 0) = \varphi_0$. The unperturbed motion U_0 is said to be stable with respect to the metric ρ if for any $\varepsilon > 0$ a $\delta > 0$ can be found such that the condition $\rho(\varphi, \varphi_0) < \delta$ implies $\rho\,[U(\varphi, t), U_0] < \varepsilon$ for any $t > 0$. Otherwise, the motion is said to be unstable. When the unperturbed motion U_0 is stable and also $\rho\,[U(\varphi, t), U_0] \to 0$ as $t \to \infty$, it is said to be asymptotically stable. A. A. Movchan (1960) pointed out the usefulness of defining stability in which two different metrics are used simultaneously.

The selection of the metrics depends on the types of problem and requirements imposed on the unperturbed motion on the basis of physical and engineering concepts. The requirement that the function and its derivative be locally close leads to metrics of the form

$$\rho_1 = \sup_x |u - v|, \quad \rho_2 = \sup_x |u - v| + \sup_x |u_{,t} - v_{,t}|,$$
$$\rho_3 = \sup_x |u - v| + \sup_x |u_{,t} - v_{,t}| + \sup_x |u_{,x} - v_{,x}|$$

etc. Metrics corresponding to closeness in the mean, form another group

$$\rho_4 = \left[\int_0^l (u - v)^2\, dx \right]^{1/2}, \quad \rho_5 = \left\{ \int_0^l [(u - v)^2 + (u_{,t} - v_{,t})^2]\, dx \right\}^{1/2},$$
$$\rho_6 = \left\{ \int_0^l [(u - v)^2 + (u_{,t} - v_{,t})^2 + (u_{,x} - v_{,x})^2]\, dx \right\}^{1/2}$$

etc. In the applications, usually not only stability with respect to the displacements and velocities, but also with respect to the stresses and strains is required. In addition, in a continuous medium, the smallness of these initial displacements and velocity does not necessarily imply the smallness of the initial energy of the system and "jumps" in the displacements and velocities may occur when $t > 0$. Therefore, metrics of the energy type occupy an important place. A. A. Movchan (1959, 1960) and A. M. Slobodkin (1962) demonstrated on examples that a metric corresponding to the total energy of the system leads to results which agree with a direct solution of the Cauchy problem for the perturbed motion.

Often engineering applications require that the definitions of stability that were presented above be generalized to the case when not only the initial conditions but also the coefficients of the differential equations, the boundary conditions and the boundary itself are perturbed. The extension of these definitions is intimately related to the concept of the correctness of boundary value problems in the theory of partial differential equations. Generally, the stability theory of deformable solids which would correspond in rigor and effectiveness to the classical Lyapunov theory is still in the initial development stage. In practice all concrete results on the stability of elastic and nonelastic systems were obtained either on the basis of methods that were formally borrowed from the theory of stability of discrete systems or on the basis of linearlized equations of perturbed motion.

§3. Equations of Perturbed Motion

In many cases, to make statements about stability, it can be assumed that the perturbations are sufficiently small and their character can be studied using linearized equations of perturbed motion. We will show how the linearized equations that are applied to stability problems in the motion of an elastic body are set up. We will use the equations of non-linear elasticity theory in the form proposed by V. V. Novozhilov (1948).

Let us consider the unperturbed motion of an elastic body characterized by the displacement vector u_j, by the stress tensor σ_{ij} and by the vectors of body and surface forces X_j and p_j. The unperturbed motion in rectangular cartesian coordinates is described by the equations

$$[\sigma_{kl}(\delta_{jl} + u_{j,l})]_{,k} + X_j - \rho u_{j,tt} = 0, \qquad (3.1)$$

where ρ is the density of the material. Here and henceforth we will use the rule of summing over dummy indices. On the loaded part of the surface of the body, the conditions

$$\sigma_{kl}(\delta_{jl} + u_{j,l})n_k = p_j \qquad (3.2)$$

must be satisfied (n_j is the normal vector to the surface of the body). We impart to the body small deviations from the unperturbed motion and we will investigate how these perturbations vary with time. The components of the perturbed motion (we will denote them by the symbol ~, and the perturbations by a bar on the top) will have the form

$$\left.\begin{aligned} \tilde{u}_j = u_j + \mu\bar{u}_j, \quad \tilde{\sigma}_{jk} = \sigma_{jk} + \mu\bar{\sigma}_{jk}, \\ \tilde{X}_j = X_j + \mu\bar{X}_j, \quad \tilde{p}_j = p_j + \mu\bar{p}_j \end{aligned}\right\} \qquad (3.3)$$

(the perturbations of the body and surface forces generally depend on time t, and μ is a small parameter). Substituting (3.3) in (3.1) and (3.2) and using the fact that the perturbations are small, we obtain, after linearization, the equations

$$[\bar{\sigma}_{kl}(\delta_{jl} + u_{j,l}) + \sigma_{kl}\bar{u}_{j,l}]_{,k} + \bar{X}_j - \rho\bar{u}_{j,tt} = 0 \qquad (3.4)$$

and the boundary conditions on the loaded surface

$$[\bar{\sigma}_{kl}(\delta_{jl} + u_{j,l}) + \sigma_{kl}\bar{u}_{j,l}]n_k = \bar{p}_j. \qquad (3.5)$$

Here $\bar{\sigma}_{jk} = \lambda_{jklm}\bar{u}_{l,m}$, where λ_{jklm} is the tensor of elastic constants corresponding to the unperturbed stressed state.

In many engineering problems, the unperturbed motion differs little from the initial undeformed state, and the deformations only increase during the transition from stability to instability. This makes it possible to identify the geometry of the unperturbed state with the geometry of the undeformed state. Equations (3.4) and boundary conditions (3.5) are considerably simplified when this is done, since the terms which involve the displacements u_j are omitted, and the elastic constants are used for the undeformed state.

When the external forces are potentials, a quadratic functional of $\bar{u}_j$ is easily constructed which is varied over the set of kinematically admissible motions to obtain the linearized equations of perturbed motion (3.4) and the boundary conditions (3.5). For example, in the case when all $\bar{X}_j = \bar{p}_j = 0$ and the displacements in the unperturbed state are negligibly small, the above functional takes on the form

$$I = \int_{t_0}^{t_1}\left[\int_V (\lambda_{jklm}\bar{u}_{j,k}\bar{u}_{l,m} + \sigma_{jk}\bar{u}_{l,j}\bar{u}_{l,k} - \rho\bar{u}_{j,t}\bar{u}_{j,t})\,dV\right]dt. \qquad (3.6)$$

Here t_0 and t_1 are arbitrarily selected instants of time. The motion is not varied at the endpoints of the time interval. The functional (3.6) coincides with an accuracy up to a constant multiplier with the second variation of the action integral calculated for the real deviations from the unperturbed motion.

Stability problems are typical of thin and thin-walled bodies. The solutions of these problems for rods, plates and shells are usually constructed on the basis of approximate equations, in which certain kinematic and dynamic hypotheses are used. These equations can be obtained in a number of ways. The first, the earliest method, is to study directly the forms of motion (equilibrium) which are adjacent to the unperturbed motions. Some normalized load is sought which is introduced in the equation of unperturbed motion. All arguments are characterized by clarity; however, in sufficiently complex problems, this clarity turns out to be deceptive. Another method is to use the nonlinear equations of the corresponding applied theories. By linearizing the equations in the neighborhood of the unperturbed motion, the sought equations are obtained. This method was used in the theory of shells by Kh. M. Mushtari (1939), N. A. Alumyae (1949), Kh. M. Mushtari and K. Z. Galimov (1957), N. A. Kil'chevskiy (1963), V. M. Darevskiy (1963) and by other authors. However, in nonlinear theory, there is less agreement in the opinions about how the fundamental equations should be expressed. Hence, when this approach is followed, all the difficulties are only shifted to another area in which there is even less agreement. The third approach is to use the general equations from the theory of elastic stability (V. V. Novozhilov, 1940, 1948). The method which is based on the corresponding variational principle was applied by V. V. Bolotin (1965). This method makes it possible to estimate the errors in various approximate variants. Her[illegible] a measure of the error is the absolute value of the ra[illegible] terms that are ignored in the expression for the densi[illegible] the quadratic functional to the remaining principal term[illegible] the "energy" error.

The equations for the stability theory of thin elastic shells were derived on the basis of the concept of the "energy" error and were subsequently simplified. For the convenience of the discussion, we present the linearized equations for the perturbed motion for certain simple problems. The small transverse deviations of a thin elastic rod in the presence of an expanding axial force N are described by the equation

$$(EIw_{,xx})_{,xx} - (Nw_{,x})_{,x} + \rho F w_{,tt} = 0. \qquad (3.7)$$

Here w(x, t) is the normal displacement of the points on the axis of the rod, EI is the bending rigidity and F is the cross sectional area. The conditions $w = w_x = 0$ must be satisfied on the fixed end and on the supported end the conditions

$w = w_{,xx} = 0$. When the end is free, the boundary conditions depend on the behavior of the force applied to this end. In the case of a force which moves with the end, retaining the original direction in the space ("dead" force), the conditions have the form $w_{,xx} = 0$, $(EIw_{,xx})_{,x} - Nw_{,x} = 0$. When the force turns and remains directed along the tangent to the deformed axis of the rod ("followup" force), the boundary conditions have the form

$$w_{,xx} = (EIw_{,xx})_{,x} = 0.$$

Small transverse deviations $w(x_1, x_2, t)$ of a thin elastic plate of constant thickness h with the initial forces in the middle surface $N_{\alpha\beta}$ are described by the equation

$$D\Delta\Delta w - (N_{\alpha\beta}w_{,\beta})_{,\alpha} + \rho h w_{,tt} = 0, \tag{3.8}$$

where $x_\alpha (\alpha = 1, 2)$ are cartesian coordinates on the middle surface, and D is the cylindrical rigidity. Rectangular cartesian coordinates are natural for a flat shell and they differ little from orthogonal curvilinear coordinates on the middle surface. The equations (V. Z. Vlasov, 1944, Yu. N. Rabotnov, 1946)

$$\left.\begin{aligned} D\Delta\Delta w - \Delta_k\chi - (N_{\alpha\beta}w_{,\beta})_{,\alpha} + \rho h w_{,tt} = 0, \\ \Delta\Delta\chi + Eh\Delta_k w = 0. \end{aligned}\right\} \tag{3.9}$$

are obtained. Here $w(x_1, x_2, t)$ is a function of the normal displacements and $\chi(x_1, x_2, t)$ is a function of the additional forces in the middle surface

$$\Delta w = w_{,11} + w_{,22}, \quad \Delta_k w = k_2 w_{,11} + k_1 w_{,22},$$

and k_1 and k_2 are the principal curvatures of the middle surface. More detailed references can be found in the textbooks and monographs by S. P. Timoshenko (1946, 1955), V. V. Novozhilov (1948), Kh. M. Mushtari and K. Z. Galimov (1957), A. S. Vol'mir (1963, 1965).

Until now we discussed the linearlized equations for the perturbed motion of an elastic body. The equations for bodies whose material has nonelastic properties can be set up analogously. Thus, equations for a linear visco-elastic material are obtained from the equations for an elastic material when the elastic constants are replaced by the appropriate visco-elastic operators. However, in the case of an elasto-plastic material considerable difficulties arise. The behavior of an elasto-plastic material is very sensitive to small changes in the deformation path, which manifests itself, in particular, in the necessity to distinguish arbitrarily small loadings and unloadings. Generally, the deformation equations of elasto-plastic systems cannot be linearized. They can only be linearized under certain additional assumptions (for example, when it is assumed that unloading occurs everywhere. Assumptions of this kind narrow down the class of perturbed motions that is studied and, therefore, the results obtained on their basis are of a limited or arbitrary character.

§4. Methods for the Determination of Critical Parameters

The basic problem in stability theory of deformable systems is to find those values of the parameters of the system and/or external conditions which correspond to the transition from stability to instability. These values are called critical values. Most frequently, the external forces are given with an accuracy up to the parameters. In this case we speak about critical forces.

Suppose, for example, that the problem is characterized by one parameter β. We can assume without loss of generality that β varies in the range $0 \leq \beta \leq \infty$ and that the motion is stable when $\beta = 0$. The upper bound for the values $\beta = \beta_*$ for which the unperturbed motion remains stable is called the critical value. In the more general case, when the number of parameters is finite, it is useful to introduce the n-dimensional parameter space $\beta_1, \beta_2, \ldots \beta_n$ and to distinguish in it the elasticity and nonelasticity regions. The surfaces $F(\beta_1, \beta_2, \ldots \beta_n) = 0$ separating the elasticity and nonelasticity regions are called critical surfaces.

When the unperturbed state is the equilibrium, the question of the simultaneous existence of other stable equilibrium states may arise. Let us again consider the case of a single parameter β. The upper bound for the values $\beta = \beta_{**}$ for which the unperturbed state is the only stable equilibrium state is called the lower critical value. For $\beta_{**} < \beta < \beta_*$ a sufficiently strong perturbation may bring the system into another stable

equilibrium state. A relatively well-known example is the phenomenon of a "crack" in thin shells subjected to compression. In problems in which the concept of the lower critical value is used, the value $\beta = \beta_*$ is called the upper critical value. When the behavior of the system depends on n parameters, $\beta_1, \beta_2, \ldots, \beta_n$, and the coordinate origin in the parameter space corresponds to stability we can introduce, analogously as before, the concept of an upper and lower critical surface.

A general method for studying the stability of elastic systems is to consider the sets of motions which are adjacent to the unperturbed motions. This method is intimately related to the general theory of stability of motion and is called the dynamic method. When the stability of various equilibrium forms is studied, the equations of perturbed motion are usually linearized. The equations which are obtained in this manner describe small oscillations of the system around the unperturbed equilibrium position. Hence, the second name, the method of small perturbations (Ye. L. Nikolai, 1928, 1929).

Let us consider as an example a thin elastic rod subjected to axial expansion by forces which are independent of time. Replacing in equations (3.7) N by βN and taking $w = \varphi \exp(rt)$, we obtain

$$(EI\varphi_{,xx})_{,x} + \beta(N\varphi_{,x})_{,x} + \rho F r^2 \varphi = 0. \tag{4.1}$$

Equation (4.1) is considered together with homogeneous boundary conditions (for example, $\varphi = \varphi_{,xx} = 0$ for a rod supported at the ends). We obtain in this way the eigenvalue problem which contains two parameters, the characteristic index r and the loading parameter β. For $\beta = 0$ all r are pure imaginaries and the oscillation frequencies are real. The critical value β_* is determined from the condition that for $\beta > \beta_*$ among the characteristic indices r there will be at least one with a positive real part. If passage to the right halfplane occurs through the value $r = 0$, the loss of stability of the unperturbed equilibrium form does not have an oscillatory character. In the remaining cases, instability of the oscillatory type will occur. In aeroelasticity problems we speak about divergence and flutter, respectively.

The method of small oscillations is not rigorous. When dissipation is not taken into account, for $\beta < \beta_*$ all characteristic indices lie on the imaginary axis. By analogy with the stability theory of discrete systems such a case should be classified as a doubtful case. When the external forces are

potentials, using the direct Lyapunov method (A. A. Movchan, 1959) stability can be proved rigorously for $\beta < \beta_*$. In the same case the introduction of any arbitrarily small full dissipation displaces all characteristic indices from the imaginary axis to the left halfplane. Then, for $\beta < \beta_*$ we obtain the analogue of asymptotic stability in the theory of discrete systems.

When the external forces are not potentials, the case of pure imaginary characteristic indices is more correctly called "quasistability" and the value of the parameter β_* is said to be "quasicritical." Also here the introduction of dissipation forces with full dissipation eleminates the doubtful case For some $\beta < \beta_{*D}$ all indices r are in the left halfplane, for $\beta > \beta_{*D}$, at least one of them is in the right halfplane. In nonconservative elastic stability problems, we encounter a very important and at first sight unexpected fact, namely, that the tending of the dissipation parameters to zero does not necessarily imply $\beta_{*D} \to \beta_*$. The limiting value β_{*D} depends on the dissipation law that was adopted (V. V. Bolotin, 1959, 1961).

We will dwell on other methods used in the study of the stability of elastic equilibrium when the external forces are potentials. Among these methods, the energy method occupies an important place. This method is based on the Lagrange-Dirichlet problem, according to which the total potential energy of the system has the minimum value in stable equilibrium. The Lagrange-Dirichlet theorem has been proved rigorously for a system with a finite number of degrees of freedom and has also been extended to elastic systems by G. H. Brian (1888) S. P. Timoshenko (1907, 1908, 1910), et al.

The application of the energy method reduces to a study of the properties of the quadratic functional of the potential energy Э, which is equal to the sum of the potential deformation energy (internal energy) and the potential energy of the external forces. When for all kinematically admissible variations of the state $\delta^2 Э > 0$, the equilibrium is stable. If for some variations $\delta^2 Э < 0$, it is unstable. The critical value of the parameter β must be found from those values for which $\delta Э = 0$, $\delta^2 Э = 0$ simultaneously. For the assumptions under which the equations of perturbed motion (3.6) were set up we have

$$\delta^2 Э = \int_V \lambda_{jklm} \bar{u}_{j,k} \bar{u}_{l,m}\, dV - \beta \int_V s_{jk} \bar{u}_{l,j} \bar{u}_{l,k}\, dV \tag{4.2}$$

$(\sigma_{jk} = -\beta s_{jk})$. The expression

$$\delta^2 Э = \int_0^l EI(w'_{,xx})^2 dx - \beta \int_0^l N(w'_{,x})^2 dx. \qquad (4.3)$$

corresponds to equation (3.7). The same notation as for small perturbation is u ed in formulas (4.2) and (4.3) for varying the stresses. We note that in textbooks and in the technical literature, the terms in the right members of expressions of types (4.2) and (4.3) are usually interpreted as the "potential energy of the deformations" and the "work of the external forces." Formulas of the type

$$\beta = \frac{\int_0^l EI(w,_{xx})^2 dx}{\int_0^l N(w,_x)^2 dx} \qquad (4.4)$$

give an upper bound for the critical parameters when the kinematically admissible states are compared. These formulas can be considered as one possible realization of the energy method (S. P. Timoshenko, 1907, and later authors).

The study of the properties of the potential energy functional can be replaced by a systematic study of the changes of equilibrium forms when the parameters of the system change. Concepts which are similar to the well-known A. Poincaré bifurcation theory (1884) lead to the static method in the stability theory of elastic systems. This method can be used to reduce the study of stability to finding branching points and limiting points. In the neighborhood of a branch point certain allied forms are found together with the equilibrium form that is investigated. Loss of stability may occur during passage through this point in accordance with the type of equilibrium form branching. A discontinuous transition from one form of equilibrium to another corresponds to passage through the limiting point. An analysis of the types of limiting points and changes in equilibrium state of elastic systems can be found in the studies of G. Yu. Dzhanelidze (1955), I. I. Gol'denblat (1965), et al. The main difficulty in applying the bifurcation method to elastic systems is the selection of parameters characterizing the state of the system. Strictly speaking, the presence of bifurcation points is neither a necessary nor a sufficient condition for a change in stability.

The reliability of the derivations based on bifurcation concepts can be improved when the number of parameters is increased. But, when this is done, the main advantage, the geometric clarity, is lost.

The analytical realization of the static method leads to the condition

$$\delta (\bar{\delta}^2 \mathcal{Э}) = 0. \tag{4.5}$$

Here $\bar{\delta}^2 \mathcal{Э}$ is the second variation of the potential energy of the system around the unperturbed equilibrium state calculated on the assumption that the variations of the displacements coincide with the real perturbations. The functional $\bar{\delta}^2$ is calculated from formulas of the type (4.2) and (4.3) and then varied over all kinematically admissible states. The corresponding Euler-Ostrogradskiy equations are the well-known neutral equilibrium equations which describe the equilibrium of the system in the state adjacent to the unperturbed states. Variation of the function (4.2) leads to the equation

$$(\lambda_{jklm} v_{l,m})_{,i} - \beta (s_{ik} v_{j,k})_{,k} = 0. \tag{4.6}$$

and variation of the functional (4.3) to the equation

$$(EIw_{,xx})_{,xx} - \beta (Nw_{,x})_{,x} = 0, \tag{4.7}$$

etc. These equations are the same as the linearized equations for the perturbed motion when it is assumed that the perturbations are independent of time. Equation (4.7) is obtained from (4.1) when the characteristic exponent r is set equal to zero.

In textbooks and in the technical literature, the statement is usually made that this method is only useful in problems in which the loss of stability takes place on the basis of the type of branching of the equilibrium forms. In fact, the neutral equilibrium equations can describe the behavior of the system in the neighborhood of the limiting points. However, to do this it is necessary to take into account the displacements and deformations in the unperturbed state, i.e., to start out with equations of the type (3.4). Generally the loading parameter will enter the equations nonlinearly.

A considerable number of special problems in the theory of elastic stability has been solved on the basis of neutral equilibrium equations of the type (4.6) and (4.7). The solution of the problems reduces to finding the eigenvalues, and selecting among them those which correspond to the transitions from stability to instability. Various methods are used, methods that were borrowed from mathematical physics, numerical analysis, theory of oscillations and the more specialized techniques of structural mechanics, the theory of shells, etc. Among these, variational methods occupy an important place, the Rayleigh-Ritz method (1873, 1889, 1908), the Bubnov method (1911) and other methods. The application of these methods is discussed in detail in the books by S. P. Timoshenko (1946), P. F. Papkovich (1939), L. S. Leybenzon (1945), Ya. A. Pratusevich (1948), et al. In problems in the stability of shells, the loss of stability as a rule is accompanied by passage through the limiting points. In addition the post-critical states of the shells are of engineering interest. Therefore, in the stability theory of shells, nonlinear equations and the corresponding energy functions are widely used. Here variational methods are almost the only means for obtaining concrete numerical results (Kh. M. Mushtari, 1946, 1955, A. S. Vol'mir, 1956, 1965, Kh. M. Mushtari and K. Z. Galimov, 1957, A. V. Pogorelov, 1962, 1966, 1967, et al.). Many problems were solved with the aid of the P. F. Papkovich (1939) procedure, according to which some of the equations are satisfied exactly and some in the variational sense. The method of reducing the stability problem to ordinary differential equations also became popular (V. Z. Vlasov, 1932, 1939).

The application of the method of successive loads to the calculation of the stability of shells during finite deflections (V.V. Petrov, 1959) is a modification of the method of successive approximations. The method, which ignores the bulging forms when the critical forces of the shells are determined, was first proposed for linear problems (V. Z. Vlasov, 1949) and was subsequently extended to nonlinear problems for homogeneous (A. V. Sachenkov, 1963, K. Z. Galimov, 1965) and layered shells (E. I. Grigolyuk. P. P. Chulkov, 1965).

A series of results in the theory of elastic stability was obtained with the aid of other analytical methods, for example, the method of a small parameter (P. Ya. Polubarinova-Kochina, 1936, S. A. Alekseyev, 1956), the method of linear integral equations (N. V. Zvolinskiy, 1937, Ya. L. Nudel'man, 1949), asymptotic methods (I. I. Vorovich, 1955, V. M. Kornev, 1967). Numerical methods and matrix methods are widely used (A. F. Smirnov, 1947, 1958, A. A. Petropavlovskiy, 1961, A. F. Smirnov and his collaborators, 1964, et al.). Structural mechanics methods are used for the calculation of rods and rod systems, namely the method of forces, the method of

deformations and the method of initial parameters (N. V. Kornoukhov, 1939, 1949, A. F. Smirnov, 1947, I. P. Prokof'ev and A. F. Smirnov, 1947, N. K. Snitko, 1952, 1956, V. G. Chudnovskiy, 1952, A. R. Rzhanits, 1955, S. A. Rogitskiy, 1961, N. I. Bezukhov and O. V. Luzhin, 1963, et al.). Qualitative methods which make it possible to obtain for the critical parameters one sided and two sided bounds are also used. These methods go back to the studies of P. F. Papkovich (1937) in which certain general properties of the critical surfaces in parameter space are determined. A development of the qualitative methods is available in the studies by A. F. Smirnov (1947), Ya. L. Nudel'man (1949), R. R. Matevosyan (1961), B. M. Browde (1964), I. I. Gol'denblat (1965), et al.

When the external forces are not potentials, the static and energy methods are generally unsuitable. The number of nonconservative problems in elastic stability for which an exact solution can be obtained is very small. The usual solution method is to pass to some equivalent system with a finite number of degrees of freedom. Such system can be obtained, for example, when the distributed mass is replaced by a finite number of concentrated masses (Ye. L. Nilolai, 1928, 1929, K. S. Deyneko and M. Ya. Leonov, 1955). Another way is to apply the Bubnov method, in which the solution is sought in the form of a series with coefficients which are unknown functions of time. Another method is to solve the Cauchy problem for a sufficiently large class of initial perturbations. This solution can be obtained on models or digital computers. By modeling various perturbed motions, conclusions can be made about the stability of the unperturbed motion. This method was used by A. S. Vol'mir and his collaborators (1959, 1960), V. V. Bolotin and his collaborators (1959, 1960), V. I. Feodos'ev (1963), et al.

§5. Stability of Elastic Rods and Rod Systems

Problems of elastic rods and rod systems loaded by potential forces are among the less developed branches in the theory of elastic stability. The study of these problems, which started already in the 18th Century, was begun by L. Euler and was continued by J. L. Lagrange, G. Kirchhoff and other major mathematicians and specialists in mechanics. The intense development of industry, transportation, shipbuilding, etc., toward the end of the 19th and the beginning of the 20th Centuries, served as an impetus which intensified the development of the practical aspect of the theory of elastic stability. The computational scheme for the majority of structures at that time were rods and rod systems. The main attention of investigators was focused, in the beginning, on rods. The studies of F. S. Yasinskiy, I. G. Bubnov and S. P. Timoshenko go back to this period.

Among the studies on the equilibrium stability of elastic systems published during the Soviet period, three trends can be singled out.

The first trend includes studies on the stability of curvilinear rods and rings. It includes the studies of Ye. L. Nikolai (1918, 1923), A. N. Dinnik (1929-1936), and I. Ya. Shtayerman (1929-1937) which were subsequently continued by A. A. Belous (1937), G. Yu. Dzhanelidze (1939), E. I. Grigolyuk (1951), V. G. Chudnovskiy (1952), Ya. A. Pratusevich (1952), A. B. Morgayevskiy (1957, 1959), V. M. Makushin (1959), et al. In a number of studies, the three-dimensional forms in the loss of stability were studied, taking into account the behavior of the load during loss of stability (A. A. Petropavolskiy, 1953, V. V. Kholchev, 1961, et al.).

The second trend is the study of the stability of thin-walled rods with an open and closed profile. The first fundamental results here were obtained by S. P. Timoshenko (1905, 1906) who constructed the stability theory of rectilinear H beams. The subsequent fundamental results were obtained by V. Z. Vlasov (1936-1940) who developed the general theory of thin-walled rectilinear rods and studied in detail the bending-torsional forms of loss of stability and introduced the concept of the stability circle, etc. The studies of V. Z. Vlasov were continued by I. F. Obraztsov (1949, 1953), S.A. Ambartsumyan (1953), Yu. D. Kopeykin (1957, 1960), V. I. Reut (1959), V. V. Meshcheryakov (1959, 1962), et al.

The third trend studies the deformation of rods after the loss of stability. A number of studies deal with the calculation of ductile elements that are encountered in instrument building. The calculation of these elements is based on the exact (nonlinearized) equations for the elastic curve. Ye. P. Popov (1948) introduced a classification for the equilibrium forms of ductile rods which initially had a straight or circular axis and proposed an efficient method for finding these forms. The postcritical deformations of elastic rods, constrained as a result of the imposed constraints, are studied in a number of studies and the load-bearing capacity after the loss of stability is estimated.

An extensive literature deals with problems in the calculation of the stability of rod systems. Statically indeterminate frames and arcs are typical computational schemes in bridge building industrial construction, machine building for transportation, etc. The calculation of the stability of such systems presents considerable computational difficulties, especially when the system consists of a large number of rods and when the degree of statical indeterminacy is

relatively high. A large number of techniques was developed to overcome these difficulties, which go back to the beginning of the classical methods in structural mechanics. Various methods are discussed in the books by A. F. Smirnov (1947, 1958), N. V. Kornoukhov (1949), A. I. Segal (1949, 1955), N. K. Snitko (1952, 1956), V. G. Chudnovskiy (1952), A. R. Rzhanitsyn (1955), I. K. Snitko (1960), R. R. Matevosyan (1961), A. A. Pikovskiy (1961), S. A. Rogitskiy (1961), N. I. Bezukhov and O. V. Luzhin (1963). The studies of V. A. Gastev (1929), I. M. Rabinovich (1932), A. P. Korobova (1934-1954), S. N. Nikiforov (1938), A. A. Kurdyumov (1941-1964), N. K. Snitko (1947-1966), et al., deal with problems in the stability of rod systems.

In addition to the classical structural mechanics method, numerical methods were developed extensively. A. F. Smirnov (1947) proposed a matrix method for the calculation of complex rod systems with an arbitrary degree of statistical indeterminacy. This method, which combined structural mechanics concepts with the idea of interpolation methods, turned out to be a very general computational means which can be conveniently implemented on electronic computers. An analogous method was proposed abroad only ten years later (the work of G. Argiris, et al.).

The method of A. F. Smirnov was applied to the calculation of the stability of complex rod systems, span structures in bridges, arcs with structures above the arc, intertwined frame systems, tall radio masts, etc. The further development of the method is presented in the books by A. F. Smirnov (1958) A. F. Smirnov, A. V. Aleksandrov, N. N. Shaposhnikov and B. Ya. Lashchenikov (1964), in the articles by A. V. Aleksandrov (1955, 1957), A. A. Petropavlovskiy (1957, 1964), V. A. Smirnov (1962), B. Ya. Lashchenikov (1963), B. P. Derzhavin (1966), and others.

Engineers who make the calculations also need qualitative methods which can be used to obtain rough estimates of the numerical values of the critical forces in order to find easily the best ways of increasing the stability and apply the results obtained for some system to a broader class of systems. The main results in this field were obtained by P. F. Papkovich (1937), A. F. Smirnov (1947), Ya. L. Nudel'man (1949), R. R. Matevosyan (1961), B. M. Browde (1963). An example of the qualitative method are the theorems of P. F. Papkovich on the convexity of the critical surface. Another example are the theorems of A. F. Smirnov on the necessary conditions for increasing the critical forces by changing the properties of the system.

§6. Stability of Elastic Plates. Stability of Elastic Shells (Linear Theory)

Linear stability theory of a plane equilibrium form of elastic thin plates has been developed in great detail. Many results were obtained in the prerevolutionary period by S. P. Timoshenko (1907-1916), A. N. Dinnik (1911), K. A. Chalyshev (1914), I. G. Bubnov (1914). The studies of the last author which dealt with structural ship elements were continued by P. F. Papkovich (1920), A. P. Filippov (1933), A. Sh. Lokshin (1935), N. V. Zvolinskiy (1938), A. I. Lur'e (1939), P. A. Sokolov (1939), and others.

At the present time many studies dealing with the stability of plates of various shapes under different types of loads have been accumulated. N. A. Alfutov and L. I. Balabukh (1967) again returned to this problem and changed the variational method used to find the critical parameter of the external forces.

Experience has shown that the plates can usually carry a considerable load, even after loss of stability. The analysis of postcritical deformations is based, as a rule, on the system of equations derived by T. Karman (1910):

$$\left.\begin{aligned} D\,\Delta\Delta w - (w_{,xx}\chi_{,yy} + w_{,yy}\chi_{,xx} - 2w_{,xy}\chi_{,xy}) = 0, \\ \Delta\Delta\chi + Eh\,(w_{,xx}w_{,xy} - w^2_{,xy}) = 0. \end{aligned}\right\} \qquad (6.1)$$

Here the same notation is used as in equation (3.9). Many studies are devoted to approximate methods used to integrate equations (6.1). P. F. Papkovich (1920) proposed a method according to which the first equation in (6.1) is satisfied approximately in the Bubnov sense, the second equation is satisfied exactly, and the tangential boundary conditions are satisfied in the mean. The development of this idea can be found in the studies of P. A. Sokolov (1932), E. I. Grigolyuk (1949), M. A. Koltunov (1953), A. S. Vol'mir (1956), A. V. Karmishin (1956), and others. Along with this, other methods were used to solve equations (6.1), the method of a small parameter (P. Ya. Polubarinova-Kochina, 1936), the method of successive approximations (S. A. Alekseyev, 1956), the asymptotic method (I. I. Vorovich, 1955), and the method of finite differences (A. S. Vol'mir and A. Yu. Birkgan, 1963), and other methods.

The needs of modern technology served as an impetus for the development of the theory of stability of anisotropic and layered plates. Stability problems in anisotropic plates were worked on by S. G. Lekhnitskiy (1941-1947) and S. A. Ambartsumyan (1961). An extensive literature is devoted to the stability of three-layer plates with soft and hard fillers (A. P. Prusakov (1951), E. I. Grigolyuk (1957, 1958), L. M. Kurshin (1958), A. Ya. Aleksandrov, L. E. Bryukker, L. M. Kurshin and A. P. Prusakov (1960), A. V. Ivanov (1964)). The stability of bimetallic plates was studied by E. I. Grigolyuk (1953). The theory of multi-layered plates consisting of alternating hard and soft layers was developed by V. V. Bolotin (1963). The theory was applied to the calculation of the global and local stability of plates by L. P. Pomazi (1965) and Ye. N. Sinitsyn (1966).

The early result in the stability theory of shells were derived on the basis of linear theory by R. Lorentz (1908), S. P. Timoshenko (1910, 1914), R. Southwell (1913), R. Mises (1914, 1929), R. Zelli (1915), L. S. Leybenzon (1917). It became evident that for certain basic types of shells and loads, the critical parameters could be determined using very simple approximate formulas. Thus, in the case of a circular cylindrical shell of radius R and thickness of the wall h, which is neither too long nor too short, loaded by axial forces p, the approximate formula

$$p_* \approx 0,6 \frac{Eh}{R}. \tag{6.2}$$

can be used. For a spherical shell under the action of external hydrostatic pressure p a simple approximate formula is also obtained

$$p_* \approx 1,2 \frac{Eh^2}{R}. \tag{6.3}$$

Many subsequent studies were devoted to improving the accuracy and to generalizing the early results. Problems in the stability of cylindrical shells under different loads were studied including a combined load (N. V. Zvolinskiy, 1935, 1937, Kh. M. Mushtari, 1938-1957, A. S. Vol'mir, 1950-1956, V. M. Darevskiy, 1957-1965, S. N. Kan, 1962-1966, V. V. Kabanov, 1963-1967, and others). Problems in the stability of conical shells were investigated (Kh. M. Mushtari, 1943, E. I. Grigolyuk, 1951, 1955, I. I. Trapezin, 1952-1960, N. A. Alumyae, 1955, 1957, and others), of toroidal shells

(P. A. Zagubinenko and I. N. Spiridonov, 1959) and shells of other shapes. Problems in the stability of shells in the presence of temperature gradients were also studied (V. V. Kabanov, 1962, and others). In the studies of Kh. M. Mushtari (1938-1943) a shell with a large number of reinforcing elements was replaced by an equivalent smooth anisotropic shell. Such a "spreading" of the reinforcing elements was later applied by many authors. The discreteness of reinforcing edges was taken into account in the studies of N. A. Alfutov (1956), V. M. Darevskiy and R. I. Kshnyakin (1960), who analyzed the conditions under which the "spreading" is admissible. The stability of structural anisotropic shells was studied by V. V. Kabanov (1964, 1967).

Rather recently attention was focused on the classical linear problem of the stability of cylindrical shells. It became evident that a relaxation of the tangential boundary conditions may lead to a considerable reduction of the critical forces in comparison with the classical boundary conditions. Thus, in the problem of the axial compression of a circular cylindrical shell, the transition from the classical support on joints to a support in which the tangential stresses at the ends vanish, reduces the critical force almost by a factor of two. The number of halfwaves in the circular direction corresponding to the loss in stability is reduced. Among the studies dealing with the effect of the tangential boundary conditions we mention the studies of A. S. Avdonin (1963), V. I. Kozhevnikov (1964), N. A. Alfutov (1965), N. A. Kil'chevskiy and S. N. Nikulinskaya (1966), Yu. M. Khishchenko (1966).

The calculation of real structures requires a study of the load-bearing capacity of shells under the action of local loads which have considerable initial imperfection, and generally shells which are in the stressed state from the instant when the loading begins. The number of studies along these lines is very large.

Problems in the stability of three-layer shells of various shapes with a filler were also developed (E. I. Grigolyuk, and P. P. Chulkov, 1963, L. M. Kurshin, 1958-1964, T. N. Vasitsyn, 1962, K. Z. Galimov, 1965, and M. A. Koltunov, 1965).

7. Stability of Elastic Shells (Nonlinear Theory)

The most important stimulus for the development of a nonlinear theory of elastic shells were the systematic discrepancies between the results obtained from linear theory and the experimental data. For many types of shells and loading conditions, the experimental critical forces were considerably

smaller than the values calculated on the basis of linear theory. The phenomenon of the loss of stability often occurs in the form of a "crack," "fissure," i.e., it is accompanied by a discontinuous increase in the deformations with a considerable change in the shape of the middle surface. The postcritical deformation pattern which is observed usually differs considerably from the bifurcation shape predicted by linear theory.

The first studies in the nonlinear theory of thin elastic shells were made in the prewar period (the studies of L. G. Donnel, T. Karman and S. S. Tsyan, Kh. M. Mushtari). They are essentially based on the creep theory of rods developed by S. P. Timoshenko (1925, 1935), K. B. Biceneau (1929) and K.Marguerre (1938). After the war these studies proceeded on a wide scale. The key idea in these studies was that the existence of stable equilibrium forms which were different from the unperturbed form for values of the loading parameters smaller than the classical critical value, was typical of problems in the stability of shells. Many studies were devoted to finding the lower critical forces for various types of shells and boundary conditions and types of loads. The stability of cylindrical shells and panels was studied by A. S. Vol'mir (1944-1956), Kh. M. Mushtari, K. Z. Galimov, M. S. Kornishin and A. V. Sachenkov (1946-1957), M. A. Koltunov (1952), N. A. Alumyae (1954), O. I. Terebushko (1956) and by many other authors. The stability of spherical shells and panels was studied by V. I. Feodos'ev (1946-1961), Kh. M. Mushtari and R. G. Surkin (1950-1956), E. I. Grigolyuk (1956, 1959), N. K. Lebedeva (1964), I. I. Vorovich and V. F. Zipalova (1966) and others. The stability of conical shells was studied by E. I. Grigolyuk (1956). The problem of the existence of lower critical forces was studied by I. I. Vorovich (1955, 1957). The cracks in bimetallic shells during heating and cooling were studied (D. Yu. Panov, 1948, E. I. Grigolyuk, 1953). The postcritical behavior of three-layer shells (cylinder, sphere, cone) was studied (E. I. Grigolyuk and P. P. Chulkov, 1965). More detailed information can be found in the books by Kh. M. Mushtari and K. Z. Galimov (1957), A. S. Vol'mir (1956, 1967), and also in A. S. Vol'mir's survey (1966).

Usually the studies were based on the theory of flat shells. We present the original system of differential equations for the problem as applied to three-layer shells with a hard filler subjected to a transverse shear (E. I. Grigolyuk and P. P. Chulkov, 1963).:

$$
\left.
\begin{aligned}
&D\left(1-\frac{vh^2}{\beta}\nabla^2\right)\nabla^2\nabla^2\chi = F_{,22}(k_{11}-w_{,11}-w^0_{,11})-\\
&\qquad -2F_{,12}(k_{12}-w_{,12}-w^0_{,12})+F_{,11}(k_{22}-w_{,22}-w^0_{,22})+q,\\
&\nabla^2\nabla^2 F = \frac{1}{2}Eh\,[(2k_{11}-w_{,11}-w^0_{,11})w_{,22}+\\
&\qquad +(2k_{22}-w_{,22}-w^0_{,22})\,w_{,11}+2w^2_{,12}-2w^2_{,12}(2w^0_{,12}-2k_{12})],\\
&\qquad\qquad \frac{1-v}{2}\frac{h^2}{\beta}\nabla^2\varphi = \varphi.
\end{aligned}
\right| \qquad (7.1)
$$

The deflection is

$$w = \left(1-\frac{h^2}{\beta}\nabla^2\right)\chi.$$

Here $\nabla^2(\ ) = (\ )_{,11} + (\ )_{,22}$, $x_i\ (i = 1, 2)$ are orthogonal coordinates in the original surface, D is the cylindrical rigidity of the packet, h is its thickness, v is the flexural rigidity parameter of the outer layers, k_{ij}(i, j = 1, 2) are the curvatures of the lines of the original surface, w^0 is the initial deflection, q is the transverse load and the Poisson ratio for all three layers is the same.

For $\beta = \infty$, $w = \chi$, $\varphi = 0$ the well-known Marguerre equation is obtained and with the additional condition

$$k_{11} = k_{22} = k_{12} = 0$$

we have the Föppl-Karman equation (6.1). System (7.1) can be treated as an improvement of the classical theory of the bending of homogeneous shells when the shear along the thickness is taken into account. Unlike in the classical theory, this system is compatible with five natural boundary conditions.

Equations of type (7.1) can also be constructed for multilayer shells which are homogeneous, orthotropic and isotropic during finite deflections. This was done in the studies of E. I. Grigolyuk and P. P. Chulkov (1965). The essence of the matter consists of the following. The shell, whether it is a layered shell or not, is broken up into a number n of fictitious shells. Next, for the displacements of the points in each

fictitious shell a linear distribution law is adopted which depends on the transverse coordinate. The conditions for the conjugation of the layers and the hypothesis of the incompressibility of the material of each layer in the transverse direction make it possible to characterize the displacement of the points in the entire packet by 2n + 3 independent functions of the coordinate parameters for the surface of the shell.

The principle of possible displacements yields exactly 2n + 3 equilibrium equations, which, on the basis of Hooke's law, are written in terms of the displacements. It is clear that such an approach makes it possible to obtain a two-dimensional system of equations of infinite order which is equivalent to a system of three-dimensional elasticity equations for a layer shell, assuming its incompressibility in the transverse direction. For this, it suffices if the number of fictitious layers tends to infinity uniformly over the entire thickness of the shell. The constraint imposed by the incompressibility of the material of the layers in the transverse direction is not essential, since it is easily removed by assuming that also the normal displacements within each layer are distributed according to a linear law, which depends on the transverse coordinate. Expressing the displacements with the aid of differential cperations in terms of three arbitrary functions, (the stress function F and the two displacement functions χ, ψ) it is possible to reduce the original system of 2n + 3 equations to three equivalent resolvent equations of the same order. The advantage of the resolvent equations in comparison with the original system is mainly that the differential operator of the resolvent system contains coefficients which decrease rapidly as the order of the derivatives increases. This makes it possible, depending on the external load and the type of boundary conditions, to retain the derivatives of the necessary order, i.e., in fact, only take into account the important boundary effects. Mathematically the introduction of the stress function F and the displacement functions χ, ψ, is equivalent to expanding the stressed-strained state in the eigenfunctions of the holoaxial positive-definitive operator which is specially adapted to the structure of the given layered shell, and the retention of the principal coefficients in the expansion corresponds to the truncation of the operators in the resolvent equations.

The subsequent development of these studies led to a substantial revision of the point of view adopted for this problem. In order to make the situation more understandable, we will consider in greater detail the problem of the stability of a circular cylindrical shell undergoing axial compression.

The classical theory, under certain simplifications, gives for this problem the approximate formula (6.2). The experimental values usually lie in ranges to which in this formula correspond the values from 0.18 to 0.60 of the numerical coefficient. The greater value is obtained for the most carefully manufactured shells and for the most accurate experimental conditions. The theoretical value of the lower critical force is calculated approximately for this problem by applying variational methods either to the system of equations of the nonlinear theory of shells or to the corresponding energy functional. The number of parameters varied in the early studies was small. Thus, expressions of the form

$$w = f_1 \sin \frac{m_1 \pi x_1}{l} \sin \frac{m_2 x_2}{R} + f_2 \sin^2 \frac{m_1 \pi x_1}{l} \sin^2 \frac{m_2 x_2}{R} + f_0,$$

were used in which the first term corresponds to the linear theory approximation and the remaining terms take into account the nonsymmetric character of the dents formed after the "crack." Here f_0, f_1 and f_2 are the parameters that are varied, l is the length of the shell, and m_1 and m_2 are positive integers. The calculations based on such expressions led to values of the lower critical force to which a coefficient which is close to 0.2 corresponds in formula (6.2). This value fully satisfied the investigators. However, recently results of more accurate computations in which electronic computers were used became available, which made it possible to increase considerably the number of parameters that were varied. It became evident that improving the accuracy of the method entails a reduced lower critical value. Thus, according to the data of N. G. Hoff (1966), when the number of terms in the series is increased from three to fifteen, the coefficient in formula (6.2) is reduced from 0.1860 to 0.0427. It is not clear by what amount this coefficient will decrease when we further increase the number of terms in the series. The situation in the stability problem of a closed spherical shell under the action of external pressure turned out to be no less dramatic. We present some data about the coefficient in formula (6.3) when it is applied to find the lower critical pressure. Kh. M. Mushtari and R. G. Surkin (1950) obtained the value 0.10, V. I. Feodos'ev (1954) the value - 0.13 (a negative value), Kh. M. Mushtari (1955) the value + 0.11, A. G. Gabril'yants and V. I. Feodos'ev (1961) + 0.06.

The explanation of these and similar results consists of the following. The forms of the middle surface which are given after the crack in the form of a finite series considerably limit the class of its possible equilibrium forms. At the same time the shell is a continuous system. The possibility that for loaded shells an infinite set of subcritical equilibrium forms exists, which are different from the unperturbed form, including a certain number of stable equilibrium forms, is not excluded. When the number of terms in the series is increased, the class of possible equilibrium forms becomes larger. Incidentally, their practical value is very limited. To realize these forms, a finite perturbation of a special class and a sufficiently large quantity are needed.

The main reason for the reduced experimental critical forces in comparison with their classical magnitudes are the initial deflections of the middle surface from the ideal form, the imperfections in the supports, the presence of residual stresses, etc. The upper critical force for real shells, as a rule, is very sensitive to a change in the parameters of the initial imperfections. This explains both the reduction in the experimental critical forces and also their large scatter. The last fact makes it necessary to take into account the random character of the initial imperfections, which can only be done by using statistical methods.

The situation with regard to the stability of flat panels supported on a sufficiently rigid profile is somewhat better. The stability of cylindrical, conical and spherical panels in a nonlinear formulation was studied by A. S. Vol'mir (1956), E. I. Grigolyuk (1956, 1960), O. I. Terbushko (1958), I. I. Vorovich and V. F. Zipalova (1966). The presence of a sufficiently rigid profile narrows down considerably the class of possible forms of loss of stability of a panel; therefore. here approximations with a few terms give usually sufficiently reliable result. A similar situation is encountered in the calculation of reinforced shells.

A new trend in the nonlinear theory of shells was developed by A. V. Pogorelov (1960, 1962, 1966, 1967), A. V. Pogorelov introduced the assumption that the form of the part of the middle surface with the cracks is isometric to its original form. The cracked part makes contact with the remaining part of the middle surface along some edges in the neighborhood of which local bending occurs. Since the method for calculating the displacements and the critical forces used by A. V. Pogorelov differs little from the usual energy method, the most important part of the assumptions made by A. V. Pogorelov is the introduction of a new large class of functions which describe approximately the deformations in thin shells.

A. V. Pogorelov made the calculations for a very large class of problems and compared the results of the computations with his original experiments. The method of A. V. Pogorelov was also applied by V. I. Babenko (1966) and V. V. Mikhaylov (1966). A discussion of the studies of A. V. Pogorelov is available in the supplement by I. I. Vorovich in A. V. Pogorelov's book (1966).

§8. Stability of Elastic and Elasto-Plastic Systems

Since stability problems are typical of thin and thin-walled bodies, they are usually formulated and solved within the frame of reference of applied theories for rods, plates and shells. Nevertheless, there are several reasons why certain stability problems should be studied from the standpoint of general elasticity theory.

First, the general equations of nonlinear elasticity theory are used to derive rigorously the stability equations for thin and thin-walled bodies. Studies along these lines (V. V. Novozhilov, 1940, 1948, V. V. Bolotin, 1956, 1965, A. I. Lur'e, 1966, and others) were already discussed in §3. Second, the solution of problems obtained on the basis of elasticity theory, can be used to estimate the accuracy and to determine the range of applicability of known approximate solutions. This trend includes the studies of L. S. Leybenzon (1917) and A. Yu. Ishlinskiy (1954). We note that in these studies it was proposed that the equations used for describing equilibrium forms adjacent to the unperturbed form be the classical elasticity theory equations. The external forces appeared only in the perturbed boundary conditions. This approach was recently analyzed by A. N. Guz' (1967). Third, the equations of elasticity theory must be used in stability problems of plates and shells making contact with an elastic material of lower rigidity. A. P. Voronovich (1948), V. N. Moskalenko (1964) and others applied this approach to layered plates with a soft filler. The stability of cylindrical shells with a soft elastic core was studied by A. P. Varvak (1966). The application of the theory of plates and shells to the load-bearing layers is typical of such problems and the application of three-dimensional elasticity theory to the filler is also typical.

If the system does not have sufficient ductility, loss in stability can occur in the elasto-plastic state. F. Engesser developed the stability theory of centrally compressed rods beyond the elasticity limit on the assumption that a loading process takes place at all points of the cross section. In this case, the critical force is determined not by the elasticity modulus like in a problem for an elastic material, but by the tangential modulus (we obtain a tangential-modulus critical

force). F. S. Yasinskiy noted with regard to this theory that unloading should be taken into account in a part of the cross section. This leads to the existence of a neutral axis in the cross section. Taking into account unloading in the cross section on the assumption that the resultant axial force does not change, F. Engesser obtained a formula for the critical force which differs from the corresponding formula for an elastic rod in that instead of the modulus of elasticity it contains a normalized modulus which depends on the shape of the cross section of the rod. For almost the entire first half of our century it was assumed that the normalized modulus load is the critical load for elasto-plastic systems and that Engesser's original result was erroneous. Many studies were published in which various problems are solved on the basis of this concept.

The normalized modulus concept was the result of extending the theory of bifurcation equilibrium forms from the theory of elastic stability to elasto-plastic problems. This extension was unjustified, which became a generally recognized fact only after the limited applicability of the normalized modulus concept was demonstrated on simple models. It became evident that the lower boundary for the critical forces is equal to the tangential-modulus load whose magnitude is calculated using an analogous formula for an elastic rod in which the tangential modulus is substituted instead of Young's modulus The conditions under which a tangential-modulus critical load is realized can be easily singled out.

This problem was studied Yu. N. Rabotnov (1952), Ya. G. Panovko (1954-1965), V. D. Klyushnikov (1957, 1964), G. V. Ivanov (1961, 1963), Yu. A. Chernukha (1966), and others. In particular, V. D. Klyushnikov studied the stability problem of the simplest elasto-plastic system in a dynamic formulation and showed that the unperturbed state of the system is stable until the tangential-modulus load is attained.

The conclusion that the tangential-modulus and normalized modulus loads limit the interval of real critical forces was very attractive, even more so, since for many systems the numerical difference between these values was small. However, an example exists in which the critical force apparently exceeds the normalized value of the modulus. A. A. Il'yushin (1960) and V. G. Zubchaninov (1960) studied the bulging of an elasto-plastic rod included in a statically indeterminate rod system. When the system has an unloading effect on the rod, the authors show that the rectilinear shape of the rod may remain stable even when the normalized modulus load is exceeded.

Until now relatively complex problems in elasto-plastic stability were studied in a restricted formulation which was analogous to the tangential-modulus or normalized modulus concept. The ultimate results depend on which variant of the theory of plasticity is used. The stability of elasto-plastic rods was studied by L. M. Kachanov (1951, 1956), A. V. Hemmerling (1952, 1959, 1965), Ya. G. Panovko (1954, 1962), A. R. Rzhanitsyn (1955), V. V. Pinadzhyan (1956), Yu. R. Lepik (1957), and B. P. Makarov (1965). A general theory of stability of plates and shells based on deformation theory was proposed by A. A. Il'yushin (1944). He also obtained a solution of a number of interesting practical problems. The general theory of stability of shells based on the tangential-modulus concept was developed by E. I. Grigolyuk (1957, 1958) within the framework of a theory of the flow and deformation type, taking into account the elastic compressibility of the material. A number of problems in the stability of plates and shells beyond the elasticity limit was studied by L. A. Tolokonnikov (1949-1955), S. M. Popov (1951, 1954), Yu. R. Lepik (1954-1957), L. M. Kachanov (1956), V. D. Klyushnikov (1957), E. I. Grigolyuk (1958) and others. A general theory of elasticity of two-layered shells beyond the elasticity limit was developed by E. I. Grigolyuk (1958), which was generalized by E. I. Grigolyuk and V. V. Kabanov (1966), who evaluated the effect of rearranging the layers on the magnitude of the critical forces. Anisotropic shells beyond the elasticity limit were calculated by V. V. Kabanov (1966, 1967). The survey by E. I. Grigolyuk (1966) is devoted to the contemporary state of the theory of stability of shells beyond the elasticity limit.

§9. Stability of Visco-Elastic and Visco-Elasto-Plastic Systems

This section discusses problems in the stability of systems whose material is damaged by creep. We will distinguish two cases: when there are no instantaneous plastic deformations and when there are instantaneous plastic deformations. In the first case we shall speak about visco-elastic systems, in the second case about visco-elasto-plastic systems. Subsequently, we will distinguish linear and nonlinear visco-elastic systems.

The nonlinearity of visco-elastic systems, at least within the framework of the majority of models proposed until now, is analytical and it can be linearized. Therefore, stability problems for visco-elastic systems turned out to be simpler than for elasto-plastic systems and the theory is more advanced. The deformation process in visco-elastic systems develops with time. The type of perturbations and the sequence with which

they act over time as well as the duration of the time interval during which stability is investigated are of essential importance. The concepts of the critical time t_*, by which is meant the duration from the instant when the load is applied until the critical state is reached in some sense is often used in the calculation of visco-elastic systems. In the general case, the time t_* is a function of the parameters of the external forces and the type of perturbations. The stability problem of systems consisting of a linear visco-elastic material is solved most easily. Let us consider, for example, a rectilinear bar compressed by a constant force P. Suppose that the material of the rod is a standard linear material in the sense that the relation between the stresses σ and strains ε is described by the formula

$$\sigma + \tau \frac{\partial \sigma}{\partial t} = E_\infty \varepsilon + \tau E_0 \frac{\partial \varepsilon}{\partial t}.$$

Here E_0 is the instantaneous elasticity modulus, E_∞ is the long-term elasticity modulus and τ is the relaxation time. The equation of perturbed motion has the form (4.7), where the elasticity modulus E must be replaced by the corresponding linear operator. It turns out that the rectilinear form of the rod is stable on any time interval provided $P < P_\infty$, where P_∞ is the Eulerian force calculated on the basis of the long-term elasticity modulus. When $P > P_\infty$, the rectilinear form of the rod will be unstable. For $P > P_0$, where P_0 is calculated from the instantaneous modulus E_0, loss of stability occurs instantaneously (with an accuracy to the dynamic transient process).

The problem that was mentioned was first studied by A. R. Rzhanitsyn (1946, 1949). The model of a linear visco-elastic body describes satisfactorily the creep in many types of polymers and concrete. Therefore, it is widely used in the calculation of structures made from these materials. We point out the studies of G. S. Grigoryan (1964) and Ye. N. Sinitsyn (1966). V. V. Bolotin and Ye. N. Sinitsyn (1967) solved the problem of the surface bulging of a halfspace from a layered material, one of whose components has linear visco-elastic properties. The general theory of visco-elastic layered shells with fillers receiving transverse shear during finite bends was developed by E. I. Grigolyuk and P. P. Chulkov (1964).

The creep in metals and alloys, as a rule, has a highly pronounced nonlinear character. The models of a nonlinear visco-elastic medium used in the theory of creep are usually such that they give for arbitrarily small stresses a creep deformation which increases without limit with time. Therefore if the stability problems for systems consisting of such materials are formulated rigorously, we will have instability in many important practical cases. At the same time the structures are used successfully under creep conditions provided the strength of the material is not disturbed and the deformations do not attain undesirable dimensions. Thus, the rigorous formulation of the stability problem turned out to be insufficiently realistic from the standpoint of engineering applications.

Many attempts were made to overcome the difficulties that were mentioned. Yu. N. Rabotnov and S. A. Shesterikov (1957) were the first to apply to stability problems of rods and plates made from a nonlinear visco-elastic material the dynamic stability criterion. They considered perturbations applied at some instant of time $t > 0$. A critical value t_* was found for which the perturbations applied at $t > t_*$ lead to a fast increase in the displacements. It became evident that the value t_* that was found agrees with certain semiempirical engineering criteria. A further analysis and development of this result is available in the articles by Yu. N. Rabotnov (1960), G. V. Ivanova (1961), L. M. Kurshina (1961) and S. A. Shesterikov (1961), I. G. Teregulova (1965, 1966).

The studies by E. I. Grigolyuk and Yu. V. Lipovtsev (1965, 1966) developed a static method for the study of the stability of visco-elastic shells based on an investigation of the branching of equilibrium forms during the creep process. Since, as a result of creep, the stresed and deformed state change continuously, at some instant of time the original equilibrium form is no longer the only possible form and contiguous equilibrium forms occur which are different from the original form. E. I. Grigolyuk and Yu. V. Lipovtsev have shown that taking into account the creep does not lead to basic changes in the stability concepts and solution methods that were developed for the study of the stability of elastic systems. Only the computation scheme is modified and made more precise. The changes are only essential in that part which is related to the determination of the stresses and strains in the initial state of the system. Here it is necessary to take into account the possible deviations of the system from the ideal state caused by the presence of initial displacements, the particular ways in which the loads are applied, etc. The neutral equilibrium equations expressed in terms of the instantaneous increments

(variations) of the stresses and strains have the same form as those for elastic systems. When they are written down it is necessary to take into account the additional deformations and stresses in the initial states which are accumulated during the creep processes.

Another way of obtaining the results is as follows: Instead of the stability problem, the problem of the behavior of the system with time for given initial conditions and other given perturbations is studied. A limiting value of the stresses, deformations, velocities, etc., is determined. The time t_* is sought in which this limiting value will be attained. Studies along these lines include the studies by V. I. Rozenblyum (1954), A. S. Vol'mir and P. G. Zykin (1962), Yu. V. Lipovtsev (1964), L. M. Kurshin and Yu. V. Lipovtsev (1964), M. A. Koltunov (1965, 1966), G. V. Ivanov and V. N. Shepelenko (1966).

Semiempirical criteria have also been used extensively, the critical deformation criterion, the tangential modulus criterion, etc. A survey of these criteria is available in the book by Yu. N. Rabotnov (1966) and A. S. Vol'mir (1967).

In the field of the stability of visco-elasto-plastic systems only the first steps were made. The problem of taking into account the instantaneous plastic deformation in the analysis of the behavior of systems made from a material damaged by creep is discussed in the book by Yu. N. Rabotnov (1966).

§10. Stability of Elastic Systems under Followup Loads

In the classical theory of stability, external potential forces were considered (mainly of gravitational origin). The development of technology led to a considerable extension of the class of loads acting on a structure. Among these, non-potential forces which do not depend explicitly on time, occupy a special position. Examples of these are forces whose vectors rotate during the deformation of the system and preserve constant angles with the orths of the local Lagrangian basis. Forces of this type are usually called follow-up forces.

Ye. L. Nikolai (1928) was the first man who studied the problem of the stability of an elastic system loaded by follow-up forces. His study investigates the stability of the rectilinear form of a ductile rod, one end of which is fixed and the other loaded by a compression force and a torque. It was established that in the case when the vector of the moment is "tangential" (i.e., it remains directed along the tangent to the bent axis of the rod) no other forms of

equilibrium except the rectilinear form exist. On the basis of this Ye. L. Nikolai concluded that the usual method of determining the critical force was not applicable to the given problem. By setting up the equations for small oscillations of the rod around the rectilinear equilibrium form, Ye. L. Nikolai established that this equilibrium was unstable for any values of the torque (if damping was not taken into account and a rod with a circular cross section was considered). In the subsequent study (1929) it was shown that in the presence of unequal flexural rigidities the rectilinear form of the rod is stable for a sufficiently small value of the torque. A critical value of the moment exists beyond which the rectilinear form is no longer stable. The results obtained by Ye. L. Nikolai were further developed by G. Yu. Dzhanelidze (1939) and I. Ye. Shashkov (1941, 1950).

The studies by Ye. L. Nikolai did not include explicit indications with regard to the nonpotential character of the external forces. In 1939, V. I. Reut formulated the stability problem for a cantilever beam with a crosspiece at the end. The rod was compressed by a form whose line of action did not change in space. It became evident that also here no other equilibrium forms, except the rectilinear form, exist. B. I. Nikolai (1939) pointed out that the force was nonconservative and studied the small oscillation of a rod around the perturbed equilibrium position and obtained the critical value of the force. The studies of Ye. L. and B. L. Nikolai apparently have not been noticed for a long time. In particular, this can be seen from the fact that G. Ziegler published in 1951-1953 a series of studies which duplicated to a considerable degree the results of Ye. L. Nikolai. On the other hand in the 50's several studies appeared in which the absence of continguous equilibrium forms was erroneously classified as a stability criterion for unperturbed equilibrium, in which the energy method was applied to nonconservative systems, etc. In the subsequent years, the number of publications on nonconservative problems in elastic stability increased sharply. We point out the studies by K. S. Deyneko and M. Ya. Leonov (1955), V. V. Bolotin (1956, 1959), G. Yu. Dzhanelidze (1958, 1965), K. N. Gopak and S. G. Krivosheyeva (1959), L. M. Zoriy and M. Ya. Leonov (1961, 1962), N. I. Ginger (1967). A survey of studies along these lines can be found in the article by G. Yu. Dzhanelidze (1965).

We will point out certain specific features in stability problems of elastic systems loaded by follow-up forces. The loss of equilibrium stability of such a system can have both an oscillatory and non-oscillatory character. In the first case the critical parameter of the external forces depends not only on the distribution of the rigidities, but also on the distribution of masses of the system. In particular, the critical parameter can be reduced by bringing closer the partial frequencies of the natural oscillations. Another

specific feature is the considerable damping effect on the value of the critical parameters when the loss of stability is of the oscillatory type. In the majority of studies on the stability of elastic systems in the presence of follow-up forces, dissipation forces are not introduced into the discussion. What is called stability in these studies is in effect quasistability in the sense of §4. When the critical parameter is calculated, taking dissipation into account and then letting the dissipation coefficients approach zero, the limiting value of the critical parameter will generally not be the same as the corresponding value found without taking dissipation into account. This effect was already discovered by G. Ziegler in 1952. Further analysis has shown (V. V. Bolotin, 1959) that the limiting value depends considerably on the relation between the partial dissipation coefficients. The quasicritical value of the parameter of the external forces is an upper bound on the critical values during dissipation, which tends to zero. Recently this problem was studied in detail by N. I. Ginger (1967).

Follow-up forces may occur as a result of idealizing the interaction of the structure with liquid or gas flows (including pressure and the reaction of jets), during the interaction of the systems with an electromagnetic field, etc., and also in the elastic branches of automatic control systems. The attempt to realize the follow-up forces using air jets was made by Yu. N. Novichkov (1967) and L. K. Parshin (1967). The stability of rods subjected to follow-up forces in a supersonic flow was studied by A. G. Gorshkov and F. N. Shklyarchuk (1966).

§11. Stability During Impact Loads

The stability problems of elastic systems also include many problems on the behavior of elastic bodies loaded by rapidly changing loads, when the latter are such that they correspond to certain equilibrium stability problems in the classical theory of elastic stability. When the dynamic loads in elastic systems are studied, their time behavior is usually determined for certain fully defined initial conditions, i.e., actually the Cauchy problem is solved. As a rule, the problem of the stability of the solutions is not formulated. Nevertheless, in applied studies references are made to "stability," "instability," "critical forces," and one meaning or another is assigned to these concepts depending on the context.

We will follow the established traditions and sometimes use these concepts in this section.

The first studies dealing with the stability of elastic systems under impact loads were made by I. M. Rabinovich, M. A. Lavrent'ev and A. Yu. Ishlinskiy. I. M. Rabinovich (1944) studied the problem of the longitudinal dynamic loading of a beam with a small initial curvature. M. A. Lavrent'ev and A. Yu. Ishlinskiy (1949) studied for the first time the effect of the magnitude of a suddenly applied force on the rate at which perturbations of various types increase. The authors used equations of type (4.7), in the first term of which w was replaced by $w - w_0$ (w_0 is the initial displacement). When the force $N = -P$ is applied at the instant $t = 0$ which for $t > 0$ remains constant, those perturbations increase most rapidly which correspond to half-wave numbers k, which are close to

$$k = \left(\frac{P}{2P_*}\right)^{1/2}.$$

Similar results were obtained for a circular ring and for a circular cylindrical shell. M. A. Lavrent'ev and A. Yu. Ishlinskiy (1949) made experiments in which the impact load was created with the aid of an explosion, which confirmed the experimental results. Among other studies dealing with the dynamic bulging of rods we mention the studies by N. K. Snitko (1944), I. M. Rabinovich (1947, 1953), V. A. Gastev (1949), I. M. Rabinovich and A. P. Sinitsyn (1956), A. S. Vol'mir (1963), A. S. Vol'mir and I. G. Kil'dibekov (1966). The dynamic bulging of an elasto-plastic rod was studied by A. K. Pertsev and A. Ya. Rukolayne (1965).

The dynamics of the bulging of plates and shells, as a rule, must be studied in a nonlinear formulation. The study reduces to the integration of equations of type (7.1) with inertial terms and nonzero initial conditions or corresponding equations with additional terms which take into account the initial imperfections, etc. In this formulation the behavior of cylindrical shells and panels was studied for the first time by V. A. Agamirov and A. S. Vol'mir (1959) as well as by G. A. Boychenko, B. P. Makarov, N. I. Sudakova and Yu. Yu. Shveyko (1959). The first group of authors studied the loading of a circular cylindrical shell by forces which increased with time. Solving the Cauchy problem on an electronic computer, they determined the value of the load corresponding to the fastest rate for the increase in the deflections. This value was called by the authors the "dynamic critical load." The second group of authors studied the sudden loading of an elastic cylindrical panel by forces whose values subsequently decrease

with time to zero. It turned out that it was possible to formulate the stability problem. For certain classes of problems, a region was constructed in the parameter plane which corresponded to the stability of the initial form of the panel. In the last few years, the dynamic bulging of plates and shells was studied extensively. A survey of these studies is available in the book by A. S. Vol'mir (1967). The effect of wave motions and plastic deformations on the behavior of shells during rapidly changing loads is of the utmost interest, but at the same time, the most difficult problem.

Problems dealing with the interaction of a shell in a gas or liquid with shock waves are allied to the above-mentioned problem. The most important results here were obtained by V. V. Novozhilov, A. D. Aleksandrin, Yu. S. Yakovlev, B. V. Zamyshlyayev, A. K. Pertsev and Yu. I. Kadashevich (1961-1964), E. I. Grigolyuk, V. L. Prisekin, L. M. Kirshin, A. G. Gorshkov and F. N. Shklyarchuk (1961, 1963, 1967), A. S. Vol'mir and M. S. Gershteyn (1965, 1966). Further studies in this field should renounce the excessively simplified assumptions about the interaction between the shells and the surrounding medium.

§12. Stability of Forced Oscillations and Parametric Resonance in Elastic Systems

For a large class of problems in the theory of elastic stability, the equations of perturbed motion involve coefficients which are periodic function of time. Problems of this kind are the stability of steady state forced oscillations of elastic systems, of a rectilinear elastic rod compressed by a periodic longitudinal force, an elastic plate or shell oscillating periodically under torqueless deformation conditions, etc. Certain problems in the theory of elastic oscillations or systems whose parameters vary periodically over time are allied with this class of problems. The phenomenon of instability in such systems is called parametric resonance.

Let us consider, for example, a rectilinear rod loaded by a longitudinal force $N = N_0 + N_t \cos \omega t$. When the axial deformations in the unperturbed state are ignored, the linearized equation of perturbed motion will have the form (4.7). Representing the solution in the form

$$w=\sum_{k=1}^{\infty} f_k(t)\,\varphi_k(x), \tag{12.1}$$

where $\varphi_k(x)$ is a complete system of basis functions, the forms of the natural oscillations of the rod, we obtain an infinite system of ordinary differential equations with periodic coefficients from the series $f_k(t)$. This important result is due to V. N. Chelomey (1938) who studied the case of an arbitrary change of the cross section along the length and an arbitrary change in the compression force N over time. In the case of a rod freely supported at the ends, the unknown functions in the system can be separated and we obtain the Mathieu equations:

$$f_{k,tt} + \Omega_k^2 (1 - 2\mu_k \cos \omega t) f_k = 0 \quad (k = 1, 2, \ldots). \qquad (12.2)$$

Here Ω_k are the frequencies of the natural oscillations of the rod loaded by the force N_0 and N_k are the Euler forces:

$$\Omega_k = \frac{k^2\pi^2}{l^2} \left(\frac{EI}{\rho F}\right)^{1/2} \left(1 - \frac{N_0}{N_k}\right)^{1/2}, \quad N_k = \frac{k^2\pi^2 EI}{l^2}, \quad \mu_k = \frac{N_t}{2(N_k - N_0)}.$$

It is known that equation (12.2) has a solution in certain domains in the parameter plane, which increases without bound with time. The instability of the unperturbed form of motion, the steady state longitudinal oscillations of the rod, corresponds to these regions. For small μ_k, the instability regions are near the frequencies

$$\omega = \frac{2\Omega_k}{n} \quad (k, n = 1, 2, \ldots). \qquad (12.3)$$

The problem that was described was first studied N. M. Belyayev (1924). In 1935 N. M. Krylov and N. N. Bogolyubov worked out the general case of fixed supports. Using Bubnov's variational method, the authors reduced, in first approximation, the problem to equation (12.2). N. Ye. Kochin (1934) studied the mathematically simpler problem of the oscillation of crankshafts. Other problems in the stability of steady state forced oscillations of rods, rod systems, plates and shells were studied by V. N. Chelomey (1938, 1939), V. A. Bodner (1938), G. Yu. Dzhanelidze and M. A. Radtsig (1940), I. S. Arzhanykh (1940), Z. I. Khalilov (1942), V. M. Makushim (1947), A. F. Smirnov (1947), A. N. Markov (1949), O. D. Oniashvili (1950), V. V. Bolotin (1951-1956), and others.

It was shown (B. Z. Brachkovskiy, 1942, G. Yu. Dzhanelidze, 1953, et al.) that a substitution of type (12.1) leads to equations of the Mathieu-Hill type if and only if the forms of the natural oscillations of the elastic system coincide with the forms for the loss of stability during static loads (the eigenvalues in the bifurcation problem). The equations for the general case were first studied by V. N. Chelomey (1938). V. V. Bolotin (1953) proposed a method for constructing instability regions in the general case. This method is based on an expansion of the solution in matrix series. V. A. Yakubovich (1958), starting with the results obtained by M. G. Crane (1955), developed a method based on the introduction of a small parameter. From the standpoint of stability, frequencies near

$$\omega = \frac{|\Omega_j \pm \Omega_k|}{n} \quad (j, k, n = 1, 2, \ldots). \tag{12.4}$$

are suspect.

M. G. Crane (1955) has shown that for Hamiltonian systems, the instability regions lie near the frequencies with the upper sign in formula (12.4). Later, V. A. Yakubovich (1957) showed that for non-Hamiltonian systems the remaining combination frequencies may also be dangerous. In certain cases (for example, in the problem of the stability of a plane-shaped strip bent by periodic moments) the combined instability regions ($j \neq k$) may be wider than the main regions ($j = k$).

In the majority of studies on the stability of forced oscillations and parametric oscillations, the dissipation forces are not taken into account. In regions which are classified as stability regions, the solutions of the linearized equations of perturbed motions are bounded. From the standpoint of Lyapunov's stability theory this corresponds to the doubtful case. Thus, more convincing results in stability require that the dissipation forces be taken into account. The high density of the instability regions found without taking into account the dissipation forces should also be mentioned. As a result of this in many problems the instability regions occupy almost the entire parameter plane. The conditions for the boundedness of the solution of the Mathieu equation with an additional term involving the first derivative of the unknown function was already studied by A. A. Andronov and M. A. Leontovich (1927). With regard to parametric oscillations of elastic systems, this problem was studied by K. A. Naumov (1946), V. V. Bolotin (1956) and K. R. Kovalenko (1959). The smallest value of the coefficient μ_k for which n-th parametric resonance of the k-th form can occur has the order

$$\mu_k \sim (\psi_k)^{1/n}. \qquad (12.5)$$

where ψ_k is the relative dissipation during oscillations with respect to this form. As a consequence of relation (12.5) when the amplitudes for the change in the external forces are not too large, only principal resonances (n = 1) and perhaps one or two side resonances are induced. It should be noted that in some cases the addition of dissipation forces may widen the instability region. An example of this is the effect of the dissipation on the regions of combined parametric resonances. On the other hand, a very important result was obtained by V. N. Chelomey (1956), who has shown that the introduction of high frequency parametric forces may stabilize the statically unstable equilibrium forms.

The problem of taking into account the displacements in the unperturbed state when the equations of perturbed motion are set up was formulated by G. Yu. Dzhanelidze and V. V. Bolotin (1956). For example, it was established that in the stability problem of a bar of rectilinear shape compressed by a periodic longitudinal force, instability phenomena can occur when the frequency of the external force is close to the natural frequency of the longitudinal oscillations of the rod. A large number of problems in the stability of rods, rod systems, plates and shells was solved taking into account the displacements in the unperturbed state. Subsequent studies were carried out by G. V. Mishenkov (1961), V. Ts. Gnuni (1961) and others. It was shown in the last study that taking into account the displacements in the unperturbed state may expand the boundaries of the instability region for a flat panel by several tens of a percent.

Nonlinear problems in parametric oscillations of elastic systems were first studied by I. I. Gol denblat (1948). A systematic study of nonlinear problems for rods, rod systems, plates and shells was made by V. V. Bolotin (1951-1956). Parametric oscillations of thin shells, taking into account geometric nonlinearity, were studied by G. V. Mishenkov (1961), S. A. Ambartsumyan and V. Ts. Gnuni (1961) and others. Nonlinear combination oscillations of elastic systems were studied by G. V. Mishenkov (1966).

Details and additional biographical references can be found in the survey articles by Ye. A. Beylin and G. Yu. Dzhanelidze (1952) and G.Yu. Dzhanelidze (1965).

§13. Stability of Elastic Systems Interacting with a Liquid or Gas

Problems in which the steady state movement of elastic bodies interacting with a liquid or a gas are studied are of great interest. In these problems, the action of the medium on the structure as well as the reverse effect of the deformations in the structure on the distribution of the velocities, pressures, etc., in the surrounding medium are important. According to the existing terminology, problems of this type are problems in aerohydroelasticity theory.

The first studies in the field of aeroelasticity were related to the calculation of the stability of wings and fins of airplanes. Aeroelastic instability phenomena (divergence of the wing, flutter of the wing and of the tail fin) were the cause of a number of failures in the early beginnings of aviation. The proper understanding and theoretical explanation of these phenomena came only much later. An important contribution to this field was made by M. V. Keldysh and M. A. Lavrent'ev (1935), Ye. P. Grossman (1937), who solved a number of problems, by modeling the structure as a beam model. From the standpoint of the theory of elastic stability, flutter and divergence are typical instability phenomena in the presence of nonconservative forces. Flutter corresponds to oscillatory instability and divergence to a loss of stability due to the branching of the equilibrium forms.

Judging by the number of publications, the most representative direction in this field is the theory of the flutter of plates and shells in a supersonic gas flow. An intense development of this theory began 10 to 15 years ago, in connection with the problem of ensuring the stability of thin platings of aircraft. The attractiveness of the theory of panel flutter to investigators is explained, to a considerable extent, by the fact that many problems could be formulated in a "pure" form, i.e., they were not complicated by engineering detail. In the last few years the applied significance of the theory became much more important.

Many studies on panel flutter were based on the theory of plane sections (piston theory) developed by A. A. Il'yushin (1948, 1956), or on equivalent theories, the formulas of Ya. Ackert, G. Lighthill, etc., which establish the local relation between the perturbed pressure of a supersonic flow and the deformed surface and the displacement of the surface at a given point. Thus, small perturbations $\bar{p}$ in the pressure on the plate which is streamlined by a supersonic flow with an unperturbed velocity U, directed along the Ox axis are determined from the formula

$$\bar{p} = \frac{\varkappa p_0}{c_0}\left(\frac{\partial w}{\partial t} + U\frac{\partial w}{\partial x}\right). \qquad (13.1)$$

Here p_0 is the unperturbed pressure, c_0 is the unperturbed velocity of sound, and $\varkappa$ is the polytropic index. The basic qualitative and quantitative results were obtained on the basis of formulas of type (13.1). The panel flutter phenomenon was predicted, the order of magnitude of the critical velocities was estimated and the effect of the curvature of the shell, the presence of forces in the middle surface, structural damping, etc. were investigated. In particular, it was shown that for a plane rectangular panel, which is not loaded, with sides of the same order a, the critical velocity U_* is on the order

$$U_* \sim \frac{\pi^4 D}{a^3}\frac{c_0}{\varkappa p_0}.$$

The main results here were obtained by A. A. Movchan and his collaborators (1956-1961), R. D. Stepanov (1957, 1960), V. V. Bolotin (1958-1960), Yu. Yu. Shveyko (1960-1966), E. I. Grigolyuk, R. Ye. Lamper and L. G. Shandarov (1963, 1964).

The study of panel flutter in a nonlinear formulation is of interest in two respects. First, it makes it possible to estimate the amplitudes of the displacement and stresses when the critical flutter velocity is increased and to provide an answer to the question to what extent this overshoot is dangerous. Second, the study of nonlinear problems is necessary in order to study the behavior of an elastic system on the instability boundary and to make a judgment about the possibility of inducing natural oscillations of finite amplitude at subcritical velocities. The theory of panel flutter in a nonlinear formulation was developed by V. V. Bolotin (1958-1961), S. A. Ambartsumyan and Ch. Ye. Bagdasaryan (1961), B. P. Makarov (1961), Yu. N. Novichkov (1961-1963), Yu. Yu. Shveyko (1961), and others. In the studies that were mentioned, a number of factors was taken into account: the geometric and aerodynamic nonlinearity, aerodynamic heating, the initial forces in the middle surface and the interaction of the panel with the reinforced structure. For a plane rectangular panel which is not loaded and which is fixed on the contour of the tangential displacements, an estimate was obtained for the amplitudes (V. V. Bolotin, 1958)

$$A \sim h\left(\frac{U^2}{U_*^2}-1\right)^{1/2},$$

where h is the thickness of the panel and U_* is the critical velocity of the flutter. In a number of studies, the effect of the structural parameters on the character of the oscillations near the instability boundary was investigated, nonstationary oscillations were studied and the rate at which the amplitudes increased when the instability regions were crossed, etc., were estimated. Many results were obtained with the aid of analog and digital computers (Yu. V. Gavrilov, B. P. Makarov and Yu. Yu. Shveyko, 1959, A. S. Vol'mir, A. Yu. Birkgan and E. D. Skurlatov, 1966, 1967).

Along with the use of simplified aerodynamic formulas, stability problems of plates and shells in a gas flow were also studied, using linearized potential theory. V. V. Bolotin (1956) studied the stability of an infinitely long circular cylindrical shell streamlined outside and inside in the subsonic and supersonic region. Among subsequent studies we point out the articles by B. I. Rabinovich (1959), Yu. N. Novichkov (1963), Ye. P. Kudryavtsev (1964), A. N. Guz' and V. N. Buyvol(1966), D. A. Derbentsev (1967). S. A. Alekseyev (1967) studied the stability of a "soft" shell in a subsonic flow. Starting in 1961, the question of the boundaries of applicability of the piston theory to stability problems in plates and shells in a gas flow was studied on a number of occasions. Among the later studies we mention the articles by K. Ye. Livanov (1965) and O. Yu. Polyanskiy (1965). Along with the condition $M \gg 1$ (M is the Mach number for an unperturbed flow), the conditions for the smallness of the perturbations and the quasistationarity, some condition connecting the variability indices of the perturbations along and across the flow must be satisfied. With regard to the upper bound for the number M, it is determined taking into account aerodynamic heating, ionization, dissociation and other phenomena taking place in the boundary layer. The effect of ionization on the stability of the panel in a flow is the subject of the articles by A. D. Lisunov (1960), L. P. Klyauz and A. M. Myakushev (1966), G. Ye. Bagdasaryan and M. V. Belubekyan (1966).

Apparently some region exists for which the application of piston theory leads to sensible results. Therefore, its application is justified when the problem is complicated by certain additional (primarily structural) factors. Many studies deal with the calculation of reinforced, layered and anisotropic shells. Among these studies we first mention

the studies of S. A. Ambartsumyan and his collaborators (1963-1967), E. I. Grigolyuk and his collaborators (1965). Several articles by S. A. Ambartsumyan and his collaborators (1964-1966) study panel flutter, taking into account the effect of the temperature on the elastic parameters of the shell. Among other studies in which the piston theory is used, we mention the article by A. D. Busilovskiy, L. M. Mel'nikova and Yu. Yu. Shveyko (1966). In this article the problem of the stability of a circular cylindrical shell of finite length is studied, which by now became a classical problem. Unlike in a number of previous studies in which the Bubnov method is used, here the exact solution of the problem is brought all the way to numerical results.

The number of studies dealing with the experimental investigation of panel flutter is not large. We mention the study by G. N. Mikishev (1959), who studied the behavior of plane panels for Mach numbers from 1.7 to 3, and the study by E. I. Grigolyuk, R. Ye. Lamper and L. G. Shandarov (1964). The last authors studied the stability of cylindrical panels at $R/h = 2250$ and $M = 1.39$. The experiments confirm the general qualitative pattern predicted by the calculation even though the phenomenon is complicated by a number of side factors.

§14. Statistical Methods in the Theory of Elastic Stability

A profound relation exists between the concepts of stability and probability. Stable states and stable motions in nature and engineering are most probable and unstable ones least probable, or even impossible. A statistical approach to the problem of stability is in some sense an extension of the classical approach. Stability, in the classical sense, is basically the property of the system to remain close to the state (motion) under consideration. The statistical approach consists of studying the distributions of the parameters of the system near the state under consideration and, thus, includes a more detailed description of the behavior of the system.

The significance of statistical methods for the theory of elastic stability is primarily due to the high sensitivity of elastic systems to small changes in a number of parameters and the random character of the change in these parameters. For thin rods, plates and especially shells, such parameters are the small initial deviations from the ideal form (the initial imperfections). The effect of these small initial imperfections explains the large scatter of the experimental critical forces for thin elastic shells (B. P. Makarov, 1962, A. S. Vol'mir, 1963, and others).

The solution of stochastic problems for distributed non-linear systems is connected with serious mathematical difficulties. Therefore, usually, the distributed system is replaced by an equivalent system which has, in some sense, a finite number of degrees of freedom. One problem is to find the distribution of the critical forces from a given distribution of the parameters of the initial perturbations. Suppose that the deterministic relation between the critical parameter β_* and the perturbation parameters $u_1, u_2, \ldots, u_m$ is known. Then, under certain constraints (V. V. Bolotin, 1958) the probability density $p(\beta_*)$ can be expressed in terms of the joint density $p(u_1, u_2, \ldots, u_m)$. This method was used to analyze the distribution of the critical forces of a flat cylindrical panel loaded by axial pressures. The expected values and the variances that were calculated were close to the experimental values. B. P. Makarov (1962, 1963) and V. M. Goncharenko (1962) studied a number of other cases: the axial and hydrostatic compression of a circular cylindrical shell, the hydrostatic compression of a cylindrical panel and other problems. B. P. Makarov (1962) and A. S. Vol'mir (1963) treated statistically the experimental data obtained from the testing of shells for stability. In particular B. P. Makarov (1962) studied the experimental data from the standpoint of the hypothesis which he postulated that the critical forces may have bimodal distributions.

The problem of the behavior of elastic systems with small random imperfections during the quasistatic increase of the external forces is, strictly speaking, beyond the scope of elastic stability theory. Suppose that the deterministic relation between the state parameters of the system $v_1, v_2, \ldots, v_n$ and the loading parameters β and the parameters $u_1, u_2, \ldots, u_m$ is given. Then, under certain constraints, the joint probability density $p(v_1, v_2, \ldots, v_n|\beta)$ for the parameters $v_1, v_2, \ldots, v_n$ can be calculated. The probability that the system will remain in the region of permissible parameter values, which includes the stable equilibrium, is determined by integrating the density $p(v_1, v_2, \ldots, v_n|\beta)$ over this region. When the maximum value of the loading parameter β is a random variable, the total probability formula is used. The probability which was calculated in this manner is a measure of the reliability of the system (V. V. Bolotin, 1958, 1965).

The practical implementation of the given scheme encounters a number of difficulties. These are both the finding of the deterministic relation between the parameters and the experimental determination of the probability densities $p(u_1,u_2,\ldots,u_m)$.

The possibility of overcoming the experimental difficulties was recently demonstrated by B. P. Makarov (1967) who measured and treated statistically on an electronic computer about 60 circular cylindrical shells and constructed the correlation matrix for the first 50 Fourier coefficients for the function of the initial imperfections.

A study of the stability of elastic systems under the action of random dynamic loads was started by I. I. Vorovich (1959). Considering the shell as a nonlinear system with a finite number of degrees of freedom under the action of slowly changing forces and random impulses of the Brownian type, he used the following equation for the function $p(v_1, v_2, \ldots, v_n)$:

$$\frac{\partial p}{\partial t} = \frac{1}{2\varepsilon} \sum_{j=1}^{n} \frac{\partial}{\partial v_j} \left(p \frac{\partial Э}{\partial v_j} \right) + \frac{c}{8\varepsilon^2} \sum_{j=1}^{n} \frac{\partial^2 p}{\partial v_j^2}. \tag{14.1}$$

Here $Э(v_1, v_2, \ldots, v_n)$ is the potential energy of the system, ε is the dissipation parameter and c is a parameter characterizing the fluctuation level. The stationary solution of equation (14.1) has the form

$$p(v_1, v_2, \ldots, v_n) = C \exp\left(-\frac{4\varepsilon}{c} Э\right), \tag{14.2}$$

where C is the normalizing constant. The distribution (14.2) coincides with the Gibbs distribution in statistical physics. It describes the explicit relation betwen the stability and probability (the minimum potential energy corresponds to the maximum of the probability density and vice versa). At the same time distribution (14.2) can only be justified under very special assumptions about the character of the external random forces and dissipations in the system.

Equation (14.1) is the Fokker-Planck-Kolmogorov equation for a continuous Markov process in configuration space. In the studies by V. M. Goncharenko (1962, 1964), M. F. Dimentberg (1962, 1964), A. S. Vol'mir and I. G. Kil'dibekov (1964, 1965), the evolution of elastic systems with a finite number of degrees of freedom was treated as a Markov process in phase space. The main content of these studies is the approximate estimate of the probability of a "crack" (the first transition past the separatrix range or the first crossing of the energy barrier for the simplest model of the shell, a nonlinear system with one degree of freedom). This problem was also studied by

B. P. Makarov (1965) using the electronic modeling method. A transition to systems with several degrees of freedom, however, is connected with great difficulties. V. V. Bolotin and B. P. Makarov (1965) proposed that the equilibrium stability be estimated on the basis of the mean time during which the system remains in some neighborhood of equilibrium and they developed an approximate method for the solution of L. S. Pontryagin's differential equation. Further results are given in the study by B. P. Makarov (1965).

The study of probabilistic stability problems in elastic systems which are treated as a distributed system has only started. V. V. Bolotin and B. P. Makarov (1967) solved the problem of the subcritical deformations of a flat elastic shell with initial irregularities. It was assumed that the scales of the irregularities and the correlations were small in comparison with the characteristic dimension of the shell and that the initial irregularities form a homogeneous ergodic random field. Formulas were obtained for the correlation functions, the spectral densities and the variances of the total and additional displacements, for additional forces in the middle surface, etc. The change in the spectral composition of the irregularities and the character of the correlation relations between various types of irregularities with increased loads were studied.

§15. Future Problems and Prospects

The theory of elastic and nonelastic stability deals with those branches of mechanics in which the development of solutions for special problems, as a rule, preceded considerably the development of general theoretical problems. Many problems which arose out of the needs of engineering were solved without a proper analysis of the fundamental concepts, the validity of the methods that were used and the boundaries of their applicability. An example are the statistical method and the normalized-modulus concept in the theory of stability of elasto-plastic systems, the unjustified application of statistical criteria to problems in elastic stability in the presence of nonconservative forces as well as other problems which were dominant for many years. Incidentally, this situation is also common in many other applied sciences. In view of this R. Bellman (1964) characterized the "stability" concept as a "highly overloaded term whose definition has not been established."

In the last decade the situation improved considerably for the better (see §2-4 where an attempt was made to throw some light on the contemporary state of the general theory). Nevertheless, a stricter definition of the fundamental concepts

and a development of general rigorous methods are the most important tasks in the nearest future. A theory of stability of deformable solids must be developed which in rigor and generality will correspond to the classical Lyapunov theory. Apparently there is great hope that the Lyapunov theory can be extended to the case of metric functional spaces. If it is possible to construct for elasto-plastic, visco-elasto-plastic systems and also for elastic systems which are loaded by forces that are not potentials, functionals which are similar to the Lyapunov functions in classical stability theory, new effective and rigorous methods for the study of concrete problems will be obtained.

In the stability of elastic systems under the action of potential forces, the most important branch is, and remains, the stability theory of thin elastic shells. The studies that were carried out in recent years definitely cast some doubt on the established pattern of concentrating on lower critical forces. From the standpoint of the calculation of thin-walled structures and also the evaluation of the experimental observation data, the true (upper) critical forces found taking into account the initial deflections of the middle surface from the ideal state, the real manner in which the boundary conditions are realized and the real loading methods, are of greatest interest. Also, in many cases, it is necessary to treat the unperturbed state as a moment state and take into account the displacements corresponding to the unperturbed state. Thus, it becomes necessary to take into account the entire complexity of the behavior of real shells as the loads increase. To overcome all these difficulties, the nonlinear theory of elastic shells must be made more precise, and effective numerical methods for the solution of concrete problems not based on very strict assumptions about the character of the deformation of the shells must be developed. Experimental methods must be proved and experimental data must be gathered. It must also be added to what has already been said, that a large part of the factors which must be taken into account for the approximation of the theoretical computational schemes for real shells have a random character. Generally, the development of probabilistic and statistical methods is one of the most promising trends in the theory of elastic and nonelastic stability. This applies in particular to the theory of stability of thin shells, since the behavior of the latter is very sensitive to small changes in the form of the middle surface, the way in which the boundary conditions are realized, and the actual loading method.

The stability of elasto-plastic and visco-elasto-plastic systems remains the most difficult and least developed branch in the theory of stability of a deformed solid. In this field, non-rigorous approximate methods are used almost exclusively. Although they fully satisfy engineers and give a correct idea about the load-bearing capacity of the structure. from the theoretical standpoint, the situation cannot be considered satisfactory.

The theory of stability of elasto-plastic systems must be constructed on the basis of the theory of stability of motion. What must be studied is not the stability of some elasto-plastic equilibrium form, but the stability of the entire deformation process which develops over time. This does not necessarily require that inertial forces be taken into account. If the external forces are conservative, in view of the dissipativeness of the elasto-plastic systems, the study of slow perturbations is sufficient. The "slow" time theory of plastic flow can be used for this purpose. Along with the unperturbed process, perturbed elasto-plastic deformation processes must be studied. The study of stability reduces to finding the conditions which ensure that the perturbed processes are close to the unperturbed processes.

The problem that was formulated is extremely difficult. The point is that an elasto-plastic system represents a nonlinear system with nonholonomic one-sided relations. Mathematical difficulties during the linearization of the equations of perturbed motion are already encounted in one-dimensional problems. These difficulties are connected with the necessity of distinguishing loading and unloading and, in a number of cases, the secondary plastic deformations must be taken into account.

In two-dimensional and three-dimensional problems, the situation is even more complex, due to the presence of corner points on the yield surface and effects connected with deformation anisotropy. In addition, as a result of the irreversibility of the plastic deformations, arbitrarily small perturbations at any stage of the deformation process may be accumulated and thus have an effect on the subsequent behavior of the system. In view of this, it becomes necessary to distinguish single and repeated loads.

Since the vast majority of structures operates under repeated load conditions, the very important problem of the stability of elasto-plastic systems under repeated loads arises. This problem is evidently intimately related to the adaptability problem. Adaptability is nothing else but the stabilization of the accumulation of elasto-plastic deformations. Thus,

adaptability and stability are allied concepts. It is possible that starting with adaptability theory, a number of results in the theory of elasto-plastic stability can be obtained. The hypothesis can be proposed, that for small vanishing perturbations, the tangential-modulus load will be the upper boundary for the forces at which adaptability occurs.

The solution of the problems that were enumerated requires a corresponding development of the theory of plasticity. The behavior of elasto-plastic systems during loss of stability may differ substantially from the proportional load. Therefore, a detailed and adequate description of the plastic deformation process during small but sharp changes from the qualitative standpoint of the loading paths is needed. Perhaps this will require taking into account the time effects.

Elasto-plastic stability problems formulated rigorously and completely may turn out to be too difficult for practical use. In addition, a rigorous formulation may turn out to be unrealistic from the practical standpoint. In this case, it is useful to replace the study of stability by a direct solution of the Cauchy problem for given perturbations. The development of computer technology opened up great possibilities for such an approach. In essence, we are speaking about the mathematical modeling of motions which neighbor to the unperturbed motion. This modeling may have a stochastic character if the perturbations are given in accordance with some probability distributions. Analogous approaches have already been used to study systems operating under creep conditions or under the action of impact loads. However, it should be noted that what is solved are not stability problems but related problems. When properly formulated, such an analysis may give more complete information about the properties of motions neighboring the unperturbed motions than an analysis of stability in the narrow sense.

Many unsolved problems also exist in the non-classical branches of the theory of stability. An example is the stability theory of elastic systems interacting with a liquid or gas. At the present time there is a tendency to use improved aerodynamic approaches and to obtain exact solutions for at least very reliable approximate solutions with the aid of electronic computers. Problems that had first priority are problems taking into account the boundary layer, the turbulence pulsations in the flow, the initial irregularities in the shell, vibrations caused by additional internal factors, etc. The complicating additional factors must be taken into account if we want to obtain theoretical results which agree fully with the behavior of the real structures under operating or experimental conditions.

Certain problems connected with taking into account the effect of damping forces on the stability of elastic systems loaded by forces which are not potentials remain unsolved. A large part of problems in the stability of elastic systems in the presence of follow-up loads was solved without taking into account damping. What has been classified as stability in many studies is in fact "quasistability" (in the sense of the definition given in §4). When the real properties of damping forces in structures are taken into account, certain solutions obtained earlier may be revised. Nonlinear problems must be studied in the future. Elasto-plastic and elasto-visco-plastic effects as well as wave processes in dynamic problems in stability theory must be taken into account in the future.

One of the most promising trends is the application of methods of probability theory and mathematical statistics. The necessity of taking into account the continuous character of elastic systems leads to a study of stochastic boundary value problems. The methods for the solution of nonlinear problems of this kind have not yet been developed sufficiently. Until now many problems are solved by reducing the elastic system to a system with a finite number of degrees of freedom which is equivalent to it in some sense. Further development in this field requires more sophisticated mathematical methods.

The demands of engineering and the internal development of the theory will facilitate the formulation of new stability problems in deformed systems. In this regard, the theory of stability is practically inexhaustible. Various structural schemes which include complex three-dimensional rod and thin-walled systems, anisotropic, reinforced and layer structures, grid and "soft" shells, etc., the great variety of the mechanical properties of the materials and the associated necessity of taking into account elastic, plastic and viscous deformations, the variety of the surrounding media (gas, liquid, plasma, complex rheological media) and the manner in which they interact with the structures (force, thermal, electromagnetic interactions) all these are sources of new interesting problems. But the interest in new problems must not reduce the attention given to fundamental concepts, general and rigorous methods.

MECHANICS OF FRACTURE[1]

V. Z. Parton, G. P. Cherepanov

(continued)

1. The authors are deeply grateful to Ye. F. Afanas'ev, R. D. Vagapov, B. A. Drozdovskiy, T. K. Zilova, L. N. Karpenko, V. P. Kogayev, N. A. Makhutov, V. N. Mosints, N. D. Sobolev, I. I. Tikhonov, M. V. Khanin, K. K. Shal'nev and R. M. Shneyderovich for the material made available for this survey.

§1. Introduction

The term "mechanics of fracture," which was coined several years ago and which quickly became popular is used in two senses. In the first, narrow sense of this term, the mechanics of fracture applies only to studies dealing with the spreading of cracks, which have been carried out on a wide scale in the last two decades both in the USSR and abroad, especially in the USA. In the broader sense, the mechanics of fracture includes that part of the science of the strength of materials which is related to the study of the strength of structures and buildings which takes or does not take into account the initial cracks in them and also the study of various laws for the development of the cracks. The mechanics of fracture is understood by the authors in the broader sense.

The origins of this science go back to such leading figures as Leonardo da Vinci, and Galileo Galilei. Probably Leonardo da Vinci was the first man who formulated the problem of making an experiment to determine the load bearing capacity (experiments with an iron wire). Although people, since ancient times, constructed various and very complex buildings, knowledge about the strength and fracture of materials was accumulated empirically and haphazardly to a large extent, and the experience was handed over from generation to generation as an art. In particular, the phenomenon which today is called the scale effect is attributed to Leonardo da Vinci. However, the achievements of Leonardo da Vinci were not known to the following generations and, therefore, had no effect on the development of the mechanics of fracture. Galileo Galilei, who determined that the destructive load in a stressed beam is directly proportional to the area of its cross section and is independent of its length, can be rightly considered the founder of the mechanics of fracture. We note that this result which is somewhat modernized to the inhomogeneous stressed state until now plays the basic role in practical engineering strength calculations.

The subsequent development of the mechanics of fracture is connected with the names of S. Coulomb, A. St.-Venant, O. Mohr, A. Griffith, S. Coulomb, A. St.-Venant and O. Mohr who laid the foundations for the theory of limiting equilibrium and A. Griffith, who is the founder of the theory of brittle rupture. Both these theories which were perfected by numerous investigators are currently the foundation of the modern theory of fracture. They describe theoretically various properties of the fracture process, which occur to one extent or another in all solids.

In nature and in human practical activity a great variety of materials with different fracture properties are encountered. These are primarily metals and their alloys which are of the greatest practical importance in engineering structures. These are followed by polymers, biological tissues and bones, rocks and soils, free flowing bodies, glasses and ceramics, porous materials, compounds, ice, etc. The external conditions, the types of loads, the configurations of the structures, the temperature, etc., are also variegated. The tendency of individual materials or certain classes of these materials to fracture under particular conditions are studied in various natural sciences and engineering disciplines and entire scientific trends have been established.

Mechanics is characterized by a striving to describe the fundamental features of the fracture phenomena within the framework of rigorously formulated and sufficiently general models that are applied to certain classes of materials. Along with this a number of very important practical fracture phenomena exist which, until now have not been interpreted from the mechanical point of view and which present an interesting field of activity for future theoretical research.

Giving preference to the most typical mathematical approach in mechanics, the authors considered it nevertheless necessary also to present the fundamental practical results. The authors are aware of the fact that certain interesting studies in the mechanics of fracture are not reflected at all in this survey or are not discussed in sufficient detail. The broadness of the selected topic can serve as a partial justification of this fact.

A number of valuable comments and ideas which considerably helped to improve the survey were made by A. A. Il'yushin, V. V. Novozhilov, G. S. Pisarenko, Yu. N. Rabotnov, L. I. Sedov and S. V. Serensen. The authors express their sincere gratitude to them.

§2. Theory of Fracture and Theory of Strength

The presence of structural formation such as grains, microcracks, dislocations, molecular bundles, etc., in all materials which are encountered in practice results in the fact that their strength is two-three orders of magnitude smaller than the theoretical strengths corresponding to the ideal molecular order. Speaking descriptively, the more defective the material (the deviation of its structure from the ideal order) the smaller the strength, all other conditions being equal.

Various types of materials are characterized by structural formations of one type or another which determine their specific deformation and fractural properties. Along with a physical study of the microstructure and microfracture of materials, it is useful to make a phenomenological analysis of the fracture phenomenon on the basis of some models which reflect the most essential aspects of this phenomenon. Since, apparently, at the present time, it is still too early to speak of the possibility of constructing a general theory of fracture, it is more advantageous to develop special theories which describe more or less adequately the behavior of certain classes of materials under certain conditions. This requires a relatively complete and general classification of the basic types of behavior of solids and of the corresponding theories.

We will first classify the rheological models that are used. Let us consider an element of volume dx dy dz, loaded on the surface by the stresses σ_{ij} as a "black box," to whose input the stresses σ_{ij} are supplied and at whose output the strains σ_{ij} are picked up. For simplicity, we will assume that if the parameters of the system also include the temperature T, the system will be closed.[1] According to the fundamental phenomenological assumption, the strains ϵ_{ij} must be completely determined by the quantities σ_{ij}, T and the evolution of their changes. Infinitesimal increments in the output quantities can be expressed in terms of the corresponding increments $d\sigma_{mn}$, dt and dT in the following form:

$$d\varepsilon_{ij} = A_{ijmn}\, d\sigma_{mn} + B_{ij}\, dt + C_{ij}\, dT \quad \text{(t is time)} \qquad (2.1)$$

Here A_{ijmn}, B_{ij}, C_{ij} are functionals of the parameters $\varepsilon_{ij}(x, y, z, t), \sigma_{ij}(x, y, z, t), T(x, y, z, t)$ in the region occupied by the body.

Let us introduce the "short-range effect" hypothesis. According to this hypothesis the parameters which define equation (2.1) for an arbitrary element of volume do not depend on the state in any other element of volume no matter how close it is. In addition to this, it is assumed that equations (2.1) do not include body forces (in particular, inertial and gravitational forces). This hypothesis is based on the physical observation that the interaction forces in elementary particles quickly decrease as the distance between them increases. Systems which satisfy this hypothesis will be called systems with a short-range effect. Many known rheological models in mechanics are systems with a short-range effect. Only such systems will be considered below. The functionals A_{ijmn}, B_{ij}, C_{ij} in t, x, y, z in the fundamental equations (2.1) in the case of systems with a short-range effect degenerate into functionals of t in the parameters $\epsilon_{ij}\sigma_{ij}$, T and only several of their derivatives with respect to x, y, z.

1. In other words, we exclude from the discussion the general case of models with internal degrees of freedom characterized by additional parameters.

Rheological models for systems with a short-range effect can be broken up into gradient and nongradient models. In the last case the fundamental equations do not include the derivatives of $\epsilon_{ij}\sigma_{ij}$, T with respect to x, y, z. The majority of models studied in mechanics are nongradient models. However, in the theory of elasticity certain gradient models have also been proposed (E. and F. Kosser, R. D. Mindlin and R. A. Tupin abroad, V. V. Bolotin, V. A. Lomakin, V. V. Novozhilov and M. E. Eglit in the USSR). In the last few years much more attention was given to gradient theories. Apparently this is explained by the fact that the physical theories of a micro-inhomogeneous elastic body make it necessary to take into account the gradient terms for derivatives of certain orders.

If the functionals A_{ijmn}, B_{ij}, C_{ij} "are not invariant" with respect to time shifts, the corresponding systems are called systems with "aging." (In fact, time does have an effect by way of the corresponding structural physical parameters which are excluded from the explicit study.) In particular, in the theory of creep, systems of this type were studied by N. Kh. Arutyunyan (as applied to creep in concrete). The noninvariance with respect to time shifts indicates that the rheological properties of the system change with time. A large part of the rheological models that were proposed is invariant with respect to a change of the time origin and, therefore, describes systems whose properties do not change with time. Below only nongradient models which are invariant with respect to a time shift for systems with a short-range effect will be considered.

It is convenient to classify naturally such systems by the character of the reaction of the system to external perturbations. We note that the strains ϵ_{ij} play the role of the reaction of the system (element of volume, and the loads σ_{ij} and the temperature T on the surface element of the volume play the role of the external perturbations. Here we do not discuss (since in the given case it is not important theoretically) the problem of the specific meaning of the finite deformations ϵ_{ij} of the element of volume. We assume that for a given state of the particle starting at a particular instant t = 0, the evolution of the external perturbations σ_{ij} and T is known exactly. We also assume that the distribution of ϵ_{ij}, σ_{ij} and T is known at the initial instant t = 0. The element of volume consists of the same material particles (x, y, z are Lagrangean coordinates). It is required to determine the reaction of the system ϵ_{ij} over time.

The reaction of the system to an external disturbance may be instantaneous or with an aftereffect (the corresponding systems will be called systems with an instantaneous reaction and with an aftereffect). For systems with an instantaneous reaction, $B_{ij} = 0$ and the functionals A_{ijmn} and C_{ij} are independent of time (this includes derivatives of the parameters of any order with respect to t). In such systems, the reaction to the instantaneous disturbance occurs immediately and remains generally unchanged, provided σ_{ij} and T do not vary. In arbitrary systems it is useful to represent the total reaction (the total increment in the deformations) in the form of a sum of the instantaneous reaction and its aftereffect. The latter is by definition that part of the total reaction which occurs with the passage of time.

We will assume that the external perturbation vanishes over time. The reaction of the system may also vanish (reversible reaction). Systems in which the reaction to a disturbing perturbation also vanishes even after an infinite time interval, will be called systems with a reversible reaction (for this definition of models in the theory of elasticity, see L. I. Sedov, 1960). Thus, the total reaction of an arbitrary system to the external perturbation which vanished at some finite instant of time can be represented as the sum of the reversible and irreversible reaction, which remains even after an arbitrarily long time interval has elapsed. In turn, each term consists of the instantaneous reaction and the aftereffect. The residual deformations characterize the "memory" of the system with respect to the external perturbation which occurred in the past and vanished.

The fundamental rheological models can be classified on the basis of the type of reaction as follows:

<u>A thermoelastic body</u> refers to systems with an instantaneous reversible reaction. The deformations ϵ_{ij} in thermoelastic bodies are single-valued functions of σ_{ij} and T. Thus, for this case, the coefficients A_{ijmn} and C_{ij} ($B_{ij} = 0$) in the fundamental equations (2.1) are the usual functions of σ_{ij} and T which satisfy, in addition, the existence conditions for the total differential. The same result can be obtained using the thermodynamic method. Equations (2.1) can be further simplified if the system has physical or geometric symmetry properties (for example, isotropy) when the deformations are small, when the relations (2.1) are linear and when the process is isothermic. An effective solution of many important problems in the deformation of solids

was obtained within the framework of such models. The corresponding trends in the mechanics of a deformable solid were investigated in many studies by Soviet authors (V. V. Bolotin, L. A. Galin, E. I. Grigolyuk, N. I. Muskhelishvili, V. V. Novozhilov, G. S. Pisarenko, I. M. Rabinovich, A. R. Rzhanitsyn, G. N. Savin, V. I. Feodos'ev, and others). The studies in these fields are discussed in other surveys in this volume.

An elasto-plastic body belongs to systems with an instantaneous reaction ($B_{ij} = 0$). The introduction of the additional hypothesis on the existence of the loading surface and the application of the quasithermodynamic Drucker postulate are probably the simplest way of obtaining the associated flow law which is the basis of the modern theory of elasto-plastic media. The following two assumptions can also be used instead of the Drucker postulate: a) the entire irreversible work is converted into heat, b) the entropy increases at a maximum rate. Other assumptions can also be used. According to the associated law, the role played by the experiment, in addition to the determination of the elaso-plastic constants, reduces to determining the loading surface and its changes during irreversible deformation processes. The use of additional physical principles makes it possible to find in a special form the functionals A_{ijmn} and C_{ij} from a smaller number of experiments. The body is said to be an ideal elasto-plastic body if the corresponding loading surface does not change during any deformation process (in this case it is also called the yield surface or the yield condition).

The different variants of yield theory which are mainly applied to metals and their alloys and also to soils (R. Mises, E. Reiss, V. Koiter, V. Prager, F. Hodge, V. V. Novozhilov, Kh. A. Rakhmatulin, S. S. Grigoryan, D. D. Ivlev, and others) are best known.

If a proportional load occurs, i.e., at each point of the body, the state parameters increase according to a known law which is directly proportional to the loading parameter, equations (2.1) can be integrated (when $B_{ij} = 0$). The same also holds for any fixed loading path of a given small particle in (σ_{ij}, T) space. This approach is used in the study of elasto-plastic media in so-called deformation plasticity theories (G. Hencky, A. Nadai, A. A. Il'yushin, V. D. Klyushnikov, V. S. Lenskiy, and others).

In the USSR, many studies in the theory of plasticity have been and are being made, both of a general theoretical character (A. A. Il'yushin, V. V. Novozhilov, L. I. Sedov, and others), as well as studies dealing with the solution of concrete problems (L. A. Galin, D. D. Ivlev, A. A. Il'yushin, L. M. Kachanov, Kh. A. Rakhmatulin, V. V. Sokolovskiy, G. P. Cherepanov, G. S. Shapiro, and others).

It should be mentioned that the basic successes in the solution of concrete problems are mainly related to the ideal rigid-plastic model or the one-dimensionality of the problem. However, some successes were also achieved in much more complex higher dimensional elasto-plastic problems.

Theories of the limiting state (an ideal rigid-plastic body, a free flowing body, not experiencing tensile stresses, etc.) can be considered as limiting cases of an ideal elasto-plastic medium in whose equations terms with the elastic deformation components are omitted.

A viscous body refers to systems with an aftereffect (with a zero instantaneous reaction) and with an irreversible reaction. Here in equations (2.1) $A_{ijmn} = C_{ij} = 0$. The B_{ij} are naturally considered as the usual functions σ_{ij}, ε_{ij} and T. In the simplest case, when the B_{ij} are linear functions of the σ_{ij} we obtain the classical model of a viscous fluid.

When the instantaneous deformation which is determined in accordance with the theory of elasto-plastic media is also taken into account and when the B_{ij} are taken as some functions of the σ_{ij}, $(\varepsilon_{ij} - \varepsilon^p_{ij})$ and T, we obtain from (2.1) the most widely used variant of the creep theory of metals (the ε^p_{ij} are the stantaneous deformations). The assumption of the existence of the creep velocity potential is fundamental in this theory. Studies in creep theory received a great stimulus in the studies of N. Kh. Arutyunyan, L. M. Kachanov, Yu. N. Rabotnov, M. I. Rozovskiy. and others).[1]

1. Detailed information on this problem is available in the monograph by Yu. N. Rabotnov (1966). See also the surveys by N.Kh.Arutyunyan (pp. [pp.155-202 missing from Russian text]) and Yu. N. Rabotnov (pp. 175-227) in this volume.

A body with a prehistory, aftereffect and a completely reversible reaction describes the behavior of the majority of polymer materials. A very general description of such systems is obtained with the aid of a slight generalization of the Volterra theory:

$$\begin{aligned}\varepsilon_{ij} = f_0(\sigma_{ij}, T) &+ \int_0^t K_{ijmn}[T, t-t', \sigma_{mn}(t')]\,dt' + \\ &+ \int_0^t\int_0^t K_{ijklmn}[T, t-t', t-t'', \sigma_{kl}(t'), \sigma_{mn}(t'')]\,dt'\,dt'' + \ldots,\end{aligned} \qquad (2.2)$$

where

$$\left.\begin{array}{ll} K_{ijmn}[T, t-t', \sigma_{mn}(t')] = 0 & {}^{a}\text{при } t' > t'_0, \\ K_{ijklmn}[T, t-t', t-t'', \sigma_{kl}(t'), \sigma_{mn}(t'')] = 0 & {}^{a}\text{при } t' > t'_0, t'' > t''_0, \\ K_{ijmn}(T, t, \sigma_{mn}) \to 0 & {}^{a}\text{при } t \to \infty, \\ K_{ijklmn}(T, t, t, \sigma_{kl}, \sigma_{mn}) \to 0 & {}^{a}\text{при } t \to \infty, \\ \ldots\ldots\ldots\ldots & \end{array}\right| \qquad (2.3)$$

Key: a. for

Here K_{ijmn}, K_{ijklmn}, . . . are continuous single-value functions of the arguments T, σ_{mn}, σ_{kl}, . . . and generally, generalized functions.

When conditions (2.3) are ignored, the residual component of the deformations will also be included and equations (2.2) can also be used to describe the irreversible reaction (creep).

The most widely used variant of a linear visco-elastic body or a Boltzman hereditary body is included in (2.2). In viscoelasticity the most important results in the USSR were obtained in the studies by N. Kh. Arutyunyn, A. A. Il'yushin, A. K. Malmeyster, Yu. N. Rabotnov, and others.

A visco-plastic body refers to a different variety of nonlinear elastic media. It is assumed that a fixed surface exists in the (σ_{ij}, T) space, such that on one side of the surface there is no response to the perturbation and that the medium behaves on the surface itself like a viscous body. The simplest models of this type describe the behavior of thick lubricants, metals at high temperatures, etc. In the USSR the theory of a visco-plastic body was developed in the studies by A. A. Il'yushin, Kh. A. Rakhmatulin, V. V. Sokolovskiy, and others.

The main types of rheological bodies (elasto-plastic, viscous bodies and bodies with a pre-history) that were mentioned, or some combination of these, describe the behavior of the system under consideration, provided the system does not have some implicit parameters (describing, for example, chemical reactions, phase shifts, electromagnetic effects, etc). In concrete studies, the main difficulty is the art of selecting appropriately the simplest model which gives the required explanation and describes the rheological phenomenon that is observed in the experiment.

The correct selection of the rheological model is of the utmost importance in the solution of fracture and strength problem in the mechanics of fracture. Without a preliminary study of the deformation processes of bodies, problems in the mechanics of fracture cannot be studied. At the same time, we emphasize, that from the physical standpoint, the plastic deformation plays the role of the damages that have accumulated i.e., the microfractural process which gradually paves the way for the macroscopic fracture.

Studies in the mechanics of fracture can be broken down into two trends. According to the first trend which goes back all the way to Galileo, it is assumed that the fracture of the body occurs only at some point of the body when a particular combination of the parameters σ_{ij}, ϵ_{ij}, T and t attains the critical value. The fracture process itself is not studied. It is clear that when such an approach is used, the strength problem is solved by selecting one rheological model or another and the fracture criteria (the selection of the latter is often called the theory of strength in the strength of materials).

This approach is the direct logical consequence of the phenomenological approach which was adopted within the frame of reference of the parameters that were mentioned. Physically it is justified by the fact that the development of faults in the

material which leads to a loss of load-bearing capacity often occurs in a narrow-near-critical region, so that a detailed knowledge of the fracture process itself is of secondary importance. The fracture criterion which is determined experimentally can often be considered to be the result of complex microphysical fracture processes occurring on the structural cell scale up to the molecular level leading to the formation of the macrodefect. In addition, the behavior of the macrodefect (which is interpreted phenomenologically as a rupture shift) depends on the type of rupture. For example, the formation of dislocations and slippage lines, even those that cut the body, as a rule, do not lead to its fracture.

The criterion quantity which is used most often is the largest principal stress, the largest relative elongation, the largest principal tangential or octahedral stress, the specific energy for the change in shape or the total specific energy of the deformation. Each criterion is applicable, under definite conditions, to some class of materials. The correct use of these criteria depends considerably on the practical experience of the investigator or engineer.

Most of the experimental strength studies presented below have been devoted to the accumulation of such experience. We note that at different historical periods, different importance was attributed to different criteria. For example, G. Lame and V. Rankine used as the strength criterion the largest principal stress and V. Poncelet and A. St.-Venant the largest strain.

We present the two most vivid examples of using, for example, the largest principal relative elongation criterion.

During the straining of a beam under the action of a constant stress σ, generally irreversible creep deformations occur (which are most substantial for metals at high temperatures and polymers). For a great part of the time before the fracture, τ, the rod develops creep at a constant deformation rate $\dot{\varepsilon}_c$ (steadystate creep). Thus, we obtain

$$\dot{\varepsilon}_c \tau = \varepsilon_0, \tag{2.4}$$

where ε_0 is the largest relative elongation. If we consider the quantity ε_0 as the material constant and take into account the

empirical relation between the steadystate creep rate and the load[1]

$$\dot{\varepsilon}_c = ce^{\lambda\sigma} \text{ (C and } \lambda \text{ are material constants)}, \qquad (2.5)$$

formula (2.4) can be used to find the time to rupture as a function of the applied stress. The formula for the time to rupture that was obtained was first proposed by S. N. Zhurkov. Subsequent experimental studies made by S. N. Zhurkov and his collaborators have shown the validity of this formula for a wide class of polymers and metals, also including materials in which irreversible deformations before rupture have not been detected, for all practical purposes, and which undergo brittle rupture. The last two concepts that were presented above lose their meaning.

When a rod is loaded by a cyclic stress with an amplitude σ which is smaller than the conventional yield point $\sigma_{0,2}$, plastic deformations accumulate in the rod and a fatigue change occurs in the structure of the material. Physically, this is explained by its microinhomogeneity and as a consequence of the impossibility of avoiding a local concentration of stresses. Suppose that during each cycle the deformation $\Delta\varepsilon_p$ is accumulated where the quantity $\Delta\varepsilon_p$ is an exponential function of the applied stress σ (by analogy with the accumulated irreversible creep deformations):

$$\Delta\varepsilon_p = Be^{\alpha\sigma}. \qquad (2.6)$$

Clearly the deformation ε_0 accumulated in N cycles before rupture will be $\Delta\varepsilon_p N$. From the above, using the assumption that ε_0 is constant, we can easily find the number of cycles to fracture as a function of the load (the Wöhler curve):

1. For many materials it is more advantageous to use other empirical formulas; for example, in the form of a power function. When this is done, the form of the relation for the time to rupture of the material also changes.

$$N = \frac{\varepsilon_0}{B} e^{-\alpha\sigma}. \qquad (2.7)$$

This relation is, in fact, observed in the case when the load σ is greater than the fatigue limit. For a smaller load, the assumption of the accumulation of plastic deformations is inadmissible, apparently, as a result of the microadaptability effect.

In the USSR many scientific groups carried out extensive investigations in the theory of strength, which are connected with the names N. M. Belyayev, V. V. Bolotin, N. N. Davidenkov, A. N. Dinnik, S. N. Zhurkov, A. A. Il'yushin, L. M. Kachanov, V. V. Novozhilov, I. A. Oding, G. S. Pisarenko, S. D. Ponomarev, I. M. Rabinovich, Yu. N. Rabotnov, P. A. Rebinder, A. R. Rzhanitsyn, S.V.Serensen, N. S. Streletskiy, G. V. Uzhik, V. I. Feodos'ev, Ya. B. Fridman, N. P. Shapov and many others. These studies made it possible to develop computational methods, safety factors, standards and norms on the basis of which various equipment and mechanisms are designed. It can be said with assurance that the gradiose success achieved in raising the construction and industry level in the USSR would not have been possible without these studies.

A study of fracture criteria (theory of strength) within the framework of the approach that was described retains its basic practical value in strength calculations. However, the studies in strength and fracture only along these lines are inadequate for a number of reasons.

Already V. Voigt made a series of experiments with brittle materials and reached a negative conclusion with regard to the possibility of using strength criteria. P. Bridgeman discovered in 1931 the "pinch-effect" phenomenon which cannot be explained from the standpoint of strength theory (G. P. Cherepanov, 1965, explained this phenomenon). In the famous study by A. F. Joffe and his collaborators (1924) a series of experiments was made in which the strength of the crystals of rock-salt was studied when the surface of the sample was in various states. It was detected that the strength of the crystal with the surface layer dissolved in hot water exceeds many times its engineering strength and attains in some cases the theoretical strength value. The effect that was discovered and also the many cases of the fracture of metallic structures under stresses that are smaller than the yield point $\sigma_{0,2}$ and many other fracture phenomena which cannot be explained in

principle from the point of view of strength theory forced certain investigators to abandon the Galilean concept of strength as a material constant (of course, for fixed external conditions). This approach, which goes back to the studies by A. A. Griffith, G. I. Taylor, E. O. Orovan, G. R. Irwin and others is based on the study of the rupture process itself.

The following initial concepts are usually used.

The fracture of a solid occurs almost always as a result of the development of several rupture displacement surfaces in it. If a displacement rupture which is normal to the surface occurs, we speak of a normal rupture (breakoff), crack or simply a crack. If a rupture displacement which is tangential to the surface occurs, we speak about a shear crack or dislocation.

The role played by the two types of ruptures that were mentioned is different under different concrete conditions. As the strength of the material is reduced due to the increased temperature during compression, as a rule, the role played by the shear cracks and dislocations increases. As the strength increases due to the reduced temperature, in the presence of cyclic loads, aggressive media, radiation, as a rule, the role played by the normal rupture cracks increases.

The development of rupture surfaces begins with the imperfections in the structure of the material which must be studied at the initial instant as certain given disturbances which are always present in the systems. These disturbances must be considered as certain initial cracks or dislocations, which agrees well with the accepted experimental observations. The subsequent development of the original disturbances under loads may assume a great variety of forms.

The simultaneous and stable development of many dislocations forming slippage bands and entire plastic regions is characteristic of the growth of dislocations. Therefore, the theory of dislocations is the physical basis for the phenomenological theory of plasticity. We note that the model of an imperfect elasto-plastic body and the theory of the limiting state (a theory of the Mohr type[1]) provide the answers to the problem of the limiting load and the load-bearing capacity of structures within the frame of reference of the rheological model without the use of any additional strength criteria.

---

1. Mohr's theory was applied on a wide scale in reinforced concrete structures.

The preferred development of one most dangerous crack is characteristic of the growth of cracks (however, there are exceptions, for example, the growth of cracks under compression conditions which are close to compression in all directions), and its tendency to grow unstably, usually causes a separation of the body into two parts. When strength criteria based on the theory of cracks were compared, it became evident that in the majority of cases the usual strength theories are obtained. However, the constants which occur in these must be considered to depend on the dimensions of the initial cracks and also on their shape and position. Incidentally, for a wide class of fracture phenomena in microinhomogeneous bodies, the strength does not depend on the magnitude of the initial perturbation (the initial crack) and it is determined by the characteristic parameters of the structure of the body, for example, the size of the grain (this fact was already pointed out in 1939 by G. Neyber, see also G. P. Cherepanov, 1967). Thus, this problem can be approached formally as the simplest generalization of the usual theories of strength obtained by introducing an additional internal structural parameter which is not used in the formulation of the rheological model. This approach is similar to the idea of introducing in the state equations additional structural parameters an idea developed by L. I. Sedov. It must also not be forgotten that the study of the rupture process is very often of independent interest, not connected with the problem of the load-bearing capacity.

Historically dislocation theory and the theory of cracks developed separately. The different formal apparatus used in these theories is explained by the fact that dislocation theory studies directly discontinuities in the displacements, and therefore in linear theory, deals with logarithmic singularities, whereas in the theory of cracks on the discontinuity surface the force conditions are usually given and, therefore, it deals with higher order singularities. However, a deep internal similarity exists between these theories which consists of the fact that the coefficients at these singularities in both theories have the meaning of the basic parameters of the system which lead the process.

In the theory of cracks, the most important problem is the formulation of the condition for the local rupture at the point on the contour of the crack. This is just as important in the solution of the problem of the development of the crack as, for example, the selection of the correct fracture criterion for a smooth sample. The local fracture condition is formulated most simply in the theory of so-called quasibrittle cracks, when the largest dimension of the plastic deformation region at the point on the contour of the crack is small compared with

the distance of this point from the nearest boundary of the body. The simplest variant of this condition based on the physical ideas of A. A. Griffith and G. Neyber was proposed in 1957 by G. R. Irwin. The coefficient at the crack singularity at the point under consideration at the local fracture instant (and the movement of the crack at this point) is considered to be equal to some material constant. The stresses are calculated on the assumption that the body is perfectly elastic. Since the coefficient that was mentioned is a function of the external loads, the length of the crack and the geometry of the body which is found from solving the elasticity problem as a whole, the local fracture condition on the contour of the crack makes it possible, in principle, to determine its development, in particular, to find the combination of external loads which separates the stability and instability regions.

When these external loads are considered as independent parameters which define completely the state of the system, the combination of loads that is obtained will be analogous to the limiting equilibrium surface for the same body without cracks made from some hypothetically perfect elasto-plastic material. However, when the loading path is changed, the rupturing combination of loads will generally be different. Thus, the analogy between the behavior of an ideal elastic body with a crack and a perfect elasto-plastic body without cracks is only valid for each given loading path (in particular, for a proportional load or when one external loading parameter increases monotonically). Figure 1 shows this analogy in a schematic diagram whose coordinates are the "generalized load p--generalized displacement v" (the arrows denote the admissible displacements along the diagram). Of course, the analogy is valid from the point of view of the external observer who knows how to measure the response of the system v to the external disturbance p.

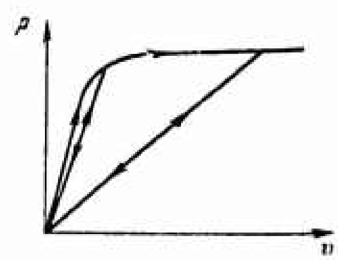

Fig. 1

Various models for the detailed rupture mechanism at the end of a quasibrittle crack have been proposed. The Leonov-Panasyuk model (1959) which was proposed independently by foreign authors is most simple and universal. According to this model it is assumed that a region of weakened bonds exists on the continued crack; the thickness of this region, in the theory of small deformations, is assumed to be zero. In addition, it is assumed that the opposite sides of this region are contracted towards one another by some stress which represents a material constant and that in the beginning of this region which coincides with the end of the crack the jump in the normal displacement at the instant of rupture is equal to some other material constant. This criterion can also be applied to cracks in elasto-plastic bodies, provided the plastic region is not small and the plastic deformations are concentrated along some thin layer where the crack continues. The last case occurs, for example, in thin plates made from low carbon steel.

Subsequently, it was shown that all known models (at the present time, about ten) differ in the detailed scheme used to describe the local rupture at the end of a brittle crack but are equivalent in the sense that they always lead to the Griffith-Irwin condition.

An approach to the description of the development of cracks in arbitrary continuous media was proposed by G. P. Cherepanov (1967). It is based on an energy concept and on the concept of a superthin structure at the end of the crack, whose dimension is small compared with the dimension of the plastic region near the apex of the crack.

The limiting equilibrium theory and the theory of brittle cracks are the basis of modern mechanics of fracture. Many concrete problems of great practical importance have been solved on the basis of these theories. These theories give an idealized description of plasticity and brittleness properties which are found to various degrees in all solids. However, phenomenological theories of strength should not be juxtaposed with the theory of cracks which interprets the phenomenological concept of resistance to direct pull and explains the reduction in the latter by a comparison with a crystal without defects giving it a static character.

Under real conditions the strength of a solid may depend on the following main factors: the material, the shape and dimensions of the body, the time, the manner in which the load is applied, the number of loading cycles, the temperature, the parameters determining the degree of agressiveness of the external medium, the velocity and the deformation prehistory.

In particular in the following sections, we will give certain generalizations of the theories that were mentioned above to the case when these factors have an effect (the simplest generalizations are, for example, to indicate the dependence of the constants that appear in the theories on certain parameters).

In practice it turns out that some transition zone in which the above-mentioned factors change exists, which separates the viscous fracture region from the brittle fracture region, where, in the latter, the use of the structure is not permitted. In the viscous fracture region, the calculation of the strength is based either on the limiting equilibrium theory or on strength theories.

The conclusion about the inadmissibility of the operation of the structure in the brittle rupture region is connected with the difficulty of detecting in advance, using control methods which do not cause fracture, cracklike defects, which couldlead to fracture and which occur in the brittle strength formulas. It should be kept in mind that there is a great variety of such defects, for example, they can consist of various types of poor penetrations in welded structures, oxidized or embrittled zones in the metal, impurities, and other types of inclusions of a metallurgical or technological kind, etc. In many structures produced in the country it is often not even possible to avoid defects whose dimensions are very large. This is primarily due to the general tendency to use stronger (as a rule more brittle) materials and the specific conditions under which certain structures operate. It must also be taken into account that the rupture of crystals which are nearly ideal has a brittle character. So far very large strength values exceeding several tens and hundreds of times the technical strength have been attained only under laboratory conditions.

The conclusion of the inadmissibility of the work of the structure in the brittle fracture region has a temporal character and apparently will have to be revised in the future. In certain structures already today, the presence of controllable cracks whose dimensions do not exceed the critical dimensions is permitted.

We note two very important classes of problems when the limiting load problem can be solved in principle without using the mechanics of fracture on the basis of the solution of the problem within the framework of a rheological model. These are cases when the body can undergo arbitrary finite deformation and problems in the loss of stability.

§3. Analysis of Stresses for Bodies with Cracks[1]

Recently the capacity of materials to form cracks was studied and a great deal of attention was given to determining the possibility of using structural elements and equipment with cracks. One of the most important special features of these types of calculations within the framework of linear elasticity theory is taking into account the redistribution of the stresses which occurs as a result of the fissures and cracks that are formed under the action of the external loads. The coefficients at the singularity of the elastic stresses at the end of the crack (which, according to the Griffith-Irwin condition determine the local fracture at the point on the contour of the crack under consideration) are called the coefficients of the stress intensity.

The field of elastic stresses in a small neighborhood of an arbitrary point O of the contour of the crack is represented in the following form:

$$\left.\begin{aligned}
\sigma_x &= \frac{K_{\mathrm{I}}}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)-\frac{K_{\mathrm{II}}}{\sqrt{2\pi r}}\sin\frac{\theta}{2}\left(2+\cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right),\\
\sigma_y &= \frac{K_{\mathrm{I}}}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(1+\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)+\frac{K_{\mathrm{II}}}{\sqrt{2\pi r}}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2},\\
\tau_{xy} &= \frac{K_{\mathrm{I}}}{\sqrt{2\pi r}}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2}+\frac{K_{\mathrm{II}}}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right),\\
\sigma_z &= \nu(\sigma_x+\sigma_y),\quad \tau_{xz}=-\frac{K_{\mathrm{III}}}{\sqrt{2\pi r}}\sin\frac{\theta}{2},\quad \tau_{yz}=\frac{K_{\mathrm{III}}}{\sqrt{2\pi r}}\cos\frac{\theta}{2},\\
u &= \frac{K_{\mathrm{I}}}{G}\sqrt{\frac{r}{2\pi}}\cos\frac{\theta}{2}\left(1-2\nu+\sin^2\frac{\theta}{2}\right)+\\
&\qquad+\frac{K_{\mathrm{II}}}{G}\sqrt{\frac{r}{2\pi}}\sin\frac{\theta}{2}\left(2-2\nu+\cos^2\frac{\theta}{2}\right),
\end{aligned}\right\} \tag{3.1}$$

(cont'd)

---

1. Additional bibliographical references on the problems discussed in this section can be found by the reader in the collection, "Applied Problems in Viscous Fracture," (1964, Russian translation, Moscow, 1968), in the survey by D. D. Ivlev (1967) and in the subsequent articles: G. N. Savin and V. V. Panasyuk (Prikl. mekh., Vol. 4, No. 1, 1968, pp. 3-24), G. P. Cherepanov (Intern. J. of Solids and Structures, Vol. 4, No. 8, 1968, pp. 811-831), Ye. M. Morozov and Ya. B. Fridman (in the collection, "Strength and Deformation of Materials in Nonuniform Physical Fields," 2nd ed., Moscow, 1968, pp. 216-253), G. G. Johnson and P. K. Paris (Engineering Fracture Mech., Vol. 1, No. 1, 1968, pp. 3-45).

$$
\left.\begin{aligned}
v &= \frac{K_I}{G}\sqrt{\frac{r}{2\pi}}\sin\frac{\theta}{2}\left(2-2\nu-\cos^2\frac{\theta}{2}\right)+\\
&\qquad+\frac{K_{II}}{G}\sqrt{\frac{r}{2\pi}}\cos\frac{\theta}{2}\left(-1+2\nu+\sin^2\frac{\theta}{2}\right),\\
w &= \frac{K_{III}}{G}\sqrt{\frac{2r}{\pi}}\sin\frac{\theta}{2}.
\end{aligned}\right\} \quad (3.1)
$$

Here σ_x, σ_y, σ_z, τ_{xy}, τ_{xz}, τ_{yz} are the stresses, u, v, w, are the components of the displacement along the axes of the xyz cartesian coordinate system, r and θ are polar coordinates in the xy plane (Fig. 2), K_I, K_{II}, K_{III} are the coefficients of the intensity of the stresses, G and ν are the shear modulus and the Poisson ratio, respectively.

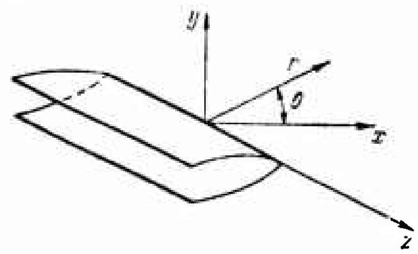

Fig. 2

When the local symmetry condition $K_{II} = K_{III} = 0$ is satisfied we speak about normal fracture cracks (or tear-off cracks). In the case when $K_I = K_{III} = 0$, $K_{II} \neq 0$, the expression "transverse shear crack" is used and when $K_I = K_{II} = 0$, $K_{III} \neq 0$, the expression "longitudinal shear crack" is used.

In the most frequently occurring and important case of a normal fracture crack, the Griffith-Irwin condition has the following form:

$$K_I \leqslant K_{IC} \left(K_{IC}^2 = \frac{2E\gamma}{1-\nu^2} \right). \tag{3.2}$$

Here K_{IC} is a constant of the brittle material, the critical coefficient of the intensity of the stresses, E is Young's modulus and γ is the dissipation energy per unit area of the growing crack.[1]

It should be noted that in the vicinity of the ends of cracks in solids goemetric and physical linearization conditions are inadmissible from the standpoint of determining the fine structure. Therefore, near the edge of the crack a region always exists in which solution (3.1) does not describe the details of the phenomenon. The elastic solution (3.1) is realized at distances which are large compared to the characteristic dimension of the region that was mentioned, but small compared to the characteristic linear dimensions of the body or the cracks. Hence, for a more rigorous formulation of the problem, solution (3.1) plays the role of an intermediate asymptote. The quantity σ is equal to the irreversible work of the external forces done to form a unit area on the surface of the crack.

Thus, the fundamental problem in the mechanics of brittle fracture reduces to an analysis of the stresses in the corresponding body with the cracks.

This section presents a survey of the basic studies in the theory of brittle cracks dealing with the determination of the stresses in bodies with cracks (including not only studies

1. The corresponding relation for the case of the dynamic spreading of cracks was obtained by G. P. Cherepanov in 1968.

in which an expression for the coefficients of the intensity of the stresses was obtained directly but also studies which are important in the theory of cracks which investigate the solutions of problems in the theory of elasticity in regions containing fissures and cuts). We will not dwell in detail on the methods used in the solution of problems in the mathematical theory of cracks. We only mention that the studies by G. V. Kolosov and N. I. Muskhelishvili are the foundation on which the solutions of the most important problems dealing with this branch of mechanics of a continuous medium have been constructed. In view of the well-known isomorphism of many phenomena, the studies by L. A. Galin, F. D. Gakhov,, M. V. Keldysh, N. Ye. Kochin, M. A. Lavrent'ev, S. G. Lekhnitskiy, A. I. Lur'e, G. N. Savin, L. I. Sedov, Ya. S. Uflyand, D. I. Sherman, I. Ya. Shtayerman and other scientists have been used on a number of occasions and can be used in the future to obtain solutions of problems in the theory of cracks. It should be mentioned that a great part of the studies in the mechanics of brittle fracture were made in the last decade.

3.1. Isotropic Elastic Body, Plane Problem

Studies in the theory of the **stressed** state near a hole which is similar to the ruptures that occur during the formation of cracks were begun by Ch. E. Inglis in 1913 and N. I. Muskhelishvili (1919) who obtained the solution of the equilibrium problem of an infinite body with an elliptical opening (in particular with a rectilinear fissure) under the action of an arbitrary stress field within the framework of the classical theory of elasticity. The fundamental studies in the mechanics of fracture are the studies by A. A. Griffith (Phil. Trans. Roy. Soc. London, 1920, A221:587, pp. 163-298, Proc. 1st. Intern. Congr. Appl. Mech. (1924)(1925), pp. 55-63), who, using the solution obtained by Ch. E. Inglis for an infinite brittle body with a rectilinear crack determined the critical values of the tearing stresses in the case of a plane deformation and a plane stressed state. He took into account the phenomenon of surface adhesion near the edge of the crack and proposed and energy criterion for equilibrium cracks.

Subsequently the problem of the development of isolated rectilinear cracks in an infinite brittle body for which various variants of the external loads were given was investigated in many studies.

D. I. Sherman (1940) and N. I. Muskhelishvili (1942) obtained an exact solution of the fundamental problems in the theory of elasticity for an arbitrary number of cuts along one straight line or circle in an infinite plane.

The problem of determining the clearance between two elastic halfplanes approaching one another, which is formed as a result of certain forces that are applied to the edges of the clearance was solved in the study by V. I. Mossakovski and P. A. Zagubizhenko (1954). Approximately at the same time (1955-1960) certain problems on cracks in rocks were formulated and solved as problems in the mathematical theory of elasticity without taking into account the strength properties of the rocks near the edges of the mine workings (see Section 3.8).

In 1959 several models for the local fracture at the end of a brittle crack were proposed (see the model of M. Ya. Leonov and V. V. Panasyuk that was mentioned above and the modeling of the adhesion forces at the end of the crack of G. I. Barenblatt, which is equivalent in its result to the Griffith-Irwin construct).

The studies by V. V. Panasyuk and L. T. Berezhnitskiy (1965) investigated the general case of the biaxial expansion of a plate with an arbitrarily oriented crack. V. I. Mossakovskiy, et al. (1968) considered the problem of the spreading of a rectilinear crack at an angle to the original direction, when a fracture point appears at the end of the crack. The study by I. A. Markuzon (1965) deals with the problem of the effect of the initial stresses on the character with which a brittle crack spreads. Here the load is selected so that in the absence of initial stresses the crack develops stably in many cases, and the initial stresses are the reason for the instability that is created.

It must be mentioned that the beginning of the growth of the crack cannot be identified with complete fracture. The latter occurs only in the case of an avalanch-like unstable spreading. Experiments and calculations have shown that in many cases cracks interact with barriers and the boundaries, and also in problems dealing with interaction of systems of cracks, the cracks develop stably on a large region in which the loads vary. Clearly the presence of stable cracks in structures and equipment often operating in definite regimes in which in the external loads change is much less dangerous and the reinforcement of such equipment by means of rivets and plates and drilling of holes on the path along which the cracks spread can considerably prolong their "life." The problem of reinforcing the cracks by transverse rigid ribs was solved in the study of Ye. A. Morozova and V. E. Parton (1961).

The problem of the mutual effect of colinear or arbitrarily oriented systems of cracks is of great importance in strength and fracture calculations. G. I. Barenblatt and G. P. Cherepanov (1960) obtained the solution for the problem of a periodic system of cuts which can be used to determine the length of a fissure in a strip. This study also investigated the effect of the boundaries of the body on the spreading of the cracks and examined the case of two cracks of the same length maintained in the open state by concentrated forces applied to their surface. The most detailed study of the limiting equilibrium problem of a plate with two colinear cracks of the same length and the derivation of the computational formulas are given in the studies by V. V. Panasyuk and B. L. Lozovoy (1961), B. L. Lozovoy (1964), and L. T. Berezhnitskiy (1965). V. V. Panasyuk and B. L. Lozovoy (1962) studied the problem of the development of two colinear cracks of different length. B. L. Lozovoy (1964) determined the critical stresses for a plate with three colinear cracks.

L. T. Berezhnitskiy (1965) studied the most general case of cracks of different lengths distributed along a straight line at an angle to the direction of tension. The results that were obtained make it possible to determine the critical stresses in problems with an arbitrary number of cracks lying on one axis. In the case of a system of cracks of different lengths which are parallel to some direction, the most dangerous crack is the crack which moves first.[1]

It is known that real materials, no matter what preliminary treatment they are subjected to, contain a large number of microdefects of various types, whose development under the action of the applied stress field leads to the occurrence of a system of cracks whose mutual effect can be highly varied.

1. The study by Ye. A. Morozov and V. Z. Parton (1968), in which as an example of different loading conditions the problem of the interaction of three cracks on the real axis was studied, where the external sections are semi-infinite is devoted to the investigation of this problem.

V. Z. Parton (1965) using the asymptotic representation of V. T. Koiter, obtained a solution of the problem for an elastic plane weakened by a doubly periodic system of cracks of the same length (checkerboard arrangement of the cracks, each of which was subjected to a homogeneous tensile stress. This study has shown that a particular mutual arrangement of the cracks leads to their stabilization (stable development).

V. M. Kuznetsov (1966) investigated an approximate method of solving the equilibrium problem for a system of semiinfinite parallel cracks with a constant load acting on a sector of finite length in the absence of tangential forces. The author compared the results with the exact solutions of similar problems that were obtained earlier. Using the same approximate method, P. A. Martynyuk (1966) studied the above-mentioned problem of determining the stresses in an infinite plate weakened by a system of cracks located on the axis at equal distances from one another. Like in the study by V. M. Kuznetsov, the simplifying assumption that $\sigma_x = \sigma_y$ on the continuation of the section and that the distance between the cracks is large in comparison with their length is introduced.

When the problem of the spreading of a curvilinear crack is studied, the additional hypothesis is used that the initial spreading of the crack occurs in the plane in which the tensile stress σ_θ attains a maximum value (see Fig. 2). This hypothesis was proposed independently by G. P. Cherepanov (1963) and F. Erdogan and G. S. Si (Trans. Amer. Soc. Mech. Engrs., 1963, D85, No. 4, pp. 519-527), and also in the studies by V. V. Panasyuk and L. T. Berezhnitskiy (1965-1966). In the last studies, expressions were obtained for the determination of the magnitude of the limiting loads in the case of one, two and a system of arc cracks with the aid of this hypothesis. We note that L. V. Yershov and D. D. Ivlev (1967) proposed in their study a formulation of the problem of determining the direction in which the crack developed on the basis of energy concepts. A preliminary determination of the field of elastic stresses in the neighborhood of the vertices of the sections was obtained with the aid N. I. Muskhelishvili's solution.[1]

1. An additional condition at the end of the crack was obtained by G. P. Cherepanov in 1968 for arbitrary curvilinear brittle cracks.

The problem of an investigation of the limiting equilibrium of plates weakened by holes with sharp ends is of particular interest. In addition to the independent value of determining the load-bearing capacity of parts with such defects, the difference in the magnitude of the critical loads for these regions and for rectilinear cracks of the appropriate length is important.

An effective solution of many problems of the type that was mentioned can be obtained with the aid of methods using the theory of functions of a complex variable, which were developed in the monograph by N. I. Muskhelishvili (1966, 1968), G. I. Polozhey (1949), G. N. Savin (1951, 1968), F. D. Gakhov (1963), S. M. Belonosov (1962). G. P. Cherepanov (1962) pointed out a class of problems in plane elasticity theory in which the corresponding boundary value problems for the analytic functions can be solved in closed form.

V. V. Panasyuk (1962) constructed the solution for the problem of the stressing of a plate with an opening in the shape of a hypercycloid and determined the elastic stresses in the neighborhood of the corner points. This problem was also studied in the study by A. P. Gres'ko (1965) with the aid of the method proposed by S. M. Belonosov (1962). V. V. Panasyuk and Ye. V. Buyna (1967) studied the problem of a brittle body weakened by holes in the shape of hypercycloid cavities which do not interact with each other. He obtained the condition for attaining the critical state at least at one vertex of the opening with the aid of N. I. Muskhelishvili's method. When they studied the problem of the limiting equilibrium of a plate with sharp stress concentrators, V. V. Panasyuk and L. T. Berezhnitskiy (1965) expressed the coefficients of the intensity of the stresses in terms of the stress function and a function mapping such a contour onto the unit circle. This makes it possible to obtain approximate solutions also in the case of a distant crack.

The difficulties connected with the nonavailability of rational mapping functions onto a halfplane and plane with a circular hole occur in problems on cracks that extend to the surface of the body. At the present time several techniques were developed for the numerical solution of problems involving cracks extending to the boundary of the body, primarily in the studies by O. L. Bowe (J. Math. & Phys., 1956, Vol. 1 No. 35, pp. 60-71), G. F. Buckner (Boundary problems in differential equations, Univ. of Wisconsin, 1960), A. A. Kaminskiy (1965 and later). Studying problems involving an arbitrary number of symmetrically located cracks, extending to the free

surface of a circular hole in an infinite body, O. L. Bowe used for the mapping of such region onto the unit circle an approximate polynomial representation for the analytic function, after which the N. I. Muskhelishvili methods could be applied. The actual calculations made by him for the simplest cases of one and two opposite cracks on the center line require a large volume of computations, since to obtain the required accuracy, about 30 terms had to be retained in the polynomial expansion. A. A. Kaminskiy perfected the Bowe method and obtained much better convergence when the mapping function was replaced by a rational function which preserved the singularity at the ends of the crack and rounded the corners at the points where the crack left the cavity. He obtained simple formulas for determining the magnitude of the limiting load in the plate weakened by a circular hole with two equal radial cracks that was mentioned. Using this method, N. Yu. Babich and A. A. Kaminskiy (1965) constructed a solution of the problem for one rectilinear crack, and A. A. Kaminskiy (1965) for two rectilinear cracks, extending to the contour of an elliptical hole (here the results of the calculations of the critical load as a function of the length of the crack are not given). Subsequently, A. A. Kaminskiy (1966) obtained a solution of the problem for the case when one or two equal cracks extend to the contour of an arbitrary smooth curvilinear hole during uniaxial stressing or stressing in all directions and determined the critical loads causing the development of the expanded cracks. G. G. Grebenkin and A. A. Kaminskiy (1967) calculated as an example the critical loads for two equal cracks extending to the contour of a square hole. V. V. Panasyuk (1965) studied the Bowe problem of a circular hole with two radial cracks of equal length extending to the boundary of the hole. An approximate method which is similar to the method of successive approximations developed in the studies of S. G. Mikhlin (1935) and D. I. Sherman (1935) is used to determine the normal stresses. A comparison with the O. L. Bowe solution for two cracks of the same length gives satisfactory agreement. Some results on the effect of the free boundary of a halfspace on the spreading of cracks were obtained earlier in the studies by Yu. A. Ustinov (1959) and V. V. Panasyuk (1960).

The numerical methods that were mentioned above give good results for infinite regions with cracks extending to the contour of a hole. The methods that were mentioned are not very effective in the study of finite regions with corner points, in which special difficulties arise. The problem of the stressing in all directions of a disc with a radial crack extending to a contour was studied for the first time in the study by Ye. M. Morozov and Ya. B. Fridman (1958), in which the rupturing stress applied along the edge of the disc when the lengths of the section were small was determined only approximately.

L. L. Libatskiy (1965) reduced the solution of the problem for a circular plate with rectilinear sections along one diameter to a singular integral equation with a regular part for which we constructed an approximate solution method which preserved the type of singularity on the edges of the crack.

The solution of the limiting equilibrium problem for a circular disc with a central crack was obtained in the study by L. L. Libatskiy and S. K. Kovchik (1967), in which the analytical solution is compared with the experimental data.

The solution of mixed problems of elasticity theory for nonclassical regions such as a strip (layer) is of interest in the study of the problem of stress concentrations, around fissures and cracks. From the mathematical point of view, these problems are very difficult. However, the systematic study of this problem, which began about 10 years ago, led to the development of effective solution methods for problems in this class (V. M. Aleksandrov, I. I. Vorovich, N. N. Lebedev, Ya. S. Uflyand, and others). These problems are easily reduced to the solution of integral equations of the first kind with a nonregular kernel with the aid of operational calculus methods. The greatest success in finding the solutions of these equations which are convenient for practical use was achieved by using specific asymptotic methods. I. A. Markuzon (1963) began the studies of the equilibrium problem of cracks in a strip (1963). V. M. Aleksandrov (1965) studied equilibrium cracks along a strip or layer in which an integral equation is constructed for the function which determines the shape of the crack. He obtained an approximate solution by expanding the kernel of the equation in a series for a large thickness to dimension ratio of the crack and obtained the load as a function of the dimensions of the crack. Using the same method and the solution of the Wiener-Hopf equations, V. M. Aleksandrov and B. I. Smetanin (1965, 1966) obtained an expression for the coefficient of the intensity of the stresses on the edges of an equilibrium crack in a thin layer. For the case of a constant load, the relation between the dimension of the equilibrium crack and the acting load is determined. A similar solution was obtained for a disc-shaped crack in a layer of finite thickness. V. M. Yentov and R. L. Salganik (1965) studied in the beam approximation, the problem of a semiinfinite crack along the center line of the strip. For loads applied to the edges of the crack, the problem reduces to a study of the lamination under the action of a normal or tangential force. In the same study, using the Wiener-Hopf method, and expression was obtained for the coefficients of the intensity of the stresses for sufficiently large and sufficiently small values of the ratio of the distance from the end of the crack to the point at which

the force is applied to the halfwidth of the strip. Using the analytical method developed by V. M. Aleksandrov and I. I. Vorovich (1960) in the study of contact problems for a comparatively thick layer, B. I. Smetanin (1968) studied the problem of a longitudinal crack in a wedge and also the plane and axisymmetric problem of a longitudinal crack in a layer under different conditions on the boundaries of the wedge and layer. For a crack whose position is symmetric with respect to the edges of the wedge (layer), and a normal load applied to the surface of the cracks, formulas were obtained to determine the surface of the crack. The coefficient of intensity of the stresses is expressed as an asymptotic series in powers of the dimensionless parameter.

The plane mixed problem for a crack was studied for the first time in the study by V. I. Mossakovskiy and P. A. Zagubizhenko (1954). Important practical lamination problems are also mixed problems in the theory of elasticity. The solution of the problem of the lamination of brittle bodies is a peculiar combination of solutions of the contact problems in the theory of elasticity and problems in the mathematical theory of cracks.[1]

A problem which can be solved exactly effectively is the problem of the lamination of an infinite body by a fixed wedge. G. I. Barenblatt (1959) obtained a solution of such a problem for a wedge of constant thickness. Unlike in the case when the position of the creep point is known, for a wedge with a rounded front edge the position of the creep point on the surface of the crack from the wedge must also be determined. G. I. Barenblatt and G. P. Cherepanov (1960) studied the problem of the spreading of a crack in front of a wedge with a small rounding and a wedge where the shape of the rounding is given by a power law. Here, the case of Coulomb friction acting on the jaws of the wedge were studied. I. A. Markuzon (1961) made the next step in the study of the problem of the lamination of brittle bodies. He obtained a relation for the length of

1. The first study in which lamination problems are studied was the study by I. V. Obreimov (1930) made in connection with experiments in the splitting of mica. Here the chip that moved was considered as a thin beam and methods from strength of materials were used to solve the problem.

the crack as a function of the length of the wedge and studied the effect of homogeneous compression or elongation stresses at infinity on the length of the free crack in the problem of the lamination of an infinite body by a wedge of finite length. Lamination problems were also studied in G. P. Cherepanov's study (1962) where the solution of a Rieman linear boundary value problem for two functions in mixed problems in plane elasticity theory was used as an example of an application.

There are no exact solutions for the problem of the lamination of a strip. Along with the study by I. V. Obreimov (1930) we mention the studies of V. D. Kuznetsov (1954), M. S. Metsik (1958), N. N. Davidenkov (196), which are also based on the beam approximation.[1]

3.2. Axisymmetric and Three-Dimensional Problems for an Elastic Isotropic Body

M. Ya. Leonov (1939) was the first man to determine the distribution of the stresses for a brittle three-dimensional isotropic body containing a plane circular crack in the plane during tension by a constant stress at infinity.[2]

1. We also mention here the studies by Ye. M. Morozov and V. Z. Parton (1968) which examine the variational principle and show that it is possible to apply it successfully to the solution of various plane and three-dimensional problems for bodies containing cracks for all possible variants of given external loads. In addition to the usual determination of the magnitude of the limiting critical loads the authors constructed an approximate technique which can be used to determine the trajectories of the cracks.

2. The classical studies by A. Sommerfeld and N. E. Kochin (1938), in which problems in the theory of diffraction and hydrodynamics whose mathematical formulation is analogous were solved should also be mentioned.

Subsequently this problem was studied by M. Ya. Leonov and V. V. Panasyuk (1961) for various given variants of the external load and recently by Ye. M. Morozov and V. Z. Parton (1968). The relations that were found here, in a number of cases, relate the length of the crack to the applied loads and they are completely analogous to the corresponding plane deformation cases and can be obtained with an accuracy up to the dimensionless constant multiplier from dimensional analysis (L. I. Sedov, 1957).

The first basic problem in the theory of elasticity for a space with a plane circular crack was solved in general form by V. I. Mossakovskiy (1955). The case of an annular (circular in the plane) crack was studied in the study by V. T. Grinchenko and A. F. Ulitko (1965). The values for the magnitude of the limiting stresses obtained by him differ from the result obtained in 1945 by R. A. Zak only by a numerical multiplier.

The axisymmetric problem in the theory of elasticity for an unbounded space containing two plane circular cracks in which the stressed state is symmetric with respect to the middle plane, was studied in the studies by Ya. S. Uflyand (1958), N. N. Lebedev and Ya. S. Uflyand (1960). The solution of this problem is constructed with the aid of an expression for the omponents involving two harmonic functions (the Papkovich-Neyber representation) with a subsequent reduction of the problem with the aid of Hankel transformations to coupled integral equations.

The Meler-Fok transformations enabled Ya. S. Uflyand (1959) to obtained a solution of the problem of the axisymmetric deformation of an unbounded body containing a plane crack in the exterior of a circle of a given radius. Here the solutions were obtained both for the case of a symmetric and antisymmetric load. V. V. Panasyuk (1962) returned to a study of this problem and determined the rupture loads that are formed in the process.

V. I. Dovnorovich (1962) using the methods that were developed for the solution of three-dimensional problems in elasticity theory (1959) determined the stressed state of an elastic body in the presence of a plane crack (slit). As an example equations were obtained for widened cracks for various given variants of the normal pressure applied to the surface of a plane crack in an unbounded elastic body. The study by Yu. N. Kuz'min and Ya. S. Uflyand (1965) considered the axisymmetric problem in elasticity theory for a halfspace weakened by a plane circular crack, and Yu. N. Kuz'min (1966) studied the case of an unbounded body with two coaxial cracks of different radii.

The Hankel transformations which reduce the problem to the problem of solving coupled integral equations can be applied effectively to axisymmetric problems for an elastic layer, in particular, to problems in the stress concentration in an elastic layer weakened by a plane circular crack (Ya. S. Uflyand (1959). The same problems were studied using different methods in the studies by V. M. Aleksandrov (1965), V. M. Aleksandrov and B. I. Smetanin (1965, 1966), B. I. Smetanin (1968) that were mentioned above. Using the apparatus of dual integral equations, N. V. Pal'tsun (1967) solved a number of problems on circular cracks in a layer.

Using the Kontorovich-Lebedev transformations, Ya. S. Uflyand (1958) studied the problem for an infinite elastic body containing a slit in the form of a halfplane under the effect of an arbitrary system of external forces.

V. V. Panasyuk (1962, 1965) studied the problem of the development of cracks whose shape in the plane was nearly circular in an unbounded isotropic brittle body. A crack, the maximum distance of whose contour from a circle is small in comparison with the radius of the circle, is said to be such a crack. Continuing the studies begun by M. Ya. Leonov and K. I. Chumak (1959), V. V. Panasyuk developed a method for the approximate solution of the class of problems that was mentioned in which the problem of the limiting load for the crack having a nearly circular shape in the plane is reduced to the determination of the elastic stresses in the neighborhood of the contour of the crack. A particular example in this class of problems is the case of a plane crack which has the shape of an ellipse in the plane. With the aid of the approximate method that was developed, V. V. Panasyuk determined the limiting critical stresses from points on the minor and major axes of the ellipse and compared them with the results from the exact solution of this problem obtained by him earlier (1962).

The expression which determines the magnitude of the limiting stresses necessary for the spreading of the crack in the direction of its smaller axis was obtained in the study by M. Ya. Leonov and K. N. Rusinko (1963) on the basis of microstress theory developed by the same authors (1961).

Yu. N. Kuz'min (1966) found the distribution of the stresses in an elastic space weakened by a system of plane cracks of equal radius which were periodic along the z axis. For a normal load applied to the surface of the cracks, the problem reduces to the solution of coupled integral equations which are further reduced to the Fredholm equations with a continuous kernel expressed in terms of known special functions.

3.3. Torsion, Bending and Longitudinal Shear

The early studies of the problem of torsion of rods with holes (in the limiting case with fissures) were made by L. N. Faylon (Phil. Trans. Roy. Soc. London, Vol. A193, 1900, pp. 309-352). A. N. Dinnik obtained in 1913 a solution for the problem of the torsion of a circular rod containing a radial crack. The methods that are discussed in the monographs by N. I. Muskhelishvili (1966), K. V. Solyanik-Krass (1949), N. Kh. Arutyunyan and B. L. Abramyan (1963) are widely used to solve problems in this class.

Ye. A. Shiryaev (1956) using the methods developed by him for the solution of torsional and bending problems (1951), studied the problem of the torsion of a homogeneous isotropic circular beam with one crack along an arc of a circle or along the radius and with two cracks located on the diameter.

A. A. Balobyan (1958) studied the problem of the torsion of prismatic rods with a box-shaped cross section and a crack.

The general formulation of longitudinal shear crack problems where the case of the so-called "nonplanar" deformation corresponds to the distribution of the displacements (stressed state in an infinite cylindrical body under the action of constant loads directed along the generatrices of the cylinder) was studied in the study by G. I. Barenblatt and G. P. Cherepanov (1961). Unlike in normal rupture cracks and transverse shear cracks, in this case, effective exact solutions of many problems can be obtained, since the unique displacement w which is different from zero satisfies, in this case, the Laplace equation. Here the highly developed methods and results of hydrodynamics can be applied directly because of the obvious analogy between problems in the theory of elasticity for a nonplanar deformation and problems in plane hydrodynamics. In the study that was mentioned, exact solutions were obtained for problems for an infinite body containing a circular hole with one or two cracks, loaded at infinity by constant tangential stresses (the analogue of O. L. Bowe's problems for normal distortional cracks) and for the mixed problem for an isolated rectilinear crack on a part of which the constant displacement is given (the analogue of the lamination problem by a wedge of finite length studied in 1961 by I. A. Markuzon). In the same study problems in the interaction of an infinite system of the identical cracks on the real axis and the case when the same cracks lie on a vertical one-row lattice have also been studied in this study. In the study of the problem of the development of curvilinear longitudinal shear cracks and also cracks whose shape differs little from the

rectilinear or circular shape, the authors used the hypothesis that the development of the curvilinear longitudinal shear crack occurs in the direction of the maximum stress $\tau_z\theta$. In the same study, using the Keldysh-Sedov formulas, a solution was obtained for the dynamic problem of the shearing of a body.

R. L. Salganik (1962) studied two problems on axisymmetric longitudinal shear cracks (a disc-shaped and infinite annular crack) in an infinite body, when the cracks are subjected to the action of tangential forces distributed on their surface.

3.4. Anisotropic Materials

The methods developed in the monographs by S. G. Lekhnitskiy (1947, 1950) can be widely used in the study of equilibrium problems and problems dealing with the spreading of cracks in anisotropic media. The problem of a rectilinear crack in an anisotropic plate was studied for the first time by A. N. Straw (Adv. Phys., Vol. 6, 1957, pp. 418-465).

G. I. Barenblatt and G. P. Cherepanov (1961) studied the problem of an isolated rectilinear crack spreading along some elastic symmetry line in an orthotropic infinite body under plane deformation conditions. The same studies considered the problem of the lamination of an orthotropic body with planes of symmetry parallel to the two axes by a rigid infinite wedge moving at a constant rate. It was assumed that on the surface where the wedge makes contact with the laminated body, Coulomb frictional forces are acting. The problem of the lamination of an orthotropic body by a stationary wedge of constant thickness in which the frictional forces are ignored has been studied in greater detail. Within the framework of dislocation theory of thin twins and cracks, the problem of the spreading of a thin equilibrium crack along an anisotropic strip of finite thickness, was studied in the study by Ye. P. Fel'dman (1967). As the external loads gradually increase, the crack grows up to some critical value, after which instantaneous fracture of the strip occurs.

Some problems on cracks on the boundary of welded halfplanes made from different anisotropic materials were studied by D. V. Grilitskiy (1963). D. V. Grilitskiy and R. M. Lutsyshin (1967) studied the stressed state of an anisotropic plate with a circular isotropic core with a soldered-in isotropic core in the presence of cuts on the seam.

O. M. Romaniv and R. S. Kosychin (1968) considered the equilibrium problem of a brittle anisotropic plate with an artibrarily oriented crack under biaxial tension-compression conditions. The anisotropy of the resistance to brittle fracture was taken into account by giving the corresponding coefficient for the intensity of the stresses and by assuming that the development of the crack in the beginning occurs along the plane where the limiting intensity of the normal tensile stresses is achieved earlier than in other directions.

3.5. Heterogeneous Materials.

The study of fractures of cemented bodies in which the cracks can spread along the cemented area is of great practical importance in the case when the strength of the latter is relatively small. If the cementing strength, for example, of two elastic homogeneous bodies is great, the crack spreads into the depth of one or both cemented bodies in accordance with the law for the development of cracks in homogeneous materials.

G. P. Cherepanov (1962) obtained the solutions for the fundamental problems in plane elasticity theory in the case when the separation line of the different elastic bodies is a straight line or a circle, and an arbitrary number of cuts lies on this line. Similar problems using other methods were solved independently by D. V. Grilitskiy (1963).

The field of stresses and displacements in the vicinity of the end of a rectilinear section which is the cementing boundary was investigated in the study by R. L. Salganik (1963). R. V. Gol'dshteyn and R. L. Salganik (1963) solved the problem of the development of cracks between plane plates along a rectilinear cementing boundary.

An analogous problems was investigated once more in the study by V. I. Mossakovskiy and M. T. Rybka (1965), in which the fracture criterion for an inhomogeneous plate consisting of two homogengeous plates but with different elastic properties weakened by a crack on the boundary was determined on the basis of the Griffith condition.

V. I. Mossakovskiy and M. T. Rybka (1964, 1965) studied the R. A. Zak problem that was mentioned above for the case of a heterogeneous brittle material consisting of two cemented half-spaces with different elastic properties. The cementing surface contains a plane circular crack under the action of homogeneous stresses applied at infinity which are perpendicular to the separation boundary of the halfspaces. The authors obtained the solution by reducing the problem to a linear boundary value problem in the theory of analytic functions with the aid of potential theory.

3.6. Bending of Strips (Beams), Stressed State of Shells with Cracks

A number of problems for the limiting equilibrium of strips containing various cracks was solved by V. V. Panasyuk and B. L. Lozov (1961-1964), using effective solution methods for the corresponding problems in the theory of elasticity developed by N. I. Muskhelishvili and G. N. Savin. The same study also considered problems on the bending of strips with rectilinear cracks, symmetric relative to the longitudinal axis of the strip and with nonsymmetric cracks (perpendicular to the lateral edges of the strip). The stressed-deformed state and the magnitude of the limiting rupturing load were determined for various conditions in which the external loads were given (constant bending moments, concentrated forces, uniform pressure).

P. Ye. Berkovich (1966), continuing the studies of V. I. Mossakovskiy and P. A. Zagubizhenko (1954), obtained a solution for the problem of the bending of a strip (beam) containing a rectilinear crack of finite width at an angle to the longitudinal axis of the beam.

The study of the stressed state of shells containing cracks is connected with great difficulties. At the present time these studies are only in the initial stage, but because of the great successes achieved in the solution of problems in the general theory of shells, we can expect that in the nearest future extensive studies dealing with the analysis of the stressed state and the limiting equilibrium conditions of shells containing cracks will be undertaken.[1]

1. We mention here the study by S. Ya. Yarem and M. P. Savruk (1967), who studied the stresed state of a cylindrical shell with a crack under a symmetric load, and also the study by Ye. M. Morozov and V. T. Sapunov (1968), who studied the problem of a spherical shell with a crack under the action of internal pressure. In the last case the stressed state is determined in the vicinity of the ends of the crack, in particular the character of the change of the stresses as a function of the thickness and curvature of the shell and the length of the crack.

3.7. Dynamic Problems in the Theory of Cracks

Recently studies connected with problems in the dynamic spreading of cracks attracted a great deal of attention. These studies were begun by N. F. Mott (Engineering, Vol. 165:4275 1948, pp. 16-18) who studied the development of an isolated rectilinear crack in an infinite body under the action of a homogeneous field of tensile stresses. The dynamic problem in the theory of elasticity for an infinite body with a rectilinear crack of a fixed length moving at a constant rate under the action of a homogeneous tensile stress applied at infinity was studied by E. G. Joffe (Phil. Mag., Vol. 42:330, 1951, pp. 739-750). G. I. Barenblatt and G. P. Cherepanov (1960) studied stationary problems dealing with the movement of rectilinear normal rupture cracks. They found the limiting speed for the spreading of a rectilinear crack in a homogeneous elastic body (which is approximately 0.6 c_2, where c_2 is the velocity of the transverse waves). This is the velocity at which the rectilinear character of the spreading is violated and the crack begins to branch due to the redistribution of the stresses near its end. However, if rectilinearity is ensured in advance, the limiting velocity of a crack spreading in a homogeneous material is equal to the velocity of the surface Rayleigh waves ($\sim 0.9\ c_2$).

Studies of the equilibrium and the spreading of a crack in an anisotropic medium (G. I. Barenblatt and G. P. Cherepanov, 1961) have shown that as in an isotropic body, the rate at which the crack spreads cannot exceed the velocity of the Rayleigh waves. In the case of an orthotropic body with two mutually perpendicular symmetry planes, for a rectilinear crack, the ratio of the critical coefficients of the intensity of the stresses in the lamination direction and in the direction perpendicular to it must not be greater than one. However, on the basis of the fundamental assumptions in problems dealing with stationary lamination at a constant rate, the end of the crack which is formed before the wedge moves uniformly at the same speed. Nevertheless, experimental studies have shown that during the development of the crack, for example, at a low speed, the speed at the end oscillates regularly about some mean value. G. I. Barenblatt and R. L. Salganik (1963) studied the self-oscillatory phenomenon of the process during lossening, assuming like A. N. Straw (J. Mech. & Phys. Solids, Vol. 8, No. 2, 1960, pp. 119-122) that the critical coefficient for the intensity of the stresses depends on the instantaneous speed at which the crack spreads, which first decreases and then increases, as the velocity increases. These authors also studied the self-oscillations during lamination by a rigid wedge moving at a constant rate in an infinite brittle body, a thin beam and a thin chip shaved off from a large body.

The problem of the transient spreading of cracks was studied by G. I. Barenblatt, R. L. Salganik and G. P. Cherepanov (1962). Taking into account several assumptions made with respect to the adhesion forces[1] acting in the region at the end and also the distribution of the tensile stresses found by K. B. Broberg (Arkiv. fys., Vol. 18, No. 2, 1960, pp. 159-192), the authors obtained a relationship between the rate at which the crack spreads and the magnitude of the applied stress. It turned out that for any material a minimum rate exists for the stable spreading of the crack which increases as the load increases and tends to the velocity of the Rayleigh waves.

B. V. Kostrov (1964) using the Smirnov-Sobolev method of invariant solutions obtained a solution for the similarity problem for the transient spreading of an axisymmetric crack under the action of a homogeneous tensile stress applied at infinity.

Continuing the study of the problem of the dynamic development of cracks, B. V. Kostrov (1966) obtained a solution for the nonstationary spreading of longitudinal shear cracks in an unbounded elastic body and calculated the distribution of the stresses outside the crack for an arbitrary time law for the displacements of the ends of the crack. Here the methods developed in problems dealing with the supersonic flow around a wing with a finite span were used.

The study by R. V. Gol'dshteyn (1966) deals with the problem of determining the stresses in the neighborhood of a stationary crack moving along the cementing boundary. The study considers the movement of a semiinfinite crack at a constant velocity in plane deformation conditions to the end of which equal concentrated forces in opposite directions are applied at a fixed distance. Using a Fourier transform and the Wiener-Hopf method, the problem reduces to the Rieman-Hilbert method for a system of functions with piecewise- constant coefficients. Continuing the study of the laws for the development of cracks in cemented bodies, R. V. Gol'dshteyn (1967) studied surface waves propagating in cemented materials along the cementing boundary under various

1. In the framework of classical elasticity theory, the introduction of adhesion forces is unnecessary and redundant in the formulation of the fracture criterion. It cannot explain the true pattern of the deformation process in the case of a detailed analysis of the phenomena at the edge of the crack (see, for example, Ye. M. Morozov and V. Z. Parton, 1968).

contact conditions along this line.

A. M. Mikhaylov (1966) considered the movement of a crack in a narrow strip in the beam approximation. Using the variational principle, he derived the equations of motion and the boundary conditions for the displacements of the neutral axis of the beam located on one side of the crack.

An exact solution of the problem of the stationary movement of a semiinfinite crack along the middle line of a strip when the velocity of the end of the crack does not exceed the velocity of the Rayleigh waves was obtained in the study by R. B. Gol'dshtey and M. Matchinskiy (1967). These authors noted that both the solution and the coefficient for the intensity of the stresses depend on the frequency of the natural antisymmetric waves in the strip propagating at the same velocity as the crack.[1]

3.8. Cracks in Rocks, Development of Cracks in Compressed Bodies

The development of cracks in rocks which are formed both naturally as a result of tectonic movements and artifically by hydraulic ruptures of strata, etc., is of considerable practical interest.

The problem of a vertical crack (a crack not completely filled with a viscous fluid under the action of lateral rock pressure) was studied for the first time by Yu. P. Zheltov and S. A. Khristianovich (1955). Only the lateral rock pressure and the pressure of the fluid were taken into account in the solution. The condition for the finiteness of the stresses at the end of the crack postulated by S. A. Christianovich (1955), was used in this study to determine the dependence of the length of the cracks on the external loads. This condition was proposed earlier in the study by G. M.Westergaard (J. Amer. Concrete Inst., Vol. 5, No. 2, 1933, pp. 93-103, J. Appl. Mech., Vol. 5, No. 2, 1939, A49-A53).

1. In 1968, R. V. Gol'dshteyn, using as in the previous case, the Wiener-Hopf method constructed a solution for the problem and for the condition that the velocity of the Rayleigh waves is exceeded and noted that the spectrum of the natural waves leads to peculiar resonance phenomena in the strip.

G. I. Barenblatt (1956) considered in his study of the hydraulic rupture mechanism of an oil layer the problem of a horizontal disc-shaped crack containing a viscous liquid located at some depth from the surface of a heavy halfspace. Using the solution of Ya. N. Sneddon and the condition for the finiteness of the stresses at the ends of the crack, an expression was obtained for the radius of the crack as a function of the volume and pressure of the pumped fluid, the depth of the deposit and the specific gravity of the rock.

Using the same formulation, the solution of the generalized problem of a vertical crack for the case of a fluid pumped through its surface was obtained in the studies by Yu. P. Zheltov (1957). He also proposed an approximate method for the solution of the problem of a horizontal crack with a variable vertical pressure field along the radius.

The development of theoretical geology, the calculation of stresses formed in rock mines, reinforced concrete structures, etc., required a study of the compressive strength of brittle bodies.

V. I. Mossakovskiy and M. T. Rybka (1965) proposed an approach for constructing the theory of strength of compressed brittle bodies with cracks based on the energy concepts of A. A. Griffith. M. T. Rybka (1966) used the Griffith criterion to determine the length of a rectilinear crack along which Coulomb frictional forces are acting in the problem of the biaxial elongation of an elastic isotropic plate, to determine the length of the rectilinear crack. Without analyzing the stressed state at the end of the crack, V. I. Mossakovskiy, et al, (1965) found the distribution of the stresses in the plane containing a crack in the shape of a three-branch broken line where homogeneous compression at infinity occurs at some angle to the middle branch of the crack.

G. P. Cherepanov (1966) studied the laws for the compression strength of brittle bodies in the idealized case of a crack with free edges. In the same study, the solution for the plane elasticity theory problem for "superimposed" cracks (a mathematical section with a given jump in the normal displacements and stresses and a tangential stress, while the interacting forces at the opposite sides are arbitrary and nonlinear), located on one line was obtained. As an application a theoretical scheme was proposed for a rock impact and certain ideas about the safest forms of the mines were predicted.

P. Ye. Berkovich (1966) determined the stress distribution in an elastic plane with a crack in an inhomogeneous field of compression stresses. Assuming that the crack consists of three sectors and that the difference in the vertical displacements of the edges is constant on the contact sector, the author reduced the problem, with the aid of complex potentials, to the solution of four linear conjugate problems in which four functions must be determined.

S. Ya. Yarema and G. S. Krestin (1966) using the method of successive approximations, determined the magnitude of the limiting load in the problem of the compression of a disc by concentrated forces containing a symmetric crack.

The critical loads caused by the development of one or two cracks which extend to the contour of the curvilinear hole when the plane is compressed by constant forces were determined in the study by A. A. Kaminskiy (1967). In the case of an elliptical hole, the author obtained simple formulas for the determination of the critical load.

3.9. Temperature Stresses

It is known that temperature problems in which the steady-state heat flow is considered with the aid of the method proposed by N. I. Muskhelishvili in 1916 can be reduced to the solution of ordinary plane problems in the theory of elasticity. The concept of the coefficient for the intensity of the stresses is basically preserved also for problems dealing with the determination of temperature stresses.

N. M. Borodachev (1966) determined the distribution of thermoelastic stresses for an infinite body containing an axisymmetric crack, and G. S. Kit and Ya. S. Podstrigach (1966) found the distribution of a stationary temperature stress field formed in the vicinity of a heat resistant fissure when the homogeneous heat flow is given at infinitely remote points on the plate.

A more general formulation of this kind of problems is available in the study by Ya. S. Podstrigach and G. S. Kit (1967). For the case of a plane, containing an arbitrary number of heat conducting cracks, between whose opposite edges no ideal heat contact occurs, they proposed a method which can

be used to determine the stationary temperature of the field.[1]

§4. Analysis of the Limiting State

The theory of the limiting state and the theory of ideal elasto-plastic media give an idealized description of the basic properties of the deformation and fracture process for the majority of solids in the viscous fracture region for a large range of times, temperatures, deformation rates, etc. These theories which were developed in the work of S. Coulomb, A. St.-Venant, A. Tresk , M. Levi, O. Mohr. L. Prandtl, were further developed in all aspects by Soviet and foreign scientists. The practical significance of the these theories extends far beyond the determination of the strength and load-bearing capacity of structures. It should be mentioned here that their most important application deals with problems in the technological treatment of metals, mechanics of soils and rocks and that recent applications solved pseudoliquificiation problems in chemical technology.

In our country the development of plasticity theory started in the 30's in the studies by S. L. Sobolev (1935), S. A. Khristanovich (1936), S. G. Mikhlin (1938), who studied certain problems for an elasto-plastic and rigid plastic body. The studies by A. A. Gvozdev (1934, 1938) in which the upper and lower bound method for the limiting loads on a rigid-plastic body was proposed was important. This method was subsequently developed intensely and it became the basis for strength calculations based on the kinematically possible velocity field and statically admissible stress fields.

An extensive development of the theory of plasticity in our country goes back to the 40's. A. A. Il'yushin (1943) proposed the theory of small elasto-plastic deformations which was widely applied in applications. He proved (1945, 1947) the simple load theorem which made it possible to use in an important special case the relation between the nonlinear elastic body model and the model of an elasto-plastic medium. L. M. Kachanov (1940), A. A. Markov (1947) and S. M. Feynberg (1948)

1. An illustration of this method is the stationary temperature and stress field in an infinite plane with a thermally insulated arc-shaped crack with a fixed homogeneous heat flow at infinity which was found in 1968 by G. S. Kit and Yu. S. Frenchko. The stress distribution in the neighborhood of the ends of the crack and the limiting value of the heat flow make it possible to determine the beginning of the crack propagation in the brittle material.

obtained the fundamental results for a nonlinear elasto- and rigid-plastic body on the basis of variational principles. L. A. Galin, A. A. Il'yushin, Kh. A. Rakhmatulin, V. V. Sokolovskiy and many others obtained the solutions for a number of interesting and difficult problems and laid down the foundation for the basic scientific schools in the theory of plasticity in the USSR.

V. V. Sokolovskiy obtained the solution for a number of elasto-plastic problems (1942, 1944, 1948) and contact problems dealing with the pressing of rigid dies into a rigid-plastic body (1940), and also developed the theory of the plane stressed (1946) and plane deformed (1945) state.

A. A. Il'yushin (1948) proposed the method of elastic solutions for hardening bodies, which reduces the solution of the boundary value problem for a nonlinear elastic body to an infinite sequence of corresponding problems for linear elastic bodies with additional body forces. Important results were obtained by A. A. Il'yushin (1944-1950) in the load-bearing theory of plates and shells made from an elasto-plastic material, in particular during loss of stability.

L. A. Galin (1944-1949) applied methods of the theory of functions of a complex variable to the solution of a number of complex, basically two-dimensional, elasto-plastic problems.

Kh. A. Rakhmatulin (1945-1948) laid down the foundations of the theory of the propagation of waves in elasto-plastic media.

In the following years, the mathematical theory of plasticity was developed in the USSR both by way of general constructions and an analysis of the initial assumptions as well as by way of an accumulation of concrete results and methods for the solution of boundary value problems.

We will only mention several general results.

L. I. Sedov (1962) developed the general thermodynamic and kinematic analysis of the basic models for a continuous medium and the most general formulation of the associated flow law for a hardening body for an arbitrary number of parameters describing the loading prehistory. In 1965 L. I. Sedov proposed the variational method for the construction of mathematical models of a continuous medium and pointed out the general form of the corresponding principle which is applicable not only in classical mechanics but also in the relativistic mechanics of continuous media and in electrodynamics. The

relations between the theory of plasticity and the continuum dislocation theory were determined using this method.

Yu. N. Rabotnov (1968) proposed a theory of plasticity which took into account the effect of a yield retardation in the general three-dimensional case. In 1958 he proved convincingly the validity of the relations of the deformation type for singular loading surfaces. In 1951 Yu. N. Rabotnov proposed an engineering theory of shells which had a great effect on the further development of the load-bearing capacity theory of shells.

V. V. Novozhilov (1947) developed the theory of finite deformations, and in 1957-1958 jointly with Yu. I. Kadashevich, proposed a variant of plasticity theory for a medium with translational hardening.

A. A. Il'yushin (1963) analyzed from a general theoretical standpoint the possible relations between the stresses and strains, formulated the isotropic postulate and derived structural formulas for the relations between the stresses and strains.

D. D. Ivlev (1958, 1966) starting with the principle of the maximum dissipation velocity of the mechanical energy proposed a derivation of the associated flow law and analyzed the equations for the most widely used variants of plasticity theory. He also investigated essential and removable discontinuities in the displacements and stresses in an arbitrary three-dimensional case. He proposed and also studied various models of complex media. In 1958, D. D. Ivlev proposed an anisotropic ideal plasticity theory based on a generalization of the Tresk plasticity condition.

V. D. Klyushnikov (1958) developed several variants of the theory of plasticity with anisotropic hardening. A. A. Vakulenko (1959) proposed an approach to the theory of elasto-plastic media from the standpoint of nonlinear thermodynamics of irreversible processes developed by him.

The model of an elasto-plastic body and the limiting equilibrium theory were applied on a wide scale in the mechanics of soils and rocks. The limiting equilibrium theory with the Coulomb yield condition is usually called the statics of a loose medium. Along these lines the most important results were obtained by V. V. Sokolovskiy, V. G. Berezantsev, S. S. Golushkevich, A. Yu. Ishlinskiy, and others.

Considerable progress was also made along the lines of constructing new models of an elasto-plastic medium which are applied to the study of a certain class of deformation and rupture phenomena in soils.

N. M. Gersevanov, V. A. Florin, N. A. Tsytovich developed the so-called filtration consolidation theory for the description of the deformation of soils saturated with a liquid. They used the concept of a doubly continuous medium and certain ideas about the properties of the skeleton and the liquid and about their interaction. A more general theory along the same lines was developed by V. N. Nikolayevskiy (1960-1962) and V. Z. Parton (1964-1968) studied the consolidation problem mathematically.

Yu. P. Gupalo and G. P. Cherepanov (1967) used a model of a body which did not withstand tensile stresses in the solution of several problems in pseudoliquification in chemical reactors.

S. S. Grigoryan (1967) proposed one variant of an elasto-plastic body which represented a generalization of the one-dimensional σ - ε curve with a fixed "tooth" (observed in experiments with soft steel but following other more complex laws) for explosions in hard rocks. The spreading boundary of the elastic and plastic zone is the discontinuity line of the stresses and strains (the fracture front). The model that was mentioned is a generalization of the model of a soft soil proposed by the same author in 1960 and also constructed by V. P. Koryavov (1962) and V. N. Rodionov (1962).

§5. Effect of the Configuration and Dimension of the Structure on the Strength

The science of strength underwent rapid development in the last half century. This is primarily connected with the progress in the development of new materials and alloys with an ever increasing strength. While, during the 19th Century technical iron was used in structures (and the strength limit was raised approximately from 30 to 40 kg/mm^2, at the present time steel alloys exist, whose strength limits are on the order 200-300 kg/mm^2 and in the last decade strengths on the order of 400-600 kg/mm^2 have been attained. At the same time the physical theories about the nature of strength and fracture have a considerable effect on the selection of the means for producing stronger alloys (this applies primarily to dislocation theory and the theory of cracks).

We will distinguish two concepts: metallurgical strength and structural strength. By the first is meant (usually in handbooks on materials) the value of the strength obtained on smooth laboratory samples of certain standard dimensions on the material in the delivery stage. The strength of the product from the same material (structural strength) is sometimes much smaller. In particular, this occurs frequently near the brittle rupture region. The effect of the dimension of the structure on the (structural) strength will be called the scale effect.

In the viscous fracture region there is no scale effect and the dependence of the strength on the configuration of the body is determined by calculations with the aid of the selected model of the body and the fracture condition at the point in accordance with some theory of strength. In the case of ideal elasto-plastic bodies, there is no need for a theory of strength and the strength is calculated within the frame of reference of the model itself. In the brittle fracture region, a scale effect always occurs, and the dependence of the strength on the configuration and dimension of the body (including the shape and dimension of crack defects) are calculated within the frame of reference of a model of an elastic body in accordance with the Griffith-Irwin theory. In this section we mainly consider the practically most important transient fracture region, in which the scale effect also occurs, which has not been studied as extensively.

5.1. Physical Nature of the Scale Effect

The evolution of the views about the physical nature of the scale effect was comparatively long and tortorous. This was due to the fact that the phenomenological concepts of brittleness and plasticity had a descriptive character, which was related to the observation of the fracture process and the shape of the rupture surfaces. Brittle rupture is characterized by a rapid fracture process, the absence of a neck, and the orientation of the break-off surface along the largest principal tensile stress area. During viscous rupture considerable plastic deformations develop and a neck is formed in the sample and the break-off surface is oriented along the maximum tangential stress area. However, in practice, combined brittle and viscous rupture occurs in various degrees in all materials.

When the scale of the sample is increased and cuts or any stress concentrations are present in it, the probability of brittle rupture tends to increase. Therefore, the first question is how to compare the plastic and brittle character of the rupture of materials (how to determine the resistance to

break-off and the tendency of the material to brittle rupture. Notwithstanding the fact that a theoretical answer to this question was already given in the classical study by A. A. Griffith (1920), for almost thirty years the answer had no effect on the solution of problems for structural metals.

The difficulty of the comparison is that an ideal plastic scale effect is not characteristic,[1] and that in the brittle rupture region, the strength also depends on the internal structural parameters and the length dimension. According to the A. A. Griffith theory the latter are the lengths of certain initial cracks which are always present in a real material, and, in fact, various types of stress concentrates may play the role of the crack (inclusions of a different kind, empty spaces, pores, etc.), which was first emphasized by A. P. Aleksandrov and S. N. Zhurkov (1933).

Only in the last few years the comparative evaluation of materials with regard to brittleness or plasticity by means of rupture tests made on samples with an artificially created crack with the smallest possible radius of curvature[2] at its end (which creates the largest relative stress concentration) was generally recognized. The studies of N. N. Davidenkov, A. F. Joffe, G. V. Uzhik, Ya. B. Fridman, B. A. Drozdovskiy played a fundamental role in the development of this point of view.

1. Theoretically the model of an ideal elasto-plastic body or limiting equilibrium cannot explain the scale effect. The same applies to any other model when the rupture is described with the aid of strength theory, and obviously in this case, the strength is completely determined by the external loading parameters.

2. In fact, a critical value of the radius of curvature at the end of the crack exists below which a further reduction of the radius of curvature is no longer meaningful. This critical value depends on the plastic properties of the metal (it is on the order of 10^{-5} - 10^{-2} cm).

When this testing method is used, the fracture point is localized in advance, which reduces to a minimum the statistical factor, and only physical causes which are the basis for the viscous and brittle rupture phenomena which explain the scale effect are left. As many experiments have shown (see, for example, the study S. V. Serensen and N. A. Makhutov, 1967), the mean stresses in the internal cross section at the instant of fracture σ_n for a sample with a cut depend in the following manner on the characteristic linear scale L (the depth of the cut-crack or the distance of its end from the opposite side of the sample): when the L are small and the geometric similarity conditions are preserved, σ_n is independent of L and it is equal to the metallurgical strength of the given alloy (viscous fracture); as the scale L increases, the strength σ_n drops and tends to the quasibrittle Griffith-Irwin asymptote (brittle rupture).[1]

A very convenient scheme which explains the transition from viscous fracture to brittle fracture with a drop in the temperature was proposed for the first time by A. F. Joffe (1924). According to this scheme, the stresses σ_B and $\sigma_{0.2}$ depend differently on the temperature T. The first stress increases as T increases, whereas the second decreases so that the point where these curves intersect (the cold brittleness temperature) divides the viscous and brittle rupture regions.

1. In 1968 G. P. Cherepanov proposed a quantitative description of the brittle and viscous rupture phenomena and also of transient phenomena (thus also the scale effect) from a unified point of view. According to this approach, the question of the degree of brittleness for the possible fracture of the structure reduces to a calculation and comparative evaluation of the dimensionless number χ. All possible values of this number lie between zero and infinity, and when $\chi \ll 1$ the fracture is brittle, and when $\chi \gg 1$, the fracture is viscous. The energy concept used here is a generalization of the well-know Griffith-Irwin-Orovan concept, which also makes it possible to determine the stable growth at the end of the crack which always occurs in an elasto-plastic material before the loss of stability and, in addition, to determine the rate of growth of the crack under a variable (for example, cyclic) load. In the presence of recesses or cuts in the structure, tests with an appropriate sharp crack on smaller samples can be used to verify directly the danger of brittle fracture by comparing the numbers χ and the model experiments (or the functions $\chi(T)$ when the temperatures T are different).

Developing further the scheme of A. F. Joffe, N. N. Davidenkov (1930-1936) introduced the concept of brittle and viscous resistance to direct pull. He proposed that the resistance to direct pull be estimated by the elongation of smooth samples in liquid nitrogen. In 1930 N. N. Davidenkov published the study by A. M. Dragomirov (made in 1917), who was the first to draw attention to the relation between the type of rupture and the character of the reduced load after the maximum during the bending of cut samples (the crystalline sectors in the rupture correspond to the stalls in the loads). N. N. Davidenkov related these observations to impact viscosity tests. In the same years, N. N. Davidenkov developed the definition of the critical (transient) brittleness temperature with the aid of "impact viscosity-temperature" curves which he constructed. He proposed that these curves also be used to determine indirectly the resistance to direct pull. N. N. Davidenkov (1938) noted that the part of the resistance work done after the maximum load is reached is most sensitive to the testing temperature and that a drop in the temperature primarily reduces this characteristic.

In 1946 B. A. Drozdovskiy divided the bending work of a cut sample into the elasto-plastic deformation work for a given cut and into the work used up in the development of the crack. He proposed that the latter be used as a quantitative estimate for the fracture viscosity of the material (which corresponded to the qualitative estimate based on the shape of the fracture). This concept is very similar to the generalization of the Griffith concept, which was developed at approximately the same time abroad by K. Zener, G. G. Holomon, G. R. Irwin and E. O. Orovan.

We mention several studies which deal with the determination of the resistance to direct pull of smooth samples. It was proposed that the circular bending of discs at a temperature of 196°C be used to estimate the resistance to direct pull (Ya. B. Fridman, 1941) in tensile tests of a thin disc welded to two rods made from a harder material (A. L. Nemchinskiy, 1950-1955).

S. I. Ratner (1959) studied the correlation between the resistance to direct pull and the magnitude of the rupturing stress during repeated loads. M. V. Yakutovich and V. A. Pavlov (1953) studied the relation between the stressed state and the direction of growth of the cracks.

P. O. Pashkov (1950) investigated the resistance of the material to brittle and viscous fracture in relation to the structure of the material and the shape of the sample. Ya. M. Potak (1955) gave an extensive analysis of brittle fractures in alloyed steel structures and pointed out the danger of brittle fracture in alloys with a large ferrite grain. Ye. M. Shevandin (1953-1965) made extensive experimental studies in the cold-brittle region of low alloyed structural steels.

T. A. Vladimirskiy (1953-1958) constructed for a series of structural steels three-dimensional "impact viscosity-sharpness of cut-temperature" curves. It became evident that when the sharpness of the cut was changed, the materials may damage in isolated places in comparison with the estimates obtained from the cold-brittle temperature.

Developing further the concepts of N. N. Davidenkov, Ya. B. Fridman (1941-1952) proposed the so-called generalized theory of strength, which was obtained from a systhesis of the theory of maximum tangential stresses and the theory of maximum elongations. The mechanical state diagrams proposed by Ya. B. Fridman take into account both the form of the stressed state and the properties of the materials (the resistance to direct pull and the resistance to yield or shear).

It was shown on samples with fissure defects using the method of knurling grids that the fracture process localizes near the end of the crack (Ya. Fridman and T. K. Zilova, 1950-1959). It follows from this theory that for the same material the fracture criterion may be, depending on the ratio of the tensile to maximum tangential stresses, either the resistance to shear or the resistance to direct pull.

In 1950 G. V. Uzhik proposed that the resistance to direct pull be estimated by tensile tests made on circular samples with a sharp annular cut. Yu. I. Likhachev (1956), who developed this method, proposed that the diameter in the cut be also changed during the stressing. A. Ye. Asnis (1947) estimated qualitatively the brittleness of steel by indicating the welded joint by the impact on the crack under the action of internal stresses. The characteristic used was the maximum temperature at which brittle rupture occurred.

An important stage in the development of experimental methods for estimating the brittle strength was the use of samples with a fatigue crack obtained previously in static or impact bending tests proposed by B. A. Drozdeovskiy and Ya. B. Fridman (1955-1960) as a universal method for estimating

the sensitivity of metals (including high strength metals) to a crack. This method made it possible to obtain easily with the aid of special vibrators almost any fatigue crack with a radius of curvature at its end smaller by several orders of magnitude than the radius of curvature of the usual tension crack, i.e., to satisfy in advance the requirement for the maximum sharpness of the crack.

We also mention here the method of W. D. Robertson, according to which the crack is initiated by the impact from the locally cooled region near the end of the cut, after which the temperature and the stoppage stresses of the crack are estimated. A slight modification of this method (the application of controlled static loads) was obtained by V. G. Cherkashkin and I. M. Rozenshteyn (1964). A. P. Gulyayev (1967) studied the impact bending of samples with cuts of different sharpness and extrapolated the "impact work-radius of cut" straight line relationship up to the zero value of the radius.

In 1965-1967 Ya. B. Fridman, B. A. Drozdovskiy and V. M. Markochev proposed that a "fracture curve" be constructed (a relationship between the increment in the size of the crack and the applied stress, the number of cycles or time) to characterize the capability of the material to brake fracture. The method developed by them for recording the development of a crack in sheet materials was used in the construction of these curves.

V. S. Ivanova (1967) proposed that the fracture viscosity on fatigue cracks be determined on the basis of the rupturing cyclic stress.

V. M. Finkel' (1964 and in later studies) studied experimentally the dynamics of the growth of cracks.

In the last few years workers engaged in applications and experiments in the USSR began to recognize more and more the approach in which the development of the cracks and the related fracture and strength problems are investigated in the frame of reference of studies dealing with the fine structure of the end of the crack, i.e., within the frame of reference of a single parameter describing the distribution of the stresses and strains near the end of the crack, the coefficient of the intensity of the stresses (for the most important practical case of normal fracture cracks). This point of view agrees well with the mathematical theory of quasibrittle cracks and in spite of its limitations, it is very progressive (it cannot be applied to a viscous or nearly viscous fracture).

Experimental studies on controlled fracture (with cracks) along these lines were made by B. M. Malyshev (1964, 1965) (lamination experiments), S. Ye. Kovchik and V. V. Panasyuk (1963-1967) (study of the growth of a crack under the action of concentrated forces, and a study of the effect of humidity and temperature on the energy on the surface of glass, etc.), V. M. Markochev (1966) (study of the rate of growth of cracks under the action of cyclic loads).

It was proposed that the controlled stable growth of a brittle crack be used to determine the brittle rupture constants and thus also the sensitivity of the materials to cracks.

The elegant method used to determine directly the effective surface energy from the hysteresis branch obtained in the "displacement-force" coordinates under loading and unloading during the stable growth of the crack should be mentioned. This method applied to the loading by concentrated forces was proposed by S. Ye. Kovchik and V. V. Panasyuk (1961). A similar method was applied abroad first to the unstable growth of cracks in metals by G. R. Irwin in 1958 (the so-called displacement or ductility method). In this approach, the theoretical solutions are not used, so that the method can be applied to a body of any shape, which can be very convenient in some practical cases.

Solutions of problems of an ideal elasto-plastic body with cuts whose thickness is zero are of interest in the theory of fracture in the transient region when the dimension of the plastic zone is comparable to the characteristic linear dimension of the body. When these solutions are supplemented by a condition for the local fracture at the end of a crack, the relationship between the strength and the shape and configuration of the body can be determined, in particular, the scale effect in the transient region can be determined. It is important to emphasize that the rigid-plastic (viscous) and brittle ruptures are always described as some limiting cases.

The jump in the displacement at the end of the crack in the case when the plastic deformations are concentrated along a line of zero thickness which continues the crack (M. Ya. Leonov and V. V. Panasyuk, 1959), and the specific energy flux near the end of the crack (G. P. Cherepanov, 1967) were proposed for the criterion magnitude which determined the beginning of the growth at the end of the crack.

We will mention several theoretical solutions that were obtained. M. Ya. Leonov and V. V. Panasyuk (1959, 1961) obtained a solution for the plane and axisymmetric elastic problem for one crack with a discontinuity in the normal displacement on the continuation of the crack. This elastic

solution can be interpreted as a solution of the elasto-plastic problem in the approximate formulation (the formulation of D. S. Dugdale, named after the British scientist who proposed in 1961, on the basis of experimental observations a similar solution as the solution of the elasto-plastic problem).

M. Ya. Leonov and N. Yu. Shvayko (1961) considered a hard body deformed everywhere except in "bad material" intermediate layers (slippage strips) which can be cut out conceptually and whose effect can be replaced by the corresponding forces. This leads to the linear elasticity theory problem on the deformation of a body with discontinuous displacements on some surfaces. P. M. Vitvitskiy and M. Ya. Leonov (1960-1962) solved several plane problems with linear Volterra dislocations. They found the values of the Kolosov-Muskhelishvili functions which determine the stressed-deformed state under the action of a linear dislocation in an unbounded plane with an elliptical hole.

The studies by P. M. Vitvitskiy and M. Ya. Leonov proposed a computational scheme for problems dealing with the development of slippage strips around sharp shell concentrations in elasto-plastic materials, with the aid [illegible] the solution was found for a plate with a narrow crack [illegible] circular hole. The last problem was also the subject of studies made by L. L. Libatskiy (1966). The relationship between the length of the plasticity strips and the load was obtained in these studies.

P. M. Vitvitskiy, M. Ya. Leonov and S. Ya. Yarema (1963) have shown that the first oblique slippage lines at the end of the cut during the elongation of thin metallic plates are formed at stresses at infinity equal to 0.66 σ_S, where σ_S is the yield point of the material and the direction of these strips subtends a 58° angle with the plane of the crack. These results were confirmed experimentally in the studies of S. Ya. Yarema (1962, 1964). The same problem was investigated in the study by K. N. Rusinko (1964).

During his study of the possible existence of a stable crack in problems dealing with the fracture of elasto-plastic plates, L. G. Lukashev (1963) developed concepts which are similar to the Leonov-Panasyuk model.

P. M. Vitvitskiy (1965) studied the problem of elasto-plastic deformations of a thin plate weakened by colinear cracks of equal length and also by two external semiinfinite cracks under stressing conditions at infinity by forces perpendicular to the line of the cracks.

B. V. Kostrov and L. V. Nikitin (1967) obtained in their study the solution of the problem for a longitudinal shearing crack with an infinitely narrow plastic zone near the ends of the crack, where the Mises plasticity condition had to be satisfied on the boundary of the plastic zone.

G. P. Cherepanov (1962) obtained the solution of the elasto-plastic problem for a longitudinal shearing crack with a plastic zone whose shape and dimensions were determined. In 1967 he proposed a solution for the elasto-plastic problem of the distribution of stresses and strains in the neighborhood of the end of a fissure. The material was assumed to be incompressible with a power relation between the second invariants of the stress and strain deviators.[1]

Defects causing the rupture of a sample or a structure can be conventionally classified into defects formed in the metallurgical process, which are created in the technological process, and defects which are formed or which develop during the operation of the structure (for example, corrosive or fatigue cracks). An alloy which is formed as a result of the metallurgical process is very complex in its structure (heterogeneous, anisotropic, with a complex distribution of internal stresses). By definition the strength is approximately $K_{IC}/\sqrt{d}$, where d is the characteristic diameter of the most dangerous crack defect and K_{IC} is a complicated function of the coordinates. The purpose of the metallurgical process, in addition to the defined chemical and temperature stability conditions of the alloy, is to create in the space uniformly distributed structural cells with minimum dimensions, whose boundaries play the role of energy strength barriers. (Most frequently such cells are the grains of the basic material and the chemically active additives formed in the crystallization centers during the hardening of the meet.) Apparently, the whole of the barriers is played by the intercrystalline films formed from the chemically inactive atoms.

1. Later when he studied the problem of the crack in D. S. Dugdale's formulation, G. P. Cherepanov (1968) obtained the growth in the length of the crack as a function of the applied load ("the fracture diagram") on the basis of the modified Griffith-Irwin-Orovan criterion proposed by him, and calculated the scale effect in the entire region (the brittle and viscous fracture are naturally limiting cases).

of the additives which are squeezed towards the boundary during the growth of the grains. The original crack defect develops during the loading process approximately up to the dimensions of the grain which are controlled in advance, so that at the instant of fracture, the quantity d is approximately equal to the diameter of the largest grain. This explains the fact that the strength of even very brittle alloys varies in a relatively small range in comparison with the strength of amorphous materials such as glass. Thus the basic method for increasing metallurigical strength from the standpoint of the linear mechanics of rupture is to increase K_{IC} (using alloying additives and thermal treatment which have an effect on the phase conversions, primarily on the boundaries of the grains) and to decrease the dimension of the largest grain (by a homogenization of the crystallization process).

Some investigators pointed out the importance of taking into account the total elastic energy margin and the ductility of the system in the explanation of the scale effect (N. N. Davidenkov, T. K. Zilova, I. A. Razov, Ya. B. Fridman, Ye. M. Shevandin, et al).

5.22 Statistical Nature of the Scale Effect

The strength of a material is always some random variable, since the exact position of all defects is not known in advance and second, even if this position were known, the corresponding mathematical problem could not be solved because of its complexity. The probability of encountering the largest and most dangerous effect in a large sample is greater. This concept is the basis of the explanation of the scale effect used in statistical theory.

The following two approaches can be used in the construction of the statistical theory:

a) one or several most dangerous defects can be selected on the basis of experience or intuition, and the remaining defects can be "spread" uniformly, assuming that the properties of the resulting continuous medium are known rigorously from the macro experiment (the length and possibly several additional parameters which determine the position of the most dangerous effect are assumed to be random variables with given distributions);

b) all defects, without exception, are "spread" over the volume and the resulting averaged medium is assumed to be continuous and "without defects" (the local strength of this medium and also the stresses are assumed to be random variables with given distribution functions at each point of the body, and the mean values of the stresses and the strength are determined, respectively, from macrotheory and macro-experiments). When this approach is used, a number of additional assumptions must be made (see below) to obtain the final expressions. The first method resembles more the theory of cracks (it stresses the physical nature of strength and of the scale effect), the second method is more formal and it resembles more the theory of the strength and resistance of materials. The approaches that were mentioned have somewhat different applications and they also play a different role in different stages of the physical rupture process.

Suppose that in the technological process defects which are more dangerous than metallurgical defects were formed in it (it is assumed that the dimensions of the grain are not large) and that the characteristic linear dimension of the structure is large in comparison with the dimension of the grain. In this case there is no doubt that the second approach must be used. It is this approach which is used in the majority of studies dealing with statistical problems in strength which will be presented below.

Now let us assume that during the technological process defects which are more dangerous than the metallurgical defects can occur in the structure. To obtain the distribution functions, according to the second approach, a representative sample from a number n of the corresponding structures is needed, and here the forecast of the strength for one concrete structure will already be probabilistic. Therefore, in practice the approach that was mentioned can only be applied to comparatively cheap mass produced articles and it cannot be used for unique or expensive structures. In this case, the only possible approach may be the first approach, which makes it possible by analyzing, for example, a comparatively small number of failures, to determine approximately the size and position of the defects causing the rupture. It must be emphasized that the technological and operational defects may completely distort even the usual character of the scale effect (for example, in larger products the strength may be higher). In what follows these defects are not included in the discussion, and by strength we shall mean the usual metallurgical strength. The conventional character of classifying the defects on the basis of their

origin should be noted. Several statistical theories were proposed for the quantitative description of the stochastic laws which govern strength. The basic principles of a statistical theory of strength for microscopic heterogeneous brittle ruptures bodies were formulated on the basis of experimental observations by A. P. Aleksandrov and S. N. Zhurkov (1933). They can be described by the following axioms. The spreading of the inhomogeneity in the properties (defects) over the volume of the brittle fractured medium is equiprobable. The instant when the weakest element of the body ruptures coincides with the rupture of the body as a whole. The strength of the sample cut out from such a body is determined by the most dangerous defect among all defects which occur in its surface layer.

The random character of the distribution of the inhomogeneity of the properties over the volume of the medium manifests itself in the scatter of the brittle strength of the samples. As the dimensions (surfaces) of the samples are increased, the frequency with which more dangerous defects occur incrases, the scatter region narrows down and the most probable strength value decreases, which is the way in which the scale effect is displayed. For a homogeneous stressed state, the lower scatter boundary is the same for samples of all dimensions and the strength of the largest samples is determined by the lowest strength of the samples with small dimensions, provided the latter are still large in comparison with the defects.

The study by A. P. Aleksandrov and S. N. Zhukov introduced the concept of a common lower scattering boundary for the strength and the dependence of the distribution of the random strength values of the body on its dimensions (1933).

These concepts are the basis for the statistical theory of strength proposed in 1939 by W. Weibull, which is based on the hypothesis of the "weakest link." This theory, in the case of homogeneous stressing, leads to a power relation for the strength as a function of the volume. Below it will be confirmed by some experimental data for metals (for example, in the studies by N. N. Davidenkov, 1943, and B. B. Chechulin, 1954-1963).

A logarithmic relation for the strength as a function of the volume was also proposed, which as G. M. Bartenev and Yu. S. Zuyev (1964) have shown, describes better the experimental data for rubbers. For glasses, the volume in the Weibull distribution must be replaced by the working surface (G. M. Bartenev and Yu. S. Zuyev, 1964).

A mathematical variant of the theory of the "weakest link" expressed in the form of a distribution for the smallest sample in a random sample was proposed by T. A. Kontorova and Ya. M. Frenkel in 1941-1943. This theory was used to determine the scale effect from the mean values of the brittle strength in the case of a homogeneous stressed state using a simplified form of the normal distribution law for the random values of the strength of elements of the body.

Subsequently a "weakest link" theory was proposed in 1949 by W. Weibull for the description of fatigue fractures, and was treated in a general mathematical formulation in the study of V. V. Bolotin (1961) and it was also used to describe the scale effect during fatigue fracture (R. D. Vagapov, 1958-1964, S. V. Serensen and V. P. Kogayev, 1959-1962).

Extensive experimental studies whose purpose was to clarify the nature of the scale effect under a single load were made by G. M. Bartenev, F. F. Witman, A. Ya. Volovik, M. Ya. Gal'perin, N. N. Davidenkov, N. A. Makhutov, N. G. Plekhanov, S. I. Ratner, S. V. Serensen, G. A. Stepanov, G. V. Uzhik, Ya. B. Fridman, B. B. Chechulin, Ye. M. Shevandin, N. P. Shchapov, and others.

The scale factor plays a particularly important role in the presence of embrittling factors (an external medium active on the surface, etc.). The fatigue strength scale effect has been studied most thoroughly.

Given the contemporary engineering development trend (the building of large power engineering structures, hydroengineering equipment, etc.) the natural fatigue tests of parts with large dimensions are impractical for all practical purposes. Therefore, an estimate of the scale effect becomes more and more important, since strength calculations are based on the characteristics of fatigue resistance obtained in laboratory samples whose dimensions can be tens or hundreds of times smaller than the characteristic dimensions of the parts.

With regard to the action of variable loads, it was established that a reduction of the fatigue limit in samples and parts when their dimensions are increased has two aspects, namely a metallurgical and mechanical aspect. In the first case, the scale effect is caused by the comparatively high degree of imperfection in the structure of the material in large casts or forgings used in the manufacture of parts with large dimensions. In the second case, the scale effect manifests itself in the reduced strength in geometrically similar samples when their absolute dimensions are increased and also when the samples are cut out from the same body (G. V. Uzhik, 1942).

The first experimental studies were devoted to explaining the relationships between the fatigue limits in geometrically similar samples during variable bending and torsion and the dimensions of their cross section. To eliminate the effect of the metallurgical factor, the samples were cut out from the same billet.

A generalization of the experimental study of the effect of the scale factor on the fatigue limit made it possible to introduce this factor along with the concentration effect into the definition of the load-bearing capacity of elements of structures under variable loads (S. V. Serensen, 1937-1945, G. V. Uzhik, 1942).

In order to make the transition from fatigue limits in laboratory samples to the strength of the part, a coefficient for the effect of the cross sectional dimensions was introduced, which was equal to the ratio of the fatigue limits of samples with a large diameter to the fatigue limit of the laboratory sample (S. V. Serensen, 1934). The experimental studies were generalized in the form of graphs for the relation between the reduced fatigue limits and the increasing dimensions of the cross section (S. V. Serensen, 1957, G. V. Uzhik, 1957).

It was established that the scale effect has an asymptotic tendency, which is more pronounced for high strength steels, and inhomogeneous cast materials in the presence of stress concentrations (S. V. Serensen, I. V. Kudryavtsev, V. P. Kogayev and L. A. Kozlov, 1949).

At the same time natural testing methods of the parts were developed which made it possible to obtain important information (N. P. Shchapov, S. V. Serensen). Test benches which can be used to fracture samples with a 150-300 mm characteristic dimension of the cross section were produced to approximate the experimental results by the natural dimensions of the parts and to estimate the metallurgical factor. It turned out that for samples with such cross sections, the fatigue limits are reduced two or three times as much in comparison with the fatigue limit of standard samples with a 7-10 mm diameter (V. A. Veller, V. P. Kogayev, I. V. Kudryavtsev, S. V. Serensen, S. I. Yatskevich).

The specific features of the scale effect in a corrosive medium were investigated (L. A. Glikman, 1953, G. V. Karpenko, 1953).

Simultaneously with the generalization of the experimental data for the purpose of an analytical extrapolation in the region of the natural dimensions of the parts, a theory of the scale effect under the action of variable loads was developed.

To describe the scale effect by the fatigue limits which depend on the dimensions of the cross section of the body and the nonuniform distribution of the macroscopic stresses in this cross section, the statistical model of a microscopic inhomogeneous polycrystalline body was used (N. P. Afanas'ev, 1940).

According to the statistical theory of N. P. Afanas'ev the limiting safe stress amplitude is determined by the presence in the body of a number of crystallites located next to one another, in which the microstresses attain the values of their strength limit during the cycling loading process. The probability of such an event depends only on the dimensions of the cross section of the body and the nonuniform distribution of the microstresses on this section, so that it is assumed that the transition from one cross section to another does not lead to a new combination in the distribution of the microinhomogeneities. Therefore, the length of the body and the distribution of the stresses along its contour are not taken into account. From this a deterministic relation is obtained between the reduced fatigue limits during tension-compression and the increased dimensions of the cross section of the body, and a stronger scale effect during variable bending. According to another variant of the same theory, the scale effect is related to the nonuniform distribution of the microstresses in the surface layer, whose depth is determined by the minimal rupturing stress which is equal to the fatigue limit during the tension-compression of a large sample. From this the relation is obtained between the fatigue limit which is expressed in terms of the amplitude of the stress at the dangerous point of the body and the gradient of the stresses in the cross section. This theory gives a unique deterministic interpretation of the scale effect during the bending of smooth samples in the presence of stress concentrations.

According to a proposal of I. A. Oding, the imperfections in a real body on the microscopic level during variable deformations which are phenomenologically nearly elastic can be represented schematically by the ideal plasticity curve with a horizontal sector which is equal to the tension-compression fatigue limit during the stressing. The fictitious fatigue limit calculated on the assumption of an elastic distribution of the stresses, turns out to be smaller, the smaller the nonuniformity of the distribution of the stresses in the dangerous cross section of the body, i.e., the larger the diameter of the sample in bending.

On the basis of decomposing the effect of the stress gradients and the dimensions of the body on the magnitude of the fatigue limit, the scale effect is represented by two components, one of which is due to the imperections of the material on the microscopic level and the other to the macroscopic heterogeneity which manifests itself in the scatter of the fatigue resistance characteristics (R. D. Vaganov, 1955). Subsequently, the scale effect was also considered in the bounded endurance region, i.e., along the fatigue curve which is represented in the "stress amplitude-endurance" coordinates.

It was shown that the scale effect can be interpreted as a reduction in the mean statistical values of the durabilities and the endurance limits as the surface of the body increases while the common lower scatter boundary is preserved when the scatter region is represented in the "stress amplitude at the dangerous point-endurance" coordinates in the cycles (there is no scale effect along the lower boundary).

As the stress distribution approximates in the cross section of the body a uniform distribution, the entire scatter region can be displaced in the direction of smaller durabilities and the limiting fatigue curve will be the curve along the lower scatter boundary for the smooth samples tested for tension-compression. This shows that it is possible to estimate probabilistically the resistance to fatigue of parts with large dimensions from the results of model samples (R. D. Vagabov, O. I. Shishorina and L. A. Khripina, 1958-1960).

In 1959 V. P. Kogayev also showed on the basis of a statistical analysis of the results of tests made on samples with various shapes and dimensions the possible existence of a common lower boundary of the minimal endurance values when the probabilites for the initial damaging stage were small and the mean endurance values depended considerably on the shape and dimensions of the body.

Ya. S. Podstrigah and M. I. Chayevskiy (1959) proposed that the temperature effect of the cyclic load caused by the imperfections in the material on the microscopic level and the nonuniformity of the stationary temperature field formed in the process in the sample be taken into account. According to this theory due to the reduced heat transfer from the internal zones of the body accompanied by an increase in its cross section, the magnitude of the tensile thermoelastic stresses in the surface layer incrases. The scale effect is treated as the effect of the asymmetry of the cycle caused by the thermoelastic stresses that were mentioned.

For the stage in which the body is separated by the main crack, it is proposed that the scale effect be estimated from the increase in the rate at which the crack develops as the dimensions of the body increase while the geometric and force similarity are preserved (R. D. Vagapov, 1960, 1961).

S. V. Serensen and V. P. Kogayev (1962) using the "weakest link" theory and the Weibull distribution function described the scale effect taking into account the non-uniform distribution of the stresses in the cross section of the body. The scale effect is determined as the reduction in the mean statistical fatigue limits as the stress gradient decreases in the dangerous cross section of the body, and the perimeter of the body increases. It was proposed that the parameters of the original distributions for elementary macrovolumes of the body and the lower scatter boundary be determined from the results of statistical tests made on two series of samples with different relations between the stress gradients and the diameter of the sample. Such information is universal in the description of the scale effect which depends on the stress gradient and the dimensions of the body.

The studies by R. D. Vagapov (1964, 1965) using the same Weibull distribution function, described the scale effect after the body was damaged by the first macrocrack in relation to the stress distribution in its surface layer. The relations between the mean statistical endurance and strength values and the surface of the body and the distribution of the stresses along its generatrix (in particular, the length of a cylindrical sample) were confirmed experimentally and the parameters of the original distribution were determined from the results of tests made on samples having the same shape.

§6. Rupture under a Cyclic Load

The phenomenon of fatigue rupture was discovered by V. Rankine and A. Wöhler more than one hundred years ago. Since then it has been studied intensely and at the present time the mechanics of fatigue failure is the basis for the design and calculation of the majority of dynamically stressed structures and machines. The increase in the operational reliability, the reduction in the weight and the increase in the economic usage indices of dynamically stressed structures in the national economy are intimately related to the development of fatigue problems. The fatigue studies of structural materials including steel for bridges, air dried timber that were begun at the beginning of the century were further developed after the Revolution. M. V. Voropayev started

the study of fatigue failure earlier than many foreign authors and introduced into the process the concept of irreversible hysteresis losses as the reason for the fatigue failure.

We will dwell briefly on certain aspects of the phenomenon of fatigue failure without discussing here such questions as the creep of metals and polymers.

In the machanics of fatigue failure, studies were made to determine the strength criteria which depend on the regime and type of stressed state; the similarity conditions and the kinetics of the accumulation of fatigue failures during a nonstationary variable load were analyzed and statistical concepts and the theory of plasticity were used for the quantitative description of the laws for the formation and development of fatigue cracks and the transition to brittle rupture.

The development of the statistical aspect of the mechanics of fatigue failure made it possible to relate studies of strength during variable loads to operational reliability theory. According to contemporary concepts, the fatigue failure of metals and polymers is physically caused by the microinhomogeneity of the structure of the material, and as a result, it is impossible to avoid a local stress concentration, which causes the accumulation of irreversible microplastic deformations.

The criteria for the resistance to fatigue failure during symmetric and asymmetric cycles and a complex stressed state were developed in the light of the analogy with plasticity and statistical strength criteria and also on the basis of statistical energy concepts.

S. V. Serensen (1937) proposed to adopt a linear approximation for the relation between the limiting amplitudes of the stresses and the mean stresses in the cycle and to express the coefficient characterizing the effect of the asymmetry of the cycle in terms of the endurance limits under a symmetric and fluctuating cycle. In 1955 L. I. Sevel'ev proposed that this coefficient be expressed in terms of the endurance limit during a symmetric cycle and the true resistance to fracture.

Taking as the criterion quantity characterizing fatigue strength, the area of the hysteresis loop, I. A. Oding (1937) obtained a quadratic relation relating the limiting amplitudes to the mean stresses. D. I. Gol'tsev (1953), using the analogy with statistical strength obtained a power relation between the limiting amplitudes and the mean stresses. He used for the criterion quantity a power function of the intensity of the tangential stresses and the mean pressure.

I. A. Oding (1948) proposed that the imperfections in the real material during variable deformations which were phenomenologically nearly elastic be represented schematically by the hysteresis loop or by the corresponding ideal plasticity curve with the horizontal sector equal to the tension-compression fatigue limit during the stressing. At the same time he adopted for the fatigue damage criterion the amplitudes of the deformations at which the width of the hysteresis loops had a certain small value.

In subsequent studies, G. S. Pisarenko and V. T. Troshchenko (1967) proposed that the equation for the fatigue curve in the region of large numbers of cycles be represented in terms of the deformation amplitudes. Beyond the fatigue limit, the authors proposed to use those stresses which correspond to some (small) value of the width of the hysteresis loop. Thus, the fatigue limit is defined as the cyclic proportionality limit.

S. V. Serensen (1941) proposed on the basis of experimental fatigue studies made on steels, pig iron and light alloys, to express the strength conditions during variable loads for the plane stressed state by the hypotheses of the largest tangential or octahedral stresses, taking into account (in linear form) the effect of the components of the normal stresses on the corresponding areas.

N. N. Afanas'ev (1940) formulated the basic statistical assumptions for the mechanics of rupture of a polycrystalline body.

To obtain the strength conditions during symmetric or asymmetric cycles and a complex stressed state, he used physical concepts for the metal as a microinhomogeneous medium characterized by the nonuniform microstressing of the crystals. It was assumed that during the cyclic loading the stresses in individual unfavorably oriented grains increased all the way to the resistance to direct pull which leads to their fracture However, the fracture of isolated grains does not yet cause the rupture of the body. The criterion quantity for the fatigue strength of a body that was used was the fracture of a particular number (which was constant for each material) of adjacent micrograins of the metal. The probability of such a situation depends on the dimensions of the cross section of the body, the nonuniformity of the microstress distribution and the macroscopic stressed state.

S. D. Volkov (1953, 1954) made an important contribution to the further development of the statistical mechanics of a polycrystalline body. Considering the material of a detail as a microinhomogeneous medium and assuming that fracture in some microvolume occurs when the tensile stresses attain the cohesion strength, he obtained the strength condition during variable loads and a complex stressed state taking into account the asymmetry of the cycle (1960).

D. I. Gol'tsev (1953), using the imperfect elasticity characteristics used the area of the hysteresis loop as the measure of the damage in one cycle without taking into account the area of the loop at the fatigue limit. The measure of the damage is independent of the number of cycles and the total rupture work, of the amplitude of the stresses. From the above follows the equation for the fatigue curve, the linear integration law for the damages during nonstationary loading regimes and the strength conditions during variable stresses and a complex stressed state in the power form.

Continuing these studies, R. D. Vagapov (1964) considered in connection with creep and relaxation phenomena (repeated load) the problem of a hysteresis loop with a variable number of parameters depending on the number of cycles describing the dependence of the measure of damage on the number of cycles, the deviation from the linear integration law for the damages and the dependence of the endurance on the type of loads.

N. N. Vasserman and V. A. Gladkovskiy (1965), studied the fatigue phenomenon from the standpoint of two mutually interrelated processes: hardening and softening. The hardening process was characterized by the magnitude of the minimal damaging stress which increases with the number of cycles. On this basis a unique interpretation of the equations for the fatigue curve and the aging phenomenon during nonstationary cycles was obtained.

I. A. Birger (1948) formulated the strength condition for a plane stressed state with asymmetric cycles on the basis of the assumption that during an asymmetric cycle only the statistical normal stresses have an effect on the fatigue strength

S. V. Serensen (1933) proposed in his study a relation for the fatigue limit as a function of the amplitude of the maximum stress and the local gradient of the stress. Continuing studies along these lines, N. N. Afanas'ev (1936) proposed a statistical explanation of the relation that was mentioned. The same effect was described by I. A. Oding (1948) on the basis of im-

perfect elasticity concepts with the aid of cyclic viscosity (equal to the product of the elasticity modulus and the width of the plastic hysteresis loop). S. V. Serensen, G. V. Uzhik and R. D. Vagapov (1955) proposed conditions for the modeling of fatigue failure.

Subsequently, the variability of fatigue characteristics which manifests itself in the scatter of the fatigue strength characteristics became the object of theoretical and experimental studies. In 1948 A. I. Kochetov and A. D. Krolevetskiy and in 1952 M. Ya. Shashin used correlation analysis in the statistical description of the sloping segment of the fatigue curve, taking into account various probabilities of failure. Here the normal distribution law was used for the logarithm of the endurance, with parameters which were independent of the amplitudes of the stresses. The possibility of applying the lognormal distribution for the rupturing number of cycles, taking into account the sensitivity threshold was investigated (V. P. Kogayev, 1957) as well as regression analysis for the description of the sloping segment of the fatigue curve which depends on the fracture, taking into account the dependence of the distribution functions on the level of the stresses (M. N. Stepanov, Ye. V. Giatsintov and V. P. Kogayev, 1959).

The fatigue damage process is separated into two stages; the stage in which microdamages are accumulated, which are scattered over the volume of the body culminating in the formation of the first microcrack, and the stage when the body is separated by the main crack. The laws were estimated on the basis of equiprobable parameters for equal damage (R. D. Vagapov, O. I. Shishorina and L. A. Khripina, 1958-1964). These studies established the analogy between the statistical rupture model of an ideal brittle body based on the "weakest link" (S. N. Zhurkov and A. P. Aleksandrov, 1933) and the proposed model for the damage of the body by the first fatigue microcrack. It was shown that the strength and endurance of parts with large dimensions could be estimated probabilistically in this manner on the basis of the results of the statistical tests made on model samples all the way up to the determination of the lower scattering boundary from the damage caused by the first microcrack.

L. G. Sedrakyan (1958 and in later studies) proposed a statistical deformation and rupture theory for brittle materials, which made it possible to clarify certain particular features of the resistance to deformation of real structural materials, such as pig iron, concrete, rocks, etc. The theory is based on the scheme of an ideal heterogeneous material and the real deformation

characteristics depend on an arbitrary function (the distribution function for the inhomogeneity of the material for the given inhomogeneity characteristic) and the material constant (the coefficient of friction) which are determined from the experiment. This model explains the gradual character of the rupture process, the fatigue and long-term strength, the increase in the volume of the material during its preferred compression and the presence of a descending branch in the compression-elongation curve and other characteristics.

To describe the statistical laws for fatigue rupture, V. V. Bolotin (1961) used the "weakest link" hypothesis and made this hypothesis more precise by relating it to the existence of a minimum value threshold.

On the basis of the hypothesis that was mentioned, S. V. Serensen and V. P. Kogayev (1962), determined the dependence of the distribution function for the random endurance and strength values for a body of a given shape and dimensions on the stress gradient in the dangerous section and the perimeter of the dangerous section. The statistical similarity criterion used was the ratio of the perimeter of the dangerous section or a part of it to the relative gradient of the first principal stress in this section. Using the criterion that was introduced, the relation between the maximum rupturing stresses in the stress concentration zone and the probability of the rupture of the body was determined.

R. D. Vagapo (1959-1965) proposed for the limiting case when the probability that the body will be damaged at some depth is negligibly small a theory of scatter for the endurances and the fatigue limit which took into account not only the transverse but also the longitudinal dimensions of the body and the distribution of the microstresses along its contour. The distribution function depends on its shape, the dimensions of the body and the loading method, i.e., it gives a probabilistic estimate of the stress concentration and the scale effect. The joint probability density for the random variables, the strength and endurance, is used in the discussion as well as the random coordinate of the damage by the first microcrack.

An intense study of the fatigue strength of parts under loads with variable amplitudes during the operating process began roughly in the 40's. The studies by S. V. Serensen (1944), D. N. Reshetov (1945) and V. M. Bakharev (1945) analyzed the linear hypothesis for the integration of fatigue failures to estimate the endurance and strength for a stress amplitude which varied over time. They proposed phenomenological treatments for the accumulation of the fatigue failures when the amplitudes were varied, which were based on

an analysis of the properties of the secondary fatigue curves for programmed loading and deviations of their parameters from the linear integration conditions for the failures (S. V. Serensen, L. A. Kozlov, 1953), the use of the hysteresis energy absorbed by the metal under stresses exceeding the endurance limit (D. I. Gol'tsev, 1955), on an analysis of the properties of the measure for the damage and the introduction of two fatigue failure stages (V. V. Bolotin, 1959-1963).

The study of fatigue strength under random external effects is of great practical and theoretical interest.

V. V. Bolotin (1963) formulated the principles for the decomposition of the random process into cycles and conditions which are needed to determine the mean endurance margin using the concept of a limiting fatigue surface (in the "stress amplitude-mean stress of cycle-endurance" coordinates).

S. V. Serensen, Ye. G. Buglov and V. P. Kogayev (1960 and in later publications) discussed the estimate for the fatigue strength under a random load in the probabilistic aspect.[1]

An evaluation of the experimental fatigue laws based on the equiprobable damage parameter, made it possible to obtain from the statistical laws deterministic laws for the nonlinear integration of the relative endurances during a nonstationary loading regime (R. D. Vagapov, 1964 and later studies). On the basis of a study of the accumulated fatigue damages from the statistical aspect, S. V. Serensen and V. P. Kogayev (1966) estimated the deterministic and random components in the sum of the relative endurances and proposed a correction for the linear hypothesis which depended on the spectrum of the stress amplitudes.

1. In 1968 M. Ya. Filatov used for the fatigue failure criterion the energy accumulated per unit volume in the metal during the entire loading period which made it possible to analyze the effect of the complex form of the cycle on the fatigue strength.

The subsequent development of this branch of mechanics is connected with the construction of stochastic models of the fatigue process. V. V. Bolotin and Kh. B. Kordonskiy (1961) proposed that the fatigue damage process be treated as a random process of the Markov type. The use of processes that are nonuniform over time, where the insensitites of the transitions diminish over time would make it possible to justify the lognormal distribution for the endurance and to explain the aging phenomenon, i.e., the increase or decrease in the endurance during the transition from one amplitude level of the stresses to another (Kh. B, Kordonskiy, 1961).

S. V. Serensen and V. P. Kogayev (1965) considered the fatigue process as a Markov process with a finite number of states which is uniform over time, with continuous time, and analyzed and quantitatively characterized the statistical laws for the accumulation of the fatigue failures during a programmed load. The distribution function for the endurance was obtained by means of stochastic matrix multiplication and the Monte-Carlo method.

By introducing the concept of a damage tensor which depends functionally on the stress tensor, A. A. Il'yushin (1966) was able to develop an approach to the study of the strength of materials, taking into account the loading history during cyclic loads.

The fatigue failure under variable contact pressures was studied by means of an investigation of the contact stresses, taking into account both the normal and tangential forces at the points where the parts made contact, and by an analysis of the strength conditions for the three-dimensional stressed state. This made it possible to obtain the relations between the limiting contact pressures and the fatigue characteristics (M. M. Saverin, 1948, S. V. Pinegin, 1967).

The extensive experimental studies of the fatigue failure laws made it possible to accumulate, starting in the mid-30's, voluminous material on the character of the fatigue curves, the types of distribution functions of the random values of the strength and characteristics of the effect of the state on the surface, the interaction of media, the field of residual stresses and the mechanical properties of the surface layer on the fatigue strength (N. M. Belyayev, M. E. Garf, L. A. Glikman, M. M. Hochberg, N. N. Davidenkov, G. V. Karpenko, I. V. Kudryavtsev, A. V. Ryabchenkov, M. N. Stepnov, V. I. Trufyakov, M. Ya. Shashin, N. P. Shchapov and others).

The results of the theoretical and experimental studies in the mechanics of fatigue failure are the basis for the improvement of the structural shapes and parts of their strength calculations and the manufacturing technology, including surface hardening.

The manufacture of powerful high parameter structures of large dimensions required the development of strength problems under a cyclic load in the elasto-plastic region. Under these conditions, in the most stressed zones of the nodes and parts, a considerable change in the deformation laws and the conditions under which the cracks are formed and spread during the cyclic load occurs. This is due to the fact that under the loads that were mentioned which correspond to a comparatively small number of cycles before rupture (up to 10^3 - 10^4), the elasto-plastic deformations are redistributed according to the number of cycles which depend on the loading conditions (the nonuniformity of the stressed state, the temperature, the deformation rate, etc.) and the cyclic properties of the materials. The formation and spreading of cracks during a small number of cycles, in the general case occurs against the background of the accumulated cyclic plastic deformations which have the same direction and the description is based on the corresponding failure criteria for a small number of cycles. The nonstationarity of the elasto-plastic deformations, during a small number of loading cycles determines the conditions for the attainment of the limiting states by the structural elements and, hence, also their load-bearing capacity.

The first studies in the USSR which studied the fatigue of aviation structural elements for a small number of cycles were made by N. I. Marin (1946). The experiments that were made on cylindrical pipes (with welded seams and without them) and plates with a hole have shown that the resistance to small-cycle fracture expressed in terms of the nominal rupturing stresses is lower than the resistance to fracture during a single static load which depends on the mechanical properties of the material and the level of the stress concentration.

In the subsequent years, the basic results in the study of resistance to small-cycle rupture were obtained in the studies by D. A. Hochfeld, V. V. Moskvitin, V. V. Novozhilov, S. I. Ratner, S. V. Serensen, Ya. B. Fridman and R. M. Shneyderovich. At the same time considerable attention was given to the construction of the state equations for the case of cyclic loading on one hand, and the fracture criteria on the other hand.

V. V. Moskvitin (1951-1965) generalizing the conditions of G. Masing and using the theory of small elasto-plastic deformations for the case of a repeated load proved a number of theorems on variable loads, secondary plastic deformations and limiting states. It was possible to use on the basis of these theorems finite relations between the stresses and strains for the solution of the corresponding problems. These relations are valid when the loads are nearly simple. The studies by V. V. Moskvitin also proved the possibility of applying the theory developed by him to the case of a complex load, when the principal stresses change sign during cyclic loading. The theory of small elastic deformations during cyclic loads was used by V. V. Moskvitin and V. Ye. Voronkov (1966) in the solution of a number of concrete problems (the cyclic bending of a beam and plates, repeated torsion of rods with a circular and oval cross section, repeated loading by internal pressure of a thick-walled cyclinder and sphere and other problems).

Along with this approach, N. N. Afanas'ev (1953), Yu. D. Sofronov (1959), N. I. Chernyak and D. A. Gavrilov (1966) developed in their studies statistical models for a polycrystalline conglomerate consisting of a large number of differently oriented elements with different mechanical properties (both without hardening and with hardening). It was possible to describe analytically the form of the elasto-plastic hysteresis loop, the Bauschinger effect, and also the change in the stresses and strains on the basis of these models. Yu. D. Sofronov obtained a relation between the stresses, strains and the number of cycles before rupture which he applied to cyclically hardening materials.

A systematic experimental study of the state equations, made by S. V. Serensen, R. M. Shneyderovich and A. P. Gusenkov (1960-1966), made it possible to determine the existence of a generalized cyclic deformation curve, which for the given material was a function of the number of cycles which was independent of the character and type of loading. It was proposed that the cycle- by-cycle relations between the stresses and strains for the loading processes be expressed in finite form on the basis of the generalized cyclic deformation curve.

It was shown in these studies that the plastic components of the cyclic deformations (the width of the loop) decreases or increases, depending on the properties of the cyclic hardening or softening of the material. In addition, the properties of the cyclic anisotropy were determined, which manifested themselves in a one-sided accumulation of the plastic deformations.

Subsequently the generalized cyclic deformation diagram was extended to asymmetric stress cycles and deformation under raised temperature conditions, using the aging hypothesis. Problems in the bending and torsion of solid rods, tension-compression of a strip with holes and rods with a circular cross section with annular recesses under cyclic deformation were solved in this formulation (R. M. Shneyderovich, A. P. Gusenkov and G. G. Medeksha, 1966, 1967).

V. V. Novozhilov, R. A. Arutyan, A. A. Vakulenko, Yu. I. Kadashevich obtained in their studies the state equations during cyclic deformation on the basis of the generalized theory of plastic flow, using a model with dry friction and taking into account the microstresses. This enabled them to study a complex load and the corresponding limiting states.

I. Z. Palley (1965 and in later studies) studied non-isothermal cyclic loading processes on the basis of a generalization of plasticity and creep theory, introducing the similarity of the stress deviators and the plastic deformation rates. The problem of a nonuniformly heated plate and disc under a cyclic load was solved in conjunction with this.

V. I. Rozenblyum, D. A. Hochfeld and V. V. Moskvitin (1958 and in later studies) investigated the limiting state during the cyclic loading of rod systems, plates, discs and shells in the light of adaptability theory.

D. A. Hochfeld (1964-1967) developed stabilization criteria for deformation processes under repeated loads and heating and described the accumulation of deformations leading to "progressive" rupture and solved the load-bearing capacity problem of discs, pipes, shells and other structural elements on the basis of adaptability theory. A. A. Chiras (1966) used linear programming methods in the solution of adaptability problems in rod systems.

S. V. Serensen, N. A. Makhutov and R. M. Shneyderovich (1964-1966) proposed a description of the conditions for rupture in a small number of cycles based on force and deformation rupture criteria. An analysis of the conditions for rupture in a small number of cycles was obtained by them on the basis of deformation criteria. The criterion for the quasi-static rupture proposed by them was the magnitude of the limiting one-sided accumulated plastic deformation which is equal to the deformation during rupture from a single load for uniform and nonuniform stressed states. The use of generalized cyclic deformation curves and deformation criteria enabled these authors (1966 and in later studies) to determine

the limiting state during fatigue processes with a small number of cycles. For the cases of small-cycle loads during which the intensities of the accumulated quasistatic and fatigue failures are comparable, the limiting number of cycles is determined on the basis of the integrability hypothesis of these damages.

For the experimental verification of the kinematics of cyclic deformation and the rupture criteria, experimental methods were developed which studied the deformation fields with the aid of optically active coatings (R. M. Shneyderovich and V. V. Larionov, 1965) the moire method (R. M. Shneyderovich and O. A. Levin, 1967) and the method of grids (N. A. Makhutov, 1964).

In their studies V. V. Novozhilov and O. G. Rybakina (1966) proposed as the fracture criterion during static loads with a small number of cycles the plastic deformation path which is proportional to the product of the intensity of the plastic deformations and the number of cycles, whose limiting value depends on the plastic friability of the material. This criterion was used to describe the cyclic ruptures during a symmetric and asymmetric deformation cycle.

A force criterion for loading with a small number of cycles in the form of the maximal local stress was developed by R. M. Shneyderovich and V. V. Larionov (1962-1965). This criterion made it possible to describe the rupture during rigid loading under uniform stressed state conditions and also the rupture during an external load in the concentration zones from the given stresses.

It should be mentioned that the energy criteria for small-cycle rupture are based on various concepts: the total energy of the plastic deformation (A. G. Kostyuk, 1966), the thermal equivalent of the elasto-plastic deformations (V. S. Ivanova, 1967) and the energy of the plastic deformations in the hardening region (N. S. Mozharovskiy, 1966).

To describe the rupture conditions during a small number of cycles, a function of the damage in the material was used, which depends on the plastic deformation path (V. V. Novozhilov and O. G. Rybakina, 1966), and the accumulated energy of the plastic deformation (A. G. Kostyuk, 1966). This function is introduced both in the state equations and in the strength conditions to determine the degree of cyclic and long-term static damage and the effect of the asymmetry of the deformation cycle.

A very important part of the general fatigue problem is the study of the laws for the development of the cracks in metals and polymers under cyclic loads. The first investigators of the fatigue phenomenon noted that the fracture is usually preceded by a lengthly stable development of a crack. Subsequent investigations have shown that about 30-60% of the time before failure a crack develops in a product subjected to a cyclic load (S. V. Serensen, R. M. Shneyderovich and N. A. Makhutov, 1966). However, the experimental data on this problem do not agree sufficiently well because of the difficulty of detecting the initial developing crack.

One of the first studies on the stable development of a fixed fatigue crack in the USSR was made by R. D. Vagapov (1959). Later in connection with the results obtained from the theory of brittle cracks, it was established on the basis of many experiments, that the rate at which the fatigue crack propagates depends only on the characterisitics of the coefficient of the intensity of the stresses at the end of the crack, when the dimension of the plastic zone on the edge of the crack is small in comparison with the dimensions of the body (D. K. Donaldson and V. E. Anderson, Proc. Crack Propagation Symp. (1961), Vol. 2, Cranfield, 1962, pp. 375-441, P. K. Paris, ibid., V. M. Markochev, 1966, K. D. Mirtov, 1968).

Thus, the dependence of the rate of growth of the crack on the geometric dimensions of the body and the external loading parameters is represented in this case by one quantity (the coefficient of the intensity of the stresses). It should be mentioned that the assumption that the plastic region is small for fatigue cracks is much less essential than in the case of fracture under a monotonic load, as a result of the lower stress level during which the fracture process occurs.[1]

1. G. P. Cherepanov derived in 1968 a theoretical relationship between the rate of growth of a fatigue crack and the characteristics of the coefficient of the intensity of the stresses and solved several concrete problems (an analogue of the Griffith problem, a crack in a layer under the action of a cyclic moment and other problems). The solution was based on a modification of the physical concepts of G. R. Irwin and E. O. Orowan on the specific energy dissipation. The empirical formula of P. K. Paris can be obtained from the relation that was derived when the stress level is not very high.

§7. Effect of Temperature on Solids

In the study of the strength and rupture of metals and polymers, thermal strength problems which consist of a study of the strength of materials and structural elements under the action of various types of force and thermal loads in a wide range of temperature changes are particularly important. These problems are particularly topical in connection with the development of such branches of modern machine building as the building of reactors, engines, rocket technology and many other branches. The tendency to increase the operating temperature of various aggregates and installations requires not only that the distribution and intensity of the temperature stresses and deformations be determined exactly, but also an investigation of their effect on the short-term and long-term strength, thermal fatigue, thermal bulging and other phenomena. On the other hand, strength criteria during comparatively low temperatures are also of interest. The development of space explorations, problems in chemical technology, the use of metallic structures under Far East conditions, etc., make it necessary to study the laws for the cold-brittleness phenomenon.

Extensive theoretical and experimental studies of various general and special problems in thermoelasticity and thermal strength were made in our country (V. V. Bolotin, I. I. Gol'denblat, E. I. Grigolyuk, V. I. Danilovskaya, A. A. Il'yushin, A. D. Kovalenko, G. S. Pisarenko, Yu. N. Rabotnov, S. V. Serensen, V. N. Feodos'ev, Ya. B. Fridman and others). In this section we will only consider thermal strength problems and ruptures at high and low temperatures. Problems in the analysis of thermoelastic and thermoplastic deformations and stresses before rupture are not discussed here. The study of strength at high and low temperatures encompasses a large class of problems of an experimental and theoretical character. The experimental studies are primarily related to obtaining the main strength and deformability characteristics of various materials (predominantly refractory materials) which depend on the temperature both during loads of short and long duration. This series of studies also includes the experimental determination of the elastic constants of the material at high and low temperatures.

Recently a large amount of data about the properties of solids and polymers at raised temperatures and various loads has been accumulated.[1] A very large number of interesting and

1. See, for example, "Handbook on Machine Building Materials," Vol. 2, (Moscow, 1959) and the collection "Plastic Layers of Organic Origin. Classification, Engineering, Nomenclature and Basic Properties (handbook material)" (1959).

important studies deal with an investigation of mechanical and thermal properties of solids and polymers at different temperatures. Systematic studies along these lines were made by N. N. Davidenkov, B. A. Drozdovskiy, I. A. Oding, G. S. Pisarenko, S. V. Serensen, Ya. B. Fridman and their students.

The studies by G. S. Pisarenko and his students worked out problems connected with the methodology and means for determining various characteristics of the materials and high and low temperatures (1958 and later studies). The studies by I. A. Oding (1945-1962), S. V. Serensen (1950 and later studies), and by Ya. B. Fridman (1952-1962) and their students are devoted to a clarification of the complex laws for the mechanical and thermal strength.

The large volume of data on the properties of different materials at raised and lower temperatures that was accumulated facilitates the problem of determining the admissible stresses during the calculations of the strength of structures from the stresses caused by the external load and the temperature. The correct selection of the admissible stresses is an exceptionally important problem, since not only the strength of the structure but also its economy and light weight depend on it. We note that almost all methods for the calculation of the admissible stresses at high and low temperatures have a very approximate character, since the material undergoes "fatigue" and "ages" with the passage of time and is subjected to a series of conditions which can only be taken into account with difficulty or not at all and are not included in the calculations. Therefore, the selection of the admissible stresses is predominantly based on empirical or statistical data.[1] The calculation methods that were constructed make it possible to determine both the short-term admissible stresses at a uniform and nonuniform temperature and the admissible stresses during the protracted action of the load at raised temperatures. Recently studies in fatigue problems and imperfect elasticity of materials at normal and high temperatures also attracted a great deal of attention.[2]

1. See, for example, Z. B. Kantorovich (1946), S. D. Ponomarev, *et al.*, (1956, 1959), I. A. Oding (1962), N. I. Bezukhov, *et al.* (1965).

2. See the collection, "High Temperature Strength Problems in Machine Building," (Kiev, 1962, 1963) and "Thermal Strength of Materials and Structural Elements" (Kiev, 1965).

S. V. Serensen and L. A. Kozlov (1958, 1965) applied to the study of the fatigue of metals at raised temperatures a statistical approach which makes it possible to estimate the reliability and the safety factors of the parts operating at a raised temperature. Similar studies for metaloceramic materials were made by G. S. Pisarenko, et al. (1962).

Particular attention should be given to the so-called temperature (heat or thermal) fatigue phenomenon, which essentially consists of the formation and development of micro-cracks as a result of the repeated action of the temperature stresses caused by the cyclic changes in the temperature. In this case failure occurs under conditions which are similar to failure for a small number of cycles at raised temperatures which, however, are not isothermal, as a result of which the thermal structural stresses do have an effect on the fracture. The thermal fatigue problem is especially important in such fields as power engineering (repeated starting of units and varying of their power aircraft construction (repeated kinetic heating), engine construction, etc. when considerable temperature stresses are formed in the structural elements.

D. K. Chernov noted as early at 1919 the basic properties of thermal fatigue. He emphasized that the reason for the formation of a grid of cracks on the walls of the bore in artillery guns and on the surface of rolling mills is the plastic deformation, which changes sign, which is formed during repeated heating and cooling.

Until the 50's the study of the resistance of structural metals to thermal fatigue rupture was made in order to compare qualitatively the behavior of materials at temperatures that varied in cycles. Samples of various shapes and dimensions were used in these studies, in which "to make more rigorous" the testing conditions, cuts in the shapes of holes and recesses were made. The samples were heated in various ways (in electrical resistance furnaces, high frequency flows, combustion product or heated air flows, gas burners, introduction in a melt), and cooling by immersion in a liquid or in air.

For the given thermal regime, the number of cycles before the appearance of a crack, the change in the shape and dimension of the samples and sometimes also the kinetics of the development of the cracks without an analysis of the stressed and the deformed state during the heating and cooling of the samples was determined in the studies that were mentioned. For example, in the study by L. A. Glikman (1937) when the number of cycles before rupture was determined in whole prismatic samples with longitudinal grooves, it was established that the cracks appear faster in samples with cuts.

V. I. Zalesskiy and D. M. Korneyev (1954) studied the thermal disintegration of cyclindrical samples made from different steels which were heated in a lead bath and cooled in running water. It was established that the cracks were formed earlier in the presence of structural conversion in a steel "which was less dense" whose surface was treated less thoroughly.

M. V. Pridantsev and A. R.Krylova (1958) testing sheet samples with holes during heating with a gas burner and cooling in air have shown that the heat resistance is reduced as the thickness of the sheet increases, and M. Ya. L'vovskiy and I. A. Smiyan (1958) developed a method for estimating the resistance of sheet materials to the effect of thermal changes.

It was already mentioned that the results of these studies were mainly used for a comparative evaluation of the thermal fatigue strength of different materials. They cannot be applied to estimate the load-bearing capacity of structural elements and buildings, the stressed and deformed states in the dangerous zones which depend, to a considerable extent, on the dimensions and shapes of the products and also the heat transfer conditions.

The development of thermal fatigue studies in the last 10-15 years is based on a temperature-time analysis of the stressed and deformed state in structural elements and a study of resistance to fracture as it pertains to the corresponding thermal and mechanical processes, which opens up certain possibilities in the solution of strength calculation problems for the parts under a cyclic thermal load.

An important stage in the experimental study of thermal fatigue were the studies by L. F. Coffin and R. P. Wesley (Trans. Amer. Soc. Mech. Engr., Vol. 76, No. 6, 1954, pp. 923-930), which made it possible to determine the stresses and strains relatively simply and reliably by testing a thin-walled tubular sample fixed at the ends and heated in cycles. Subsequently it was proposed (S. V. Serensen and P. I. Kotov, 1959) that tests be made with a rigidity that was varied by way of controlling the elastic loading system, i.e., varying the boundary conditions in the case when the magnitude of the force deformation is smaller than the temperature deformation.

The extension of the possible variants of the relations between the mechanical and temperature cycles (the deformation from the external loads may be larger than the temperature deformation and at the instant when the maximum temperature

is attained, tension rather than compression may occur; the phase shift of the temperature and mechanical cycles, (various combinations of thermal and mechanical loads, etc.) were obtained by N. D. Sobolev and V. I. Yegorov (1962) by synchronizing the additional temperature loading cycle during heating-compression and cooling-tension), by A. V. Strizhalo (1967) by using an installation with a programmed change of the load and temperature, by A. I. Ivanov and B. F. Trakhtenberg (1968) by developing a method for independent mechanical and thermal loading.

A. A. Platonov and N. M. Sklyarov (1962) and A. V. Ratner (1964) proposed that the resistance of the material to thermal fatigue be estimated by testing samples during the one-sided accumulation of plastic deformations in tension halfcycles at the instant when the sample is cooled.

R. A. Dul'nev, V. I. Yegorov, Ye. N. Pirogov and N. D. Sobolev proposed more exact methods for determining the magnitude of the elasto-plastic deformations during the tension-compression testing of the samples (1962 and later studies).

The great variety of variants of the stressed and deformed state under real cyclic temperature loading conditions made the corresponding studies necessary. In the experiments made by V. N. Kuznetsov (1957) in a tubular sample a plane stressed state with changing sign was formed as a result of the radial temperature gradient which changes cyclically. Yu. F. Balandin (1967) tested cyclically heated and cooled tubular samples fixed at the ends which were uniformly loaded by constant internal pressure.

It should be mentioned that the thermal fatigue mechanism is similar in many respects to the fatigue mechanism during mechanical interaction, since in both cases the reasons for the fracture are the same factors: the interaction of multiple variable stresses and the plastic deformations which change sign. Therefore, to determine the thermal fatigue laws, often auxiliary data on the behavior of the material that is studied during an isothermal cyclic load are used (Ya. B. Fridman, 1962). However, differences also exist which do not allow us, in a number of cases, to replace the thermal fatigue tests by mechanical fatigue tests. The point is that due to a change in the temperature during each cycle a constant change in various physical properties of the material occurs (the elasticity modulus, the yield point, etc.) which in turn leads to a change in the resistance of the material to the action of thermal stresses. Thermal fatigue is characterized by a localization of the deformation in zones with the largest

temperature drop even in a homogeneous stress field (thermal concentration) due to the nonuniformity of the temperature field which occurs in the parts. We also note that the resistance to mechanical fatigue at temperatures that are not high and loading frequencies that are not too small depend little on the loading frequency while thermal fatigue is essentially related to the duration of the loading cycle and also to the endurance time of the material in the high temperature part of the cycle.

Multiple cyclic loads often lead first to the rupture of individual grains or the boundaries between them and then to the complete rupture of the sample (thermal fatigue from the thermal structural stresses). The studies by V. A. Likhachev (1958), N. N. Davidenkov and V. A. Likhachev (1960) investigated the dependence of the microstructural stresses on the temperature.

The facts that were mentioned above indicate the necessity of a systematic study of thermal fatigue, in particular of obtaining for the given temperature intervals curves which relate the magnitude of the deformation to the number of cycles before rupture.

Serious theoretical studies of this aspect of the problem were made by L. Coffin (see above) and S. Manson (Mach. Design, Nos. 12-13 and 16-18, 1958). The last author proposed a single fatigue strength curve.

V. N. Kuznetsov (1957) studied experimentally strength problems during thermal fatigue and proposed a relationship between the numbers of cycles before rupture, the intensity of the plastic deformations and the maximum amplitude of the linear plastic deformation.

S. V. Serensen and P. I. Kotov (1960) obtained in their studies deformation curves in heating and cooling halfcycles.

N. S. Mozharovskiy (1966) obtained the relations between the stresses and strains for any thermal loading cycle on the basis of experimental studies of the tension of rods made from refractory hardening materials.

Subsequently S. V. Serensen, Yu. F. Balandin, V. I. Yegorov, P. I. Kotov, N. S. Mozharovskiy, N. D. Sobolev (1960 and in later studies) obtained the empirical relations between the endurance (the number of cycles before fracture) and various parameters of the stressed and deformed state and the thermal cycle (the magnitude of the change in the plastic and total deformation, the stresses, the temperature drop per cycle, etc.).

The results of tests made on a number of refractory steels in the same temperature interval in the uniaxial stressed state and during pure shear enabled N. D. Sobolev and V. I. Yegorov (1963) to propose an energy theory for the change in shape for thermal fatigue. The energy rupture criterion that was set up and the existence of a single cyclic deformation curve made it possible to obtain the relations between the endurance and the change in the intensity of the stresses, the deformations and the energy of the plastic deformation per cycle and to find the relation among them.

N. S. Mozharovskiy (1967) has shown in his study that the basic rupture criterion that can be used for plastic hardening materials under thermal cyclic loads caused by plastic deformations which change sign is the value of the total irreversibly absorbed energy used up in the hardening deformation process, which is determined from the corresponding noniso-thermal deformation curves.

Yu. F. Balandin (1964) and N. S. Mozharovskiy (1967) investigated the effect of an applied static load on the thermal cyclic stress.

The character of the temperature cycle (the temperature level, the duration of the cycle) determine the size of the deformation, the shape of the cyclic deformation curve, the relaxation of the stresses and the resistance to rupture. In this connection the results obtained in the studies by Yu. F. Balandin (1966) and R. A. Dul'nev (1967) who studied the problem of the effect of the time endurance at a maximum temperature of the cycle are of practical significance.

P. I. Kotov (1961) and N. S. Mozharovskiy (1963) have shown in their studies that the characteristics of the refractory materials at different temperature regimes can be represented in the form of a single thermal fatigue curve. Many studies studied the "damage" in the material resulting from a thermal fatigue load. A considerable change (reduction) in the resistance of the material to deformations and fracture during successive single static loads has been observed (Ya. B. Fridman and V. I. Yegorov, 1960).

N. D. Sobolev and Ye. N. Pirogov (1967) studied the laws for the accumulation of the damages during nonstationary regimes, dividing the loading process into two stages, one of which is related to the time until the macrocrack is formed and the second is related to the development of this crack. It was established that for the same rupture probability in the first stage, the transition from the higher load to the

lower level yields a greater damage than predicted by the linear law for the summation of the damages and vice versa; it is smaller when the order of the loading is reversed. The accumulation of the damages in the first stage is described by the linear law and the rate at which the crack develops at a given instant is independent of the loading prehistory. Problems dealing with the summation of the damages were studied by V. M. Filatov (1967) who has shown the applicability of the linear summation with respect to the number of cycles under the conditions of his experiment.

At the present time an important problem is the possibility of comparing different materials according to their thermal stability, taking into account the effect of many changes in their physical and mechanical properties.

G. N. Tert'yachenko (1964) obtained an expression which determined the minimum value of the temperature drop for the heat transfer boundary conditions, at which plastic deformations which are responsible for the thermal fatigue failure are formed on the surface of the cylinder. This temperature drop is a function of the yield point, the Poisson ratio, the modulus of elasticity and a quantity which depends on the Bio criterion ($Bi = hb/\lambda$, where h is the heat transfer coefficient, b is the characteristic dimension of the body and λ is the heat conductivity coefficient).

The study by V. I. Yegorov and N. D. Sobolev (1963) gives for the uniaxial stressed state, on one hand, a relative evaluation of the materials based on their endurance for the same values of the deformations and stresses in a fixed temperature interval, and on the other hand, a comparison is made for the same boundary conditions, when the deformation for a fixed temperature drop is a function of the linear expansion coefficient. One method of estimating the thermal stability of parts is to test them under conditions which model natural conditions.

G. N. Tert'yachenko, R. N. Kuriat and L. V. Kravchuk (1963, 1964) tested along these lines real nozzle plates in gas turbines and modeled the temperature flux and the boundary conditions for the heat transfer on a gas dynamic bench.

Various methods are used to increase the resistance to thermal fatigue strength including those used to increase mechanical strength (improve the quality of the surface and reduce the stress concentration, etc.), as well as specific methods used for equalizing the temperature field (heat conductive coatings, etc.).

We will mention another aspect of the direct effect of the temperature on the strength of materials and polymers. In some cases the parts, the structural elements and the equipment are subjected to the effect of temperature stresses formed in a very short time interval (almost instantaneously) as a result of a rapid change in the temperature. Such a stress which is called a temperature (heat or thermal shock) causes dynamic thermal stresses and leads to the brittle rupture of the material.

The temperature shock is most dangerous for materials in the brittle state. In the plastic state the thermal shock is usually not dangerous, since the stresses cannot exceed much the yield point and they decrease with time.

It is well known that the stressed and deformed state of the body caused by the thermal shock can be determined in a number of cases by solving jointly the heat conductivity and thermal elastic equations.

For superrapid heat processes (explosions, heat systems with large heat fluxes) the correct pattern for the propagation of the thermoelastic stresses is obtained from the solution of dynamic thermoelastic problems, taking into account inertial terms, whereas the fields of the temperature stresses during slower thermal effects are determined with a sufficient degree of accuracy from solving quasistatic thermoelasticity problems.

The analytical solutions of certain dynamic problems in thermoelasticity which determine the character of the propagation of the dynamic thermoelastic stresses have been obtained relatively recently (V. I. Danilovskaya, 1950, 1952, 1960). However, in spite of the importance of dynamic problems dealing with various types of explosive rapid processes, it must be mentioned that the most practical applications in many branches of engineering are the solutions of stationary thermoelasticity problems with nonstationary temperature fields. In this case, it is assumed that the stressed state at each instant of time corresponds exactly to the temperature drop which occurs at this instant and the inertial terms are ignored. In practice, even these theoretical results are considerably simplified, and in many cases the resistance of the materials during thermal shock is determined directly from the experiment.

The main methods for the experimental study of the resistance of materials to thermal shock consist of the following. A sharp change in the temperature field is obtained by placing the sample in a liquid bath and by blowing a gas stream or a liquid over the sample. The difference between the temperatures of the medium and the sample is selected on the basis of the fracture conditions.

Samples can also be heated rapidly with the aid of a low-inertial heater or with the aid of internal heat released in the material of the sample. The heat release can be obtained by nuclear radiation or by passing electric current or high frequency current through the sample.

G. N. Tret'yachenko and L. V. Kravchuk (1964) used a gas dynamic bench to generate the thermal shock, where annular samples or parts of different shapes were subjected to heating in combustion products and cooling in an air stream. As a result of this, the values of the rupturing temperature differences (the temperature of the gas-initial temperature of the sample) were found for a number of high temperature metaloceramic materials.

A method for the study of the resistance of brittle materials to thermal shock with a variable heat transfer coefficient was proposed in the study by N. I. Tikhonov, et al., (1963). According to this method, the samples are heated in a continuous furnace which is then cooled and the sample is cooled by thermal radiation. The thermal stresses are determined by calculations during the experiments, using the temperature on the surface of the sample that is measured during the experiment.

The study by V. I. Dauknis, et al. (1967) describes an installation for the study of thermal shock in which a moving collection of samples is used.

The proposals for the use of beam and electron heating in thermal shock installations that appeared recently in the literature also merit attention.

One useful appliation of thermal fracture is flame boring or thermal boring, in which a high temperature gas jet is used to destroy the rock. A theoretical model of the flame boring phenomenon was proposed by G. P. Cherepanov (1966).

The effect of the temperature on the strength of polymer materials will be discussed below. Here we will mention the studies dealing with the strength of rubbers at raised temperatures. In the study of problems dealing with the effect of temperature on the rupture rate of hollow rubbers, G. M. Bartenev (1958-1964) has shown that the rate at which the cracks and cuts are formed and grow increases as the temperature increases. The same studies investigated the effect of the temperature on the time curve for the strength of rubber in the interval from 20 to 140°C. The complex effect of the temperature on the endurance was determined and the range of

practical safe loads was found. It was shown that the temperature-time curves for rubber differ from those for solid polymers and that at high temperatures (90-140°C) the time curves for the strength deviate from a linear curve in the region of large endurances (in the coordinates log τ - log σ) which is apparently connected with the change of the structure in the surface layer of the samples under the action of the destructive processes. In addition, unlike in solids (G. M. Bartenev, 1964), the stress has an insignificant effect on the activation energy, which for rubber, has a sufficiently low value. This is apparently connected with the fact that the kinetics of the destruction of rubber are mainly determined by the intermolecular bonds.

Recently in connection with the extensive use of low temperatures (liquid oxygen, hydrogen, helium) it became necessary to investigate the mechanical and other properties of metallic materials under low temperature conditions in many branches of modern engineering. Many studies were published dealing with the behavior of structural metallic and nonmetallic materials at temperatures up to - 253°C (20°K). These data are useful in the selection of materials used in building various machines which are used, for example, as the working body or the working medium or reduced gases. Tests that are carried out at low and very low temperatures made it possible to study the transition process from the viscous to brittle rupture and to determine the limiting brittle failure strength.

The cold-brittleness problem is inseparably connected with the names of F. F. Vitman, N. N. Davidenkov, A. F. Joffe, Ye. M. Shevandin, N. P. Shchapov, M. V. Yakutovich, and others.

Studies in the strength of soldered, welded and rivetted joints, as well as materials with stress concentrations, were very important and made it possible to foresee and to avert sudden disasters that could possibly occur in low temperature conditions.

Studies connected with evaluations of the cold brittleness were begun by N. N. Davidenkov (1930-1938), who defined the critical (transient) brittleness temperature and proposed the use of curves relating impact viscosity and the temperature for the indirect determination of the resistance to brittle rupture. N. N. Davidenkov (1938) noted that that part of the work which is used up after the maximum load is reached is most sensitive to the testing temperature (during the bending of a cut sample) and that this characteristic is reduced when the temperature is lowered.

Subsequently, Ye. M. Shevandin (1953-1965) studied low alloyed structural steels in the cold-brittleness region and Ya. M. Potak (1955) analyzed the brittle fracture of structures made from alloyed structural steel. He also noted the tendency to brittle fracture of parts containing large ferrite grains.

The series of studies made by T. A. Vladimirskiy (1953-1958), led to the construction of three-dimensional curves (impact viscosity-sharpness of the cut-temperature) for several structural steels. He showed that when the sharpness of the cut is changed the materials can change in places on the basis of an estimate of their critical temperature.

It was noted in the study by G. V. Uzhik and Yu. Ya. Voloshenko-Klimovitskiy (1962) that brittle rupture is a form of violation of strength that can be overcome at low temperatures. He determined the laws for the change of the yield point in metals at high loading rates and at low temperatures. He also pointed out in the same study the importance of these parameters in the evaluation of the danger of brittle failure.

The data on the effect of low temperatures on the mechanical properties of metallic alloys were systematized in the study by P. F. Koshelev and S. Ye. Belyaev (1967).

In conclusion we draw attention to the fact that the majority of investigators noted an increasing sensitivity of the material to stress concentrations and a drop in the strength of cut samples with a drop in the temperature (Ya. B. Fridman, 1952, Ya. M. Potak, 1955, G. V. Uzhik, 1957).

§8. Long-term Strength Problems

One of the simplest forms of loading is static loading up to a certain value taken on by the stress tensor with subsequent maintenance of the material at the load values that were attained. In this case the deformations increase (decrease) and after the passage of a certain time the material ruptures. However, fracture does not occur until the stress tensor exceeds a certain value which is called the limiting long-term strength of the material.

Systematic studies of various laws related to the long-term strength of metals and polymers led to the development of several trends in this branch of mechanics of rupture (A. P. Aleksandrov, G. M. Bartenev, S. N. Zhurkov, V. A. Kargin, P. P. Kobeko, B. P. Konstantinov, Yu. S. Lazurkin, A. K. Malmeyster, A. N. Orlov, Yu. N. Rabotnov, and others).

Studies of the dependence of the strength on time began when the role played by time during the loading process in silicate glasses was clarified. A. A. Griffith has already shown in 1920 that the newly prepared glass rods have much greater strength than those that were left in the air for some time. A similar phenomenon was noted by A. P. Aleksandrov and S. N. Zhurkov (1933) during their studies of the strength of quartz fibers, which served as the beginning of the theoretical and experimental studies in this field made by S. N. Zhurkov and his collaborators (1953-1961). A study, mainly of the uniaxial stressing of materials with different mechanical properties has shown that for metals, plastics, and polymer fibers, the relation between the stresses and the endurance can be expressed by an exponential relation

$$\tau = Ae^{-\alpha\sigma}, \tag{8.1}$$

where A and α are constants depending on the temperature. Relation (8.1) is valid in a sufficiently wide range of temperature changes and the "endurance lines" at different temperatures form (in log τ - σ coordinates) a bundle emanating from one point which evidently corresponds to the critical stress σ_k. At relatively low temperatures the time dependence is weak, and if, for example, the tension occurs at a fast rate, the fracture has a character which is similar to the critical rupture. In this case time has a small effect on the magnitude of the rupturing stress and rupture does not occur for all values $\sigma < \sigma_k$, no matter how long the material is in the stressed state. Accordingly, the concept of the "strength limit" or the engineering concept "time resistance to brittle fracture" is introduced. The dependence on time does not manifest itself for all practical purposes at all for plastics at temperatures - 200°C and below, while for metals and inorganic glasses with a high melting temperature, the usual temperatures are already low. Except the case of relatively low temperatures, the problem of the strength is solved as a function of the time until the rupture during which the sample is in the stressed state.

In the early 30's studies connected with the investigation of the mechanical properties of amorphous and high molecular solids began to develop intensely. Progress along these lines is connected with the names of A. P. Aleksandrov, P. P. Kobeko, M. O. Kornfeld, Ye. V. Kuvshinskiy, and others. Approximately at the same time the concepts on the leading role of thermal movement in the determination of the mechanical properties of solids were about to be developed. This approach was mainly based on the ideas of Ya. I. Frenkel of the thermal

fluctuating mechanism for the motion of particles which was universal for all liquids and solids. According to this concept a change in the configuration of the atoms in a solid occurs at the thermal fluctuation instant which increases for a certain time the local energy, and the external stress leads only to the orientation of such changes and macroscopic plastic deformation and rupture processes.

An extensive series of studies of the effect of the temperature on the deformation of polymers was made mainly in the 40's (A. P. Aleksandrov, P. P. Kobeko, Ye. V. Kuvshinskiy, Yu. S. Lazurkin, N. I. Shishkin, and others) and metals (N. N. Davidenkov, F. F. Vitman, N. A. Zlatin, V. A. Stepanov, L. M. Shestopalov, and others).

The kinetic concept in which the determining factor during a plastic deformation are the thermal fluctuations was also extended to the rupture of all solids. S. N. Zhurkov obtained on the basis of an empirical study of the endurance of solids under a load the following temperature-time relation:

$$\tau = \tau_0 \exp\left(\frac{U_0 - \gamma_* \sigma}{kT}\right). \tag{8.2}$$

Here τ_0 is a constant which is close to the period of the thermal fluctuations (for solids $\tau_0 \approx 10^{-12} - 10^{-13}$ sec), $k = 1.37 \times 10^{-16}$ is the Boltzman constant, $U = U_0 - \gamma_* \sigma$ is the activation energy, U_0 is the activation energy in the absence of a stress which is close to the sublimation energy for metals and the energy of the chemical bonds for polymers and γ_* is a correction factor which depends on the nature and structure of the material.

Relation (8.2) turned out to be useful for a sufficiently large class of materials, including polymers, in a large range of testing temperatures and times. We note that the largest scatter in the experimental data is observed at very long and very short unloading times, and the smallest scatter at medium times for the long-term strength, for which relation (8.2) is most justified. The tests made in a high vacuum (G. M. Bartenev, 1955) have shown that the external medium does not have the most important direct effect on the time curve with the exception of certain special cases of strong media which are active on the surface (see V. I. Lichtman, Ye. D. Shchukin and P. A. Rebinder, 1962).

The problem of obtaining the long-term strength characteristics of various materials in the necessary working temperature range is connected with a very large number of experimental studies which often cannot be made for materials intended for long-term service. Therefore, the many attempts to construct a long-term strength theory based on the extrapolation of the results of short-term tests where the long-term tests at low temperatures are replaced by tests of short duration at high temperature are natural. The physical models which were constructed taking into account these experiments are based on idealized materials, and probably absolutely universal formulas do not exist at all, since various materials behave generally differently during the tests. Dislocations and plastic deformations play the fundamental role in the fracture of crystalline bodies and various types of defects and microcracks for brittle amorphous bodies.

Ya. I. Frenkel predicted already in the 20's on the basis of a study of thermal fluctuations the occurrence of point defects-vacancies in bodies in thermodynamic equilibrium. At the present time the theory of vacancies is one of the basic trends in the theory of solids. In particular, the condensation of the vacancies which are no longer in equilibrium as a result of rapid cooling during plastic deformation may lead to the formation of pores, whose growth and combination cause the rupture. The theory of coagulation of the vacancies was developed in the studies by V. I. Vladimirov (1960), V. I. Vladimirov and Sh. Kh. Khannanov (1967).

It was already mentioned above that the plastic deformation always precedes rupture and subsequently often accompanies it, so that the study of the dislocation properties is important in this connection. The first mathematical model of a moving dislocation was constructed by Ya. I. Frenkel and T. A. Kontorova (1938). Subsequently studies on the kinetics of dislocation structures and the fine structure of the dislocation center were continued by A. N. Orlov, et al. (1950 and in later studies).

B. Ya. Pines (1955, 1959), L. E. Gurevich and V. I. Vladimirov (1960), A. N. Orlov (1961), and others proposed time dependent theories for crystalline bodies.

These theories are based on assumptions of one kind or another about the character of the origin of the microcracks which coagulate in the growth process into one main crack leading to rupture. The studies by B. Ya. Pines (1955, 1959) proposed the idea of a selfdiffusion mechanism for the growth of cracks which led the author to a relation which is similar to (8.2).

L. E. Gurevich and V. I. Vladimirov (1960) paid attention to the role played by the plastic deformation in the origin and development of the cracks.

The problem of the connection between the formation of cracks and creep and long-term strength has not been studied sufficiently so far. However, the empirically established fact (in particular, for pure metals and single-phase alloys) that the product of the steady-state creep rate $\dot{\varepsilon} = d\varepsilon/dt$ and the time until rupture is constant

$$\dot{\varepsilon}\tau = \lambda \tag{8.3}$$

is well known (S. N. Zhurkov and T. P. Sanfirova, 1958, B. Ya. Pines and A. F. Sirenko, 1960). (Here λ is independent of the stresses and the temperatures.) This fact[1] made it possible to use the same relations for the extrapolation of data on creep and long-term strength.

A. N. Orlov (1961) has shown that as a result of the plastic deformation, creep develops in parallel and prepares the material for rupture. The author obtained this result on the basis of the experimental relation (8.3) that was mentioned above, which relates the creep rate to the time before rupture. A. N. Orlov proposed a rupture model during the coagulation of a large number of microcracks formed on different slippage lines. In the following years these studies were continued by A. N. Orlov, V. I. Vladimirov and Sh. Kh. Khannanov, who have shown that when the discreteness of the dislocation accumulations and their displacements are taken into account during the development of the crack, the possible origin of the microcrack under local stresses which are considerably smaller than the theoretical strength can be explained on the basis of the thermal fluctuations.

A number of general theories on dependence of the strength on time for brittle bodies and brittle solid polymers which are similar were proposed both in our country (G. M. Bartenev, 1955), as well as abroad (P. Gibbs and I. B. Cutler, J. Amer. Ceram. Soc., Vol. 34, No., 1951, pp. 200-206, D. A. Stuart and O. L. Anderson, ibid., Vol. 12, No. 36, 1953, pp. 416-424). These theories are based on the kinetics for

1. See the monograph by Yu. N. Rabotnov (1966).

the growth of the crack which is considered as the successive rupture of the bonds at the apex of the crack under the action of the stresses and thermal fluctuations of the atoms or molecules. The studies of Yu. N. Rabotnov, based on the state equations, taking into account cracking are important for the kinetics of the long-term static rupture (see also the studies by S. T. Mileyko, V. L. Mirkin, I. A. Oding, and others).

Recently several studies appeared in which an attempt was made to take into account the time effects in the macroscopic theory of cracks. These are taken into account in various ways.

In the model of an ideal elastic body with a constant energy surface, the crack cannot develop under a fixed load. Therefore, the observed growth of the crack during a constant load must be related to the elastic imperfections, in particular, to the yield phenomenon in solids.

L. M. Kachanov (1961, 1963) studied the case of linear creep corresponding to a Maxwell medium scheme and established a linear relationship between the critical coefficient of the intensity of the stresses and the time, and introduced a new material constant called the coefficient of damage. In this formulation he studied the problem of the development of a crack under the action of concentrated forces in an infinite plane and in a strip of finite width. L. M. Kachanov noted that the qualitative pattern is generally preserved also in other linear media with the yield property (for example, a medium obeying the Boltzman integral relations).

G. I. Barenblatt, V. M. Yentov and R. L. Salganik (1966, 1967), have shown that the constant value of the critical coefficient of the intensity of the stresses in the theory of equilibrium cracks becomes a function of the rate at which the crack propagates when the fracture kinematics are taken into account. It is assumed that all effects for sufficiently large stresses (visco-elasticity, microstresses, etc.) are concentrated in a small region at the end, and as before the material outside the crack is assumed to be elastic. The form of the functional relationship for this critical coefficient can be determined for a particular concrete model of the relations from the system of basic equations set up by the authors. As an example, the authors consider the case of a Griffith crack which was nearly an equilibrium crack where the relation between the critical coefficient of the intensity of the stresses and the rate of movement at the end of the crack were selected for the two cases of a pure fluctuation and pure rheological mechanism. When they investigated

the fracture conditions and problems related to the long-term strength, the authors have shown that a generalization of the well-known static fracture condition is the possibility of determining the fracture in the case under consideration as the nonexistence of the system of differential equations defining the length of the crack (for the given propagation path). It was also shown in the studies that the critical coefficient of the intensity of the stresses depends on the character of the loading and that a large loading rate interval must exist in which the critical coefficient corresponding to the rupture instant is constant for all practical purposes.

The pure fluctuation rupture which occurs under relatively small loads was subjected to a special study. In this case the time effects occur only as a result of the random ruptures of the bonds under the action of the thermal fluctuations and the movement of the crack is so slow that the relaxation processes can be ignored. Assuming that the initial cracks have a dimension on the order of the end region of a crack, the authors obtained the value of the characteristic time in the endurance formula (8.2).[1]

It was already mentioned that the most important strength characteristic of the material is its endurance during arbitrary loading regimes. Usually this characteristic is determined directly from the experiment. However, it can also be calculated theoretically in the case, when, for example, the time relation for the strength under constant elongating loads is known. If we assume as S. N. Zhurkov and B. N. Narzullayev (1953) did that the fracture is an irreversible process and the rate of growth of the crack depends only on the stress σ, the G. Bailey condition is satisfied (Glass Ind., Vol. 20, No. 1-4, 1939, Ceram. Abs. Vol. 19, No., 1940, p. 89)

1. In 1968 G. P. Cherepanov obtained the following equation for the rate of growth of the crack dl/dt as a function of the coefficient of the intensity of the stresses K:

$$\frac{dl}{dt} = A \exp \frac{\alpha K}{T}.$$

on the basis of the G. Neuber concept and the experimental data on long-term strength. Here α and A are constants and T is the temperature.

$$\int_0^{\tau'} \frac{dt}{\tau(\sigma)} = 1. \qquad (8.4)$$

Here τ' is the endurance of the sample for any given testing regime, $\tau(\sigma)$ is the endurance during a constant elongating stress which is found from the known time curve for the strength.

To estimate the fracture time during a load which varies over time, the Bailey rule for the integration of the damages is used in the form (8.4) where now $\sigma = \sigma(t)$, and $\tau(\sigma) = \tau(\sigma(t)$, T are found, for example from relation (8.2). Many experimental tests agree well with relation (8.4) especially when the rates at which the loads vary are small.

The problem of the effect of time on the deformation and strength of polymers is extremely important. It is known that depending on the structure, temperature, the thermal prehistory, polymers may either be in the structural-liquid (viscous flow and highly elastic) or in solid (crystalline and glass-like) states. The strength of a polymer depends not only on its structure and deformation properties but also on its physical state, which is intimately related to the deformation time, temperature, etc.

P. P. Kobenko, Ye. V. Kuvshinskiy and G. I. Gurevich (1937) were the first to propose the relaxation theory for the deformation of polymers and V. A. Kargin and G. L. Slonimskiy (1941, 1948, 1960), starting with the general Boltzman-Volterra theory and concepts on the molecular structure of polymers developed the mathematical theory of three deformed states (glass-like, highly elastic and viscous flow) which occur under small stresses. Under large stresses a series of interesting features occur, for example, the oriented structure during the elongation of the solid polymers, which has an effect on the strength and fracture and considerably hardens the material.

V. A. Kargin and T. I. Sogolova (1953, 1964) studied the effect of the orientation, structure and relaxation time on the strength of polymers (a temperature-time relation for the relaxation time was proposed and studied by A. P. Aleksandrov, G. I. Gurevich, et al., 1945).

It was already mentioned above that according to S. N. Zhurkov the rupture process is a process in which the bonds that were ruptured by the thermal fluctuation accumulate. S. N. Zhurkov, E. Ye. Tomashevskiy, et al. (1964) observed directly such an increase in the number of ruptured bonds using the barometric resonance method. S. N. Zhurkov, A. I. Slutsker, V. I. Betekhtin, et al. (1962-1967) determined in their studies the relation between the dislocation structure of the material and the structural-sensitive coefficient.[1] The new concepts about the kinetic nature of the fracture were extended to the case of the complex stressed state in the study by V. A. Stepanov, et al. (1964).

The initial development of microcracks as a result of the thermal fluctuation ruptures of individual bonds in oriented polymers was studied by A. I. Gubanov and A. D. Chevychelov (1963 and later publications). They have shown that as a result of the redistribution of the stresses along the bonds, depending on the length of the polymer chains in polymers the ruptures of the bonds begin immediately after the load is applied. This leads to the formation of microcracks with dimensions ~100-600 Å (on the order of the dimension of fibrilla).

Thus, according to modern concepts, the microcracks are formed both in crystals and in polymers in the earliest stages of the plastic deformations, and their concentration increases over time, they interact among themselves, conglomerate, appear as macroscopic cracks and finally lead to rupture.

Experimental studies of the dependence of the strength on time for organic and inorganic glasses were made both in our country (G. M. Bartenev, 1950, 1951, 1960, B. Ya. Pines and A. F. Sirenko, 1960), as well as abroad (A. Holland and W. Turner, J. Soc. Glass Technol., Vol. 24:101, 1940, pp. 73-93, and Vol. 32:144, 1948, pp. 5-20), and led to the derivation of a relatively large number of empirical formulas, among which the power approximation

1. The increase in the number of microcracks with dimensions ~100-600 Å in oriented polymers under a load was observed in 1969 by V. S. Kuksenko, A. I. Slutsker and S. N. Zhurkov. It was shown that a definite concentration of microcracks $\sim 10^{13}-10^{17}$ 1/cm^3 corresponds to the microscopic rupture of the sample (depending on the type of polymer and mean dimension of the cracks).

$$\tau = B\sigma^{-b}. \quad (8.5)$$

turned out to be most useful. Here, as before, τ and σ are the values of the endurances and the tensile stresses and B and b are constants. Several theoretical schemes for the dependence of the strength on time were proposed on the basis of these data (T. A. Kontorova, 1946, G. M. Bartenev, 1960).

Studies of the strength and rupture mechanism of various polymers (crystalline, amorphous solid polymers, linear and three-dimensional structured polymers) made it possible to clarify the dependence of the reduction in the strength of polymers on time (static fatigue).

A. V. Tobol'skiy (1960) proposed the method of generalized coordinates for the construction of universal endurance curves for polymers. Usually it is extremely difficult to obtain the curves for various properties of polymers in a large time interval. To obtain such relations, the curves for various temperature values are shifted on the graph to obtain the generalized curve at the chosen temperature. This method, which is used extensively, which was formulated by A. V. Tobol'skiy is based on the use of the temperature-time superposition, in particular the temperature-frequency relation in the deformation of polymers which was first detected by A. P. Aleksandrov and Yu. S. Lazurkin in 1939. This method can also be used to construct endurance curves under conditions which cannot be studied directly in the experiment.

A. P. Aleksandrov, G. M. Bartenev, V. A. Kargin, A. I. Kitaygorodskiy, Yu. S. Lazurkin, A. K. Malmeyster, G. L. Slonimskiy, and others studied extensively the effect of time on the strength of crystalline polymers and amorphous solid polymers. Under low stresses, the crystalline polymers, plastics at the usual, and rubbers at low temperatures, behave like ordinary solids. However, after the stress reaches a certain value, a "neck" is formed at the weakest point, into which the entire sample passes with the passage of time, after which elongation occurs again until total rupture. We note that in spite of the similarity, the mechanism for the formation of a "neck" in a crystalline polymer and an amorphous solid polymer is different. The mechanism for the rupture of amorphous polymers in a glass-like state was studied by G. M. Bartenev (1960, 1964, 1966).

In the analysis of rupture of any materials, it is important to take into account the time effects. This problem is especially important in the study of polymer materials which are characterized by a pronounced dependence of the rupture on the external conditions and the presence of relaxation processes.

G. N. Savin and A. A. Kaminskiy (1967) studied the growth of cracks under solid polymer fracture conditions (polymer glasses) at a fixed temperature in the case of a constant protracted external load. Having considered the development of a crack in an elasto-plastic material, the structure of whose contour takes into account the specific structural features of the crack in polymer materials (the opposite edges of the crack in the end region on a sector of finite length are connected by thin fibers), the authors did not require that the condition for the smallness of the end region be satisfied like in their previous studies. According to this scheme in some time interval $0 \leq t \leq t_*$ the crack expands but does not elongate and from the instant t_* the entire crack begins to grow.

According to the concepts of G. M. Bartenev (1960), the action of vibrating loads creates an inhomogeneous distribution of the temperature in the sample which leads to the activation of the fracture process in the region where it is localized. The effect of vibrational heating on the propagation of cracks in polymers was investigated in the study by G. I. Barenblatt, V. M. Yentov and R. L. Salganik (1967). These authors proposed that the fracture time is determined by the development of the main crack and that the fracture has a fluctuating character. The authors did not take into account the direct force action of the vibration loads and restricted themselves only to taking into account the heating induced by them (the distribution of the stresses and deformations was calculated on the basis of the equations of elasticity theory). This approach is dictated by the fact that in experiments with cyclic loading of polymers at the surrounding medium temperature, the fracture time that is observed in reality is smaller than that calculated from the Bailey condition (8.4), which is explained either by the effect of the relaxation processes or by the heating of the material during the cyclic deformation as a result of the mechanical losses (V. R. Regel' and A. M. Leksovskiy, 1965).

G. P. Cherepanov (1967) studied a quasistatic isothermal process for the propagation of cracks in an isotropic homogeneous viscoelastic body. He derived for the general case a non-linear integro-differential equation which he used to determine the time and along with it the law for the propagation of the crack over time.[1]

1. In 1968 A. A. Kaminskiy used in the study of the propagation of cracks in elasto-plastic media the δ_K-theory of M. Ya. Leonov-V. V. Panasyuk. He obtained the solution of the problem for a crack which weaked a thin elastic plate when concentrated forces of equal magnitude were applied to the edges of the cut and using the Volterra principle, obtained the equation of motion for the ends of the fracturing crack, replacing the Young modulus by the appropriate time operator. A. A. Kaminskiy investigated the special cases of exponential and rational-exponential heredity kernels for a Maxwell body. From the last two examples it follows that during transient creep, when the effect of the creep is damped, the crack grows at a damped rate and ceases to grow altogether after some time. At the same time in the case of stationary creep the growth of the crack is not damped and takes place at a constant rate. These conclusions agree with the results of L. M. Kachanov (1961, 1963) and G. P. Cherepanov (1967).

   V. M. Yentov and R. L. Salganik (1968) studied, taking into account the distribution of the stresses in a visco-elastic body with a propagating crack, the problem of the rupture of a beam made from a visco-elastic material with a crack to which symmetric forces were applied in a material with "memory." Using the relation that was obtained, which relates the length of the crack $l(t)$ to the applied load $P(t)$, the work done to form the new surface was determined, which was calculated in a similar way as that used by I. V. Obreimov (1930) in the case of the splitting of an elastic beam. The authors also studied the distribution of the stresses and deformations near the end of a semiinfinite crack under an arbitrary (symmetric) load in a Kelvin-Foigt material.

We note that the term "viscoelasticity" includes a large class of physical processes, for example, such as relaxation, caused by physical-mechanical, thermoelastic, electrical, mechanical, and other phenomena. It is known that a deep relation exists between the theories of elasticity and viscoelasticity and that the equations of linear elasticity theory (with linear boundary conditions) can also be extended to the viscoelastic case by substituting instead of the elastic constant the operators which depend on time (the Volterra principle).

Along with the energy approach, which has the advantage of simplicity and generality, there is a tendency to set up fracture models in which the stressed bonds in the body are ruptured successively. The first model for such brittle rupture was proposed by L. Prandtl (Z. angew. Math. und Mech., Vol. 13:2, 1933, pp. 129-133) who considered two elastic bodies (beams) secured along their entire length containing a crack, when the transverse elastic bonds undergo brittle rupture after a certain elongation is obtained.[1]

Recently studies of the endurance of rubbers have become more and more important. According to S. N. Zhurkov and B. N. Narzullayev (1953), equation (8.2) can be applied not only to solids, but also to all rubbers except those that are being crystallized. At the same time the experimental studies made by G. M. Bartenev and Yu. S. Zuyev (1964) lead to relation (8.5) for a particular value of the long-term strength of rubbers. Now in relation (8.5) B is a constant which depends on the thickness of the sample and the temperature, b is a constant which characterizes the slope of the endurance curves which depends on the stiffness of the rubber ($3 \leq b \leq 12$ for real rubbers). The largest deviation from equation (8.2) (when it is assumed that the constants do not depend on the

1. V. M. Yentov and R. L. Salganik studied in 1968 within the frame of reference of this model a semiinfinite crack in an infinite body in which the bonds were assumed to be ideally brittle. They also examined the relation between the microscopic and macroscopic approach in the theory of fracture. In the analysis of the kinetics of fracture in the pure fractured case, unlike in their previous studies, the authors did not make any simplifying assumptions about the form of the region at the end of the crack. The problem of the stationary propagation of the crack at a rate nearly equal to the velocity of the Rayleigh waves was also studied in the same study.

stress and the time for such materials as rubber, silk and glass, certain plastics and ebonite) is explained by the change in the structure in the polymer during the deformation process. The time curve for the strength of rubber is intimately related to the type of latex and the degree of its transverse cross-linking as well as the effects of the temperature. For certain latex-like polymers, the following temperature-time relation for the strength was proposed together with (8.2)

$$\tau = C\sigma^{-b} \exp\left(\frac{U}{kT}\right). \tag{8.6}$$

Here b and C are constants which depend on the type of latex and vulcanizate.

Using relations (8.2), (8.4), (8.5) and (8.6) the endurance of plastics and rubbers under cyclic loads were calculated (S. N. Zhurkov and E. Ye. Tomashevskiy, 1955, B. I. Panshin, G. M. Bartenev, et al., 1960, V. P. Regel' and A. M. Leksovskiy, 1962).

Problems dealing with the effect of aggressive media on the endurance of plastics and rubbers are presented in the following section. A detailed study of these problems and a presentation of methods for increasing the endurance of rubbers in aggressive media, are available in the study by G. M. Bartenev and Yu. S. Zuyev (1964).

We will dwell briefly on another aspect related to the study of artificial stones, in particular concrete. It is known that concrete hardens after it is prepared and that its elastic, nonelastic and strength properties change with the passage of time. Various rheological equations, both in differential and integral form in which the rheological coefficients are functions of time are used to describe the deformation process in concrete. In particular, we mention the studies by N. Kh. Arutyunyan, A. A. Gvozdev, A. K. Malmeyster, Yu. N. Rabotnov and A. R. Rzhanitsyn along these lines.

Concrete is a set of crystallizing and coagulating structures which have an effect on the strength characteristics of the concrete. However, in the study of problems related to long-term strength the crystallization structure plays a determining role. The property of concrete to rupture with the passage of time under smaller loads than the value of the short-term loads has been known for a long time (G. R. Schenk, J. Amer. Concrete Inst., Vol. 27, 1935 p. 2). A. K. Malmeyster

and his collaborators (1957) proposed methods for determining the rheological coefficients and made detailed studies of the long-term strength of systems which exhibited twinning and which model qualitatively well the rupture phenomena in real materials during rupture and shear. An experimental verification of the results that were obtained (A. M. Skudra, 1956, Ye. K. Shkerbelis, 1957) confirmed these results well. The method that was developed for calculating the long-term strength of concrete during stressing makes it possible to foresee the development of cracks in reinforced concrete over time and to make a judgment about the redistribution of the stresses occurring in concrete during its fracture with the passage of time.

Studies of the long-term strength under complex stressed state conditions are extremely important. Depending on the conditions (temperature, stress, material), the fracture occurs during large or small deformations, i.e., it has a viscous or brittle character. Thus, we can speak about the viscous fracture time and the brittle fracture time. The calculation of the viscous fracture time is a problem in the theory of creep, in which experiments are useful rather in the verification of creep laws of one kind or another than for obtaining fracture criteria (Yu. N. Rabotnov, 1966).

In many cases the brittle rupture region is more important in the evaluation of the endurance of structures and buildings. Most frequently the limiting deformation is not very large and the service life is limited by this limiting deformation, not by the viscous fracture, in the region of large deformations. Modern materials, for example, those in turbines (are characterized by the brittle character of the fracture at a relatively small deformation at the fracture instant. With regard to the brittle fracture criterion, it is logical to assume that the rate at which the crack develops depends on the magnitude of the normal stress in the planes in which the crack is formed. The tests made by V. P. Stobyrev (1958, 1959) on EI-437B alloys made from rod materials and from pressed intermediate materials for a gas turbine disc have shown that the estimate based on the largest normal stress is more accurate. It was proposed that the quantity $\sigma_Э = 1/2(\sigma_K + \sigma_0)$ be used as the equivalent stress, where σ_K is the maximal stress during brittle rupture and σ_0 is the localized rupture stress. In the $\sigma_Э$ -log τ coordinates (τ is the time before rupture), the experimental data were near the standard curve. A treatment of the experimental data by Sh. N. Kats (1955, 1957) for tubular samples from carbon and austenitic

steel and the data of B. V. Zver'kov (1958) for the EI-496 alloy have shown that the best results are obtained when the above expression is taken as the equivalence stress. The studies by V. P. Sdobyrev and also by I. Ye. Kurov and V. A. Stepanov (1962) and I. I. Trunin (1963) have shown that the endurance values of the metals during torsion are determined, as before, in accordance with relation (8.2), which, however, is slightly smaller than the value of the endurance during tension.

The experimental studies of the long-term strength in the complex stressed state that were made make it possible to determine the time until rupture of products of various shapes under complex and inhomogeneous stressed state conditions. The usual approach is to find the magnitude of their largest normal stress on the basis of some theory of creep, which is then compared with the long-term strength curve found during the experiment. The time until rupture is determined from the long-term strength curve. This method is clearly arbitrary, since it does not take into account at all the formation of cracks. The calculations based on aging theory take this only partially into account.

L. M. Kachanov (1958, 1960) proposed a scheme for determining the endurance in which the general creep equations are used, and in which the cracks are formed only on the surfaces which are perpendicular to the largest stress σ_1, where the equation for the kinetics of the cracking have the form

$$\omega = \varphi(\sigma_1, \omega). \tag{8.7}$$

For fixed external loads while the distribution of the stresses remains unchanged, ω increases according to (8.7), assuming the value $\omega = 1$ over time, which corresponds to the boundary of the rupture front separating the zone which has the capacity to resist ($\omega < 1$) from the zone where rupture already occurred.

The possibility of constructing more general theories of long-term strength with the aid of various creep theories was pointed out by Yu. N. Rabotnov (1959, 1966).[1]

1. See the surveys by N. Kh. Arytyunyan and Yu. N. Rabotnov, in this volume (pp. 175-227).

§9. Effect of the External Medium on the Rupture of Solids

The effect of the external medium on the mechanical properties of solids, in particular metals, has been known for a long time. In the beginning studies of this aspect of the problem were made predominantly from the standpoint of the chemical (corrosive) action of the medium (the change in the mechanical properties of metals during electrochemical corrosion or etching).

Continuing the studies begun by A. F. Yoffe and his collaborators (1924) on the elastic properties and strength of crystals (rock-salt) in various media, P. A. Florenskiy, et al. (1932) have shown that the technical strength varies as the medium changes (studies of the strength of mica in air, oil and a number of organic liquids). S. N. Zhurkov (1932) derived the condition for obtaining samples of higher strength from glasses of various types whose surfaces were etched with hydrochloric acid. At the same time they studied the effect of the medium on the strength of quartz.

However, the effect of the surrounding medium was observed not only during its chemical action. It was shown that the adsorption of the particles active on the surface from the surrounding medium facilitates the deformation and fracture of the solid, often to a much greater degree than during the chemical conversions.

The physical-chemical effect of the external medium on the deformation and fracture processes is based on the effect of the reduced strength resulting from adsorption. The initial effect of the adsorption is that the particles are active on the surface and facilitate the beginning of plastic displacements and the development of various defects under smaller stresses. Thw work used up to form such "defective" surfaces is reduced when the free surface energy on the boundary of the solid with the surrounding medium is reduced in comparison with its values observed in vacuum. Hence, the presence of the medium active on the surface leads to the result that the interaction with the adsorption-active molecules (or atoms) facilitates the reconstruction and rupture of the bonds between the atoms in the given material. The effect of the adsorption facilitating the deformation or the adsorption reduction in the strength is sometimes called the P. A. Rebinder effect.

As a result of the studies made in this field which belongs to the boundary between molecular physics, the physics of solids and physical and colloidal chemistry, it was possible

to determine several new phenomena caused by the absorption interaction of deformed solids with the surrounding medium. Among these new phenomena we should first mention such phenomena as structural changes in the deformed materials, a reduced yield point under the effect of adsorption, an increased creep rate in the metals, and the electrocapillary effect enhancing the deformation of metals and a reduced fatigue strength.

The studies and use of the strength reduced as a result of adsorption have led at the present time to a new independent branch of science and engineering. The study of the negative effect of particles acting on the surface on the strength of materials led, for example, to a new method of obtaining stronger materials which was proposed recently, which is based on the use of active adsorption particles.

In the beginning the effect of the surrounding medium on the mechanical properties of metallic monocrystals such as tin, lead, zinc, aluminum, which were grown using the P. L. Kapitsa, I. V. Obreimov method and the recrystalization method were mainly studied. It was established that the intensity of the action of the particles acting on the surface on the mechanical properties of metallic monocrystals depends essentially on the temperature and the deformation rate (V. I. Lichtman, P. A. Rebinder and L. P. Yanova, 1947). At the same time for equal temperatures and deformation rates, the mechanical properties of solids, especially metals may change in a sufficiently wide range, depending on the distribution of the stresses inside the sample. It is well known that the usual deformation curves represent the average values of the forces and deformations and give a very indirect idea about the true distribution of the stressed and deformed state inside the body. The quantitative aspect of this problem is very complex, but the qualitative pattern of the phenomenon has been investigated sufficiently completely, starting mainly with the studies by N. N. Davidenkov (1936). The point is that during the deformation process the homogeneous mechanical system is converted into a heterogeneous system and that this conversion consists mainly of the development of defect zones in the structure which are always found in a real solid. Experiments have shown (V. I. Lichtman and Ye. K. Venstrem, 1949) that the three-dimensional stressed state depends essentially on the magnitude of the adsorption effect (for example, it increases in proportion to the deviation of the stressed state near the surface from the compressed state in all directions see P. A. Rebinder, L. A. Shreyner, et al, 1944, 1949).

Subsequent studies of monocrystals made it possible to clarify the effect of the medium active on the surface in the initial plastic stage to the yield point (V. I. Lichtman and Ye. P.Zakoshchikova, 1949). The relations for the hardening coefficient as a function of the number of loading cycles in the inactive and active medium were obtained. The phenomenon of the redistribution of the deformations and stresses under the action of the adsorbed particles which is observed here is explained by the activation of the relaxation processes.

At the same time the studies by P. A. Rebinder, Ye. K. Venstrom, et al., investigated the interesting phenomenon called the electrocapillary effect, which consists of the fact that during the polarization of the surface of brittle bodies having electronic conductivity and also of metals in aqueous electrolyte solutions, the hardness of the metals varies and depends on the jump in the potential on the "solid-solution" boundary.

It is known that the magnitude of the surface tension determines several properties of the solid, such as, for example, the hardness, the creep, the coefficient of friction and other properties, which serve as the basis for the determination of the zero charging points in metals. The adsorption region of a bounded substance can be determined according to the change in this dependence of the mechanical properties on the potential and a judgment can be made about the adsorption degree of the latter (P. A. Rebinder and N. A. Kalinovskaya, 1934, P. A. Rebinder and Ye. K. Wenstrom, 1944, 1945, 1949, V. I. Lichtman, Ye. D. Shchukin and P. A. Rebinder, 1962). The studies by P. A. Rebinder and his collaborators determined the relation between the hardness of the metal and the potential in the form of the electrocapillary curve obtained for liquid metals.

It should be mentioned that the hardness of the metal reflects the degree of its dispersion which leads to the formation and expansion of the microcracks and that the rate at which these processes take place increases as the surface tension of the metal is reduced. From here when the electrode potential is displaced (in the positive or negative direction from the potential with a zero charge) and during the adsorption of the organic substances on the "electrode-solution" boundary the hardness of the metals is reduced.

In connection with the interest shown in the role of oxidation films (B. V. Deryagin, 1937) in the adsorption effect which facilitates the deformation, the studies of the electrocapillary effect were continued during the study of the creep in metallic monocrystals (Ye. K. Benstrem and P. A. Rebinder, 1952). For metals with a cubic lattice, the

differences in the mechanical properties between the mono- and polycrystals is negligible. However, this difference becomes very pronounced for metals which have the same basic system of slippage planes (for example, metals with a hexagonal lattice or β-tin). The studies that were made (V. I. Lichtman and P. A. Rebinder, 1947, S. Ya. Weiler and L. A. Schreiner, 1949, 1950, S. Ya. Weiler and G. I. Yepifanov, 1953) have shown the considerable effect of the particles active on the surface in the elastic deformation region of polycrystalline metals.

Considering this aspect from another point of view, S. W. Weiler and V. I. Lichtman (1960) established the effect of adsorption layers on the elastic deformation of metals when lubricants active on the surface are used during the treatment of metals by pressure. Studies along these lines led to the development of the theoretical basis and methods for the application of lubricants. It was shown that in the presence of substances acting on the surface, the surface layer of the metal becomes more fluid, it is plasticized, and when it is treated by pressure, it receives the main part of the redundant shear deformation. A kind of selflubrication occurs (the metal is not lubricated by the lubricant, but by its own thin layer which is plasticized by this lubricant). The action of this thin, easily deformed layer, is intensified by the chemical interaction of the metal with the active molecular groups in the substances acting on the surface, which leads to the formation of peculiar metallic soaps bonded with the surface, which intensify its plastification process. Clearly, it is difficult to evaluate this phenomenon quantitatively, but S. J. Weiler (1949, 1950, 1953), for example, proposed a method for estimating the lubricating action of the medium during the deep drawing of metals. The considerable reduction in the forces in the presence of active lubricants turned out to be characteristic for various methods of treating the metals (pressing, setting, drawing, cutting).

Subsequent experiments made on polycrystalline metals made it possible to determine the effect of the increase in the degree of the hardening (strengthening) that occurs during periodic deformations in the presence of particles acting on the surface (T. Yu. Lyubimova, P. A. Rebinder, et al, 1948, 1950).

Continuing the studies of this aspect, G. V. Karpenko and his collaborators (1949-1953, 1962) extended the concepts of various forms of the adsorption and corrosion effects of the medium on the fatigue strength of metals. It is known that the fatigue strength of metals may be considerably

reduced under the action of agents which reduce the strength (for example, a corrosive medium) and that this reduction depends on the time during which the part was in the corrosive medium and on the number of loading cycles (I. A. Oding, 1949). It was shown that during corrosion fatigue leading to a considerable loss of fatigue strength, only limited endurance exists and that there is no true fatigue limit (G. V. Karpenko, 1952).

Generally, corrosion fatigue involves two processes, the first being the facilitated formation of microcracks under the action of a cyclic load as a result of adsorption, and the second the electrochemical corrosion inside the microcracks that are formed, facilitating their further growth. It is interesting to note the particular orientation of the fatigue microcracks and the preferred saturation of the surface by fracture foci when the values of the cyclic load coefficient are small (G. V. Karpenko, 1951).

The corrosion fatigue phenomenon shows that a medium which is acting chemically on the metal has an effect on its fatigue strength. However, in the absence of a chemical action, the fatigue strength is reduced when the medium contains particles that are active on the surface. This phenomenon was called adsorption fatigue, and, unlike in the corrosive action, the reduction in fatigue strength under the action of media that are activing on the surface, it does not depend on the time spent by the part in the medium and on the number of loading cycles.

Sh. Ya. Korovskiy (1948) began the study of the effect of media acting on the surface on the fatigue strength. Later the cooling effect of liquid media and generally the adsorption and corrosive effect of liquid media on the fatigue strength of steel was also studied (G. V. Karpenko, et al., 1949, 1952, I. V. Kudryavtsev, 1949). We note that a considerable role in the reduction of the fatigue strength under the action of particles acting on the surface is played by the concentration of these particles in the solution and the nature of the solvent (A. B. Taubman, 1930, G. V. Karpenko, 1950).

Studies of the rupture of metals in the stress concentration region under the action of an agressive medium are important in the mechanics of rupture. Experimental studies have shown both the catastrophic drop in the fatigue strength of samples with stress concentrations under the action of liquid metals (M. I. Chayevskiy, 1961) and also the absence of the softening effect under the action of a corrosive medium (G. V. Karpenko and F. P. Yanchishin, 1955, M. I. Chayevskiy, 1959). Thus,

in the fatigue loading process the adsorption, diffusion and corrosion factors may both reduce and increase the fatigue strength of samples with stress concentrations or not have a noticeable effect at all (M. I. Chayevskiy and G. V. Karpenko, 1962). I. A. Oding (1959) has shown that during cyclic loading, the dislocations that are generated, their movement coagulation and the annihilation of vacancies are related to the diffusion and movement of the dislocations which are more intense, and that the change in the crystal lattice prevents the return of the dislocations during unloading. The stresses from the cyclic load are superimposed on the stresses formed as a result of the directed movement of the dislocations and their accumulation around barriers (the formation of a constant stress gradient in the volume of the grain).

M. I. Chayevskiy (1962, 1965, 1968) using the results of the studies of the adsorption, diffusion and corrosive effects of aggressive media under static loads reached the following conclusions with regard to the character with which the deformed metal interacts with the aggressive medium (samples with stress concentrations whose cross-sectional dimension exceeds many times the dimension of the grain): 1) the fatigue strength can be considerably reduced as a result of diffusion of the medium into the defective part of the metal at the apex of the stress concentrator, 2) the operational capabilities of the samples with stress concentrators are increased as a result of a protective diffusion layer formed by the medium resulting from the diffusion and interaction with the defective volume of the metal, 3) the medium dissolves the metal at the bottom of the concentrator (see M. S. Hochman, A. M. Datsishin, et al, 1968). A regular process smoothes the stress concentrators and an irregular process (along the boundaries of the grains) weakens the sample with the stress concentrator and a reduction in the performance of the sample occurs only when the test base is large (G. V. Karpenko and F. P. Yanchishin, 1955, M. I. Chayevskiy, 1959).

Thus, along with the important role played by adsorption effects, the chemistry of the process is also important in the softening of a polycrystalline metal which is deformed in an agressive medium.

The problem of the effect of lubricant oils (which are practically corrosion safe) on the strength of steel became recently important. Experiments have shown that during the cyclic loading of steel, the adsorption fatigue phenomenon occurs in the oils which depends on the adsorption activity of the oil (G. V. Karpenko, 1953). Some aspects of the studies dealing with the effect of liquid media on the fatigue of steel, the qualitative changes in the steel under the action of the

adsorption-fatigue and corrosion-fatigue processes are discussed in the monographs by V. I. Lichtman, P. A. Rebinder and G. V. Karpenko (1954) and G. V. Karpenko (1963). The results pertaining to the changes in the cyclic viscosity of the steel in various media, problems dealing with the effect of the frequency with which the stresses change, the effect of residual stresses on the adsorption and corrosion fatigue of steel and the scale effect are also discussed here.

The effect of the external medium manifests itself differently depending on the structure and composition of the metal (for example, in soft steel with a low carbon content the fatigue strength limit in an aggressive medium is reduced by 3-7%, and in steels with a higher carbon content by 15-20%). The study of the noxious effect of particles acting on the surface on the fatigue properties of metals led to the development of methods for increasing the resistance of the metals (particularly steel) to fatigue in aggressive media. A detailed study of problems in the strength of prestressed structural elements and building subjected to corrosive action, of the corrosive fatigue of steel and the cracking of metals, are available in the studies by A. V. Ryabchenkov (1953), V. V. Romanov (1960, 1967), Ya. M. Potak (1955), G. V. Karpenko (1963, 1967), E. M. Gutman (1967).

Another important trend is related to the strength of metals in the presence of dissolved metallic coatings. Many cases of rupture when a small amount of the liquid metal was present on their surface and when the applied stresses are below the limiting admissible stresses have been known in engineering for a long time. The interest in the problem of preserving the strength of structures and buildings in the presence of melted metals increased, in particular in conjunction with the construction of energy, nuclear and rocket installations in which the heat carriers that are used are liquid metals. This phenomenon was explained for the first time with the aid of the Rebinder effect in the studies by S. T. Kishkin and Ya. M. Potak (1955), and the studies by P. A. Rebinder, V. I. Lichtman, Ye. D. Shchukin (1962) and their collaborators have shown that the greatest fracture of metals may occur in the presence of liquid metals. The studies that were made have shown that the adsorption activity of a liquid metal depends on the degree of its solubility in the solid metal, where the solution of the problem of the reduced strength by adsorption (N. V. Pertsov and P. A. Rebinder, 1958) the melting diagram corresponding to a binary system (a diagram with a narrow solubility region for a light alloyed metal in a refractory metal indicates the possible changes in the strength, unlike in diagrams with a wide region of solid solutions or chemical

bonds) is used for the diagnosis. The study of the electron structure of atoms in both metals has shown that a sharp reduction in the strength occurs mainly when both metals are of the non-transition category (Yu. V. Goryunov, N. V. Pertsov and P. A. Rebinder, 1959). The subsequent study of the adsorption-active effect of melted metals detected an interesting analogy between the brittle temperature loss without coatings and the brittle rupture loss at the usual temperatures under the action of melted metals leading to rupture at considerably lower values of the stresses (Yu. V. Goryunov, N. V. Pertsov, P. A. Rebinder, B. D. Summ, Ye. D. Shchukin, et al, 1963 and later publications).

In addition to the sharp reduction of the strength and the brittleness which occurs, a liquid adsorption-active metal at high temperatures and relatively low deformation rates leads to a lower yield point and a smaller hardening coefficient of the metal (plasticizing action), which was already mentioned above when the effect of lubricants was considered in the treatment of metals by pressure. We recall that the plastic flow in crystals represents the origin and movement of the dislocations in the slippage plane and their rise to the surface of the crystal. Ye. D. Shchukin (1962) has shown that the adsorption of the particles acting on the surface has an effect on the interaction of the dislocations with the surface. As a result of the reduced surface energy of the deformed solid, due to adsorption, the material is plasticized (the dislocation which extends to the surface occurs at a smaller total stress, and when the external stress is constant a larger number of dislocations reaches the surface per unit time, i.e., more plastic shifts occur). The plasticizing action of the melts is similar to the action of organic adsorption-active particles; in both cases the rise of the dislocations to the surface is facilitated.

One important problem in the mechanics of a solid is the problem of the development of macroscopic cracks, where the presence of the adsorption metal is reflected essentially in the entire character of the rupture. The rate of growth of the crack depends on the "flow-around" rate on the edges of the crack, in particular on the rate at which the metallic melt arrives at the apex of the crack. Together with the propagation of the melt along the edges, the liquid metal penetrates the walls of the crack that is being formed, and the finite length of the crack depends on the particular type of competition in these processes. Ye. D. Shchukin has shown that the faster the adsorption metal propagates and the slower it penetrates the walls, the greater the length of the crack all other conditions being equal (the mass of the solvent, the

tensile stresses, the geometry of the plate, etc.). The following relation was obtained for the length of the crack l as a function of the mass of the adsorption-active melt m (Yu. V. Goryunov, N. V. Pertsov, B. D. Summ, Ye. D. Shchukin, et al, 1962, 1963)

$$l = \beta m^{2/3}. \tag{9.1}$$

Thus, the development of the crack is intimately related to the loss for the spreading of the liquid metal along the unoxidized metallic surface film in which the surface diffusion must be distinguished (the migration displacement of multiple layers) and the viscous dissolution of the phase layer.

A study of the laws for the propagation of adsorption-active metals (Yu. V. Goryunov, B. D. Summ, N. V. Pertsov, P. A. Rebinder, Ye. D. Shchukin, et al., 1963) made it possible to explain a number of specific features in the development of microcracks in the presence of mercury and gallium. It is interesting to note that the laws for the spreading of liquid metals can be successfully applied not only to the study of the problem of the development of microcracks, but also to welding, soldering processes, the application of protective metallic coatings, the behavior of the liquid under weightless conditions, etc. A further study of the deformation processes of polycrystalline metals in the presence of adsorption-active liquid metals made it possible to obtained the computational equations used to determine the amount of the melt necessary to obtain the limiting adsorption reduction in the strength (Yu. V. Goryunov, G. I. Den'shchikova, B. D. Summ, V. Yu. Traskin, 1965, 1967). As Yu. V. Goryunov, B. D. Summ, N. I. Flegontova (1964) have shown, the reduction in the strength in definite ranges depends on the ratio of the amount of the liquid metal to the volume of the sample.

Until now it was emphasized that the Rebinder effect manifests itself mainly when an external load and a medium active on the surface act simultaneously on the solids (in the unstressed state, no noticeable changes occur in the mechanical properties of the solid). However, for example, a polycrystalline zinc plate in the presence of gallium begins to flow at a very small load, its own weight, which occurs usually with metals at very high temperatures which are close to the melting temperature. The sharp rise in the plasticity in this case is related to the structural changes that occur in the layers between the grains as the surface energy is reduced due to the adsorption of the gallium atoms.

The changes in the structural properties of monocrystals are even more stunning (which are especially pronounced in a "gallium-tin" pair). It became evident that a monocrystal without structural defects similar to the grain boundaries is converted in the presence of a gallium film in a polycrystalline sample. The phenomenon of spontaneous internal dispersion of the metal which is accompanied by a sharp drop in the surface energy can be used to raise the strength of the metals, and in the process, the adsorption-active melts are no longer agents which reduce the strength but rather facilitate the rise in the strength of metals. Along with perfecting the methods for growing tiny nearly defect-free fibrous crystals, a method was proposed (V. I. Lichtman, P. A. Rebinder and Ye. D. Shchukin, 1960, P. A. Rebinder, 1968) for freezing the samples in which internal dispersion occurred, leading to a homogeneous and fine-grained structure. The strength of such samples exceeds several times the strength of the initial undispersed sample.

Recently it became possible to use the effect of the external medium to raise the strength of catalysts, which play a very important role in modern chemical industry, and to investigate the adsorption reduction in the strength of solids during irradiation. The reduced strength of non-metallic bodies under the action of particles acting on the surface was applied to problems in the reducing strength of minerals, which led to a considerable intensification of drilling processes.

As was already mentioned above, an important fact in the mechanics of fracture is that the particles on the surface of the samples can considerably change the critical stresses at which the cracks begin to grow (for example, the strength of glass that was dried well increases 4-5 times as much).

Ye. D. Shchukin and V. I. Lichtman (1958, 1959) made the following assumption about the brittle fracture mechanisms of bodies with arbitrary dislocation inhomogeneities. Two fundamental stages are observed during the fracture of metals. In the first stage, equilibrium cracks are formed and they develop under the action of shearing stresses at points with a high stress concentration. In the second stage, the cracks make the transition from the equilibrium state under the action of normal stresses to the spontaneous propagation along the entire cross section of the monocrystal. Both these processes are naturally facilitated when the free surface energy is reduced as a result of the particles active on the center, which penetrate inside the crystal along the defective sectors of the structure. Such a model may serve as the theoretical basis for the well-known experimental fact that the product of the

normal and shearing stresses during brittle rupture is constant, which allows us to select this product as a measure of the strength of the monocrystal.

It was already mentioned that a study of the effect of the media active on the surface on the development of the cracks is of interest. The great variety and the complex physical-chemical interactions of the media which occur under high local stress conditions at the end of the crack can be reduced to the fact that the dependence of the physical (adsorption) and chemical (corrosion) factors on the stressed-deformed state at the end of the crack in the brittle fracture region is completely determined by a single natural parameter, the coefficient of the intensity of the stresses (G. G. Johnson and P. K. Paris, Engng. Fracture Mech., Vol. 1, No.,1, 1968, pp. 3-45). The rate of growth of the crack, in the phenomonological sense, depends only on the coefficient of intensity of the stresses and on the physical-chemical constants of the "deformed body-external medium" pair. The important experimental result, namely that the effect of the external medium begins to be felt only at a certain ratio of the physical-chemical parameters of the medium to the coefficient of the intensity of the stresses (which manifests itself in the growth of the crack for a constant coefficient for the intensity of the stesses) should be mentioned.

Apparently the best studied mechanism for the reduction of the effective surface energy in a solid in a medium acting on the surface is the pure adsorption mechanism. The subcritical growth of the crack may be negligibly small in a number of cases, so that the change in the effective surface energy that was mentioned completely describes the effect. For example, for silicate glass, the surface energy in the presence of moisture is reduced approximately by 20%. In metals, the effective surface energy exceeds approximately by three orders of magnitude the free surface energy estimated on the basis of physical concepts.[1] However, it is interesting to note that the rupturing stress is determined by the effective surface energy, whereas the adsorption effect has primarily an effect on the free surface energy.

1. See, for example, G. G. Hillman, Splitting, plasticity and Viscosity of Crystals (1959, Russian translation in the collection: Atomic Fracture Mechanism, Moscow, 1963).

2. G. P. Cherepanov gave an explanation of this seeming contradiction (1968), who found for the case of a thin plate from the energy equation the following bound: the ratio of the effective surface energy to the free surface energy has the same order of magnitude as the ratio of the Yung modulus to the yield point.

§10. Permanent Fracture Problems

Together with local fracture processes (for example, at the end of a crack) and three-dimensional fracture (for example, during absolute viscous fracture, when the load bearing capacity of the sample is used up uniformly over the entire dangerous section), gradual unloading processes on the surface of the body or on a sector of the body are of great practical and theoretical interest. The most important phenomena in this class of problems are the erosion rupture of the surface layer of a solid as a result of the force effect of a gas flow or liquid (the pairs "solid-liquid," "solid-gas"), the wear of solids during friction (the "solid-solid" pair), the wear of a solid in a liquid flow with hard particles (the pair "solid-liquid with hard particles") and also some other phenomena.

The short fracture times for structural materials as a result of erosion, after which the use of the industrial objects becomes economically unfeasible, is the reason for the many studies of this phenomenon. As a result of the difficulties connected with the study of erosion fracture in pure form, almost all investigators were forced to consider erosion fracture under the simultaneous action of a number of factors having an effect, to a greater or smaller degree, on the erosion fracture process itself, which made its study more difficult. Among these studies the following can be singled out: the chemical interaction of the materials with gas flows or liquids, the chemical conversions in the material itself, sublimation, melting, thermal stresses, adsorption phenomena, the effect of various kinds of radiation on the properties of the materials, etc.

In this section we will consider gas erosion, hydroerosion, wear during friction on the edges and abrasive erosion. We will cover not only those studies which, in the opinions of the authors, had an important effect on the evolution of the views on the problems that were mentioned but also those which had an important effect on the contemporary state of this problem. An acquaintenance with this branch of knowledge which borders physical-chemical mechanics and in which, for the time being, predominantly only qualitative results have been obtained, is fruitless for theoreticians in mechanics, since it is a new field of future quantitative research.

The majority of investigators who studied gas erosions saw the reason for the mechanical fracture of the surface of materials in various processes accompanying the erosion. These views were supported in no small degree by the fact that the frictional stresses on the surfaces of materials even under such difficult conditions as, for example, when spacecraft

enter the dense layers of the atmosphere are relatively small and considerably smaller than the limiting shear strength of the material.

In the 50's K. K. Snitko, et al., proposed the so-called oxidation theory, a variant of the chemical theories of gas erosion. According to this theory, the main reason for the erosion fracture of metals is the oxidation of iron (and the burned up carbon) and other elements which oxidize iron more easily during the direct oxidation by free nitrogen and also during the indirect oxidation by means of the carbon dioxide and water vapors present in the gases. These conclusions were based on the results of studies of mechanism and kinematics of the decomposition of dust at high pressures.

According to chemical theories of gas erosion, the surface of the metal around which hot gases flow, under pressure undergoes both structural-chemical (under the effect of oxidation, cementing, anisotropy) as well as mechanical changes, which result in the fracture of the thin surface layer of the metal.

A. F. Golovin (1941) made a systematic study of fractured bores in artillery guns resulting from erosion and determined the presence of hardened sectors below the fields of the cuts caused by the dynamic effects leading the nose of the projectiles. The conclusion was reached that the thermal factor has the dominant effect on the hot gas erosion process and that the basis for the rupture mechanism is the "spreading" or "blowing" of gas jets of the melted or softened no longer solid surface layer of the metal (as a result of small thermal fatigue cracks).

I. S. Gayev (1950), et al., obtained some experimental data which confirmed indirectly the idea of the vaporization of metals during erosion fracture. It was established that the vaporization rate for steel increases as the temperature and the carbon content increase. A comparison of the reduction in the weight of the sample during vaporization under the effect of a high temperature with the erosion tests of the samples made from the same alloys has shown that the strength of the materials in both types of tests follow the same squence. It was established that together with the diffusion and recrystallization, the vaporization rate may characterize the strength of the bonds maintaining the atoms in the crystal lattice during heating. Apparently these parameters often characterize the endurance of the metals and alloys at high temperatures also in the case of erosion tests. I. A. Oding (1949, 1963) assumed that the erosion fracture process represents a pure mechanical action of the vapor flowing on the metal containing drops of water and various hard particles.

N. S. Alferov (1952), who studied the erosion fracture of plates in gas turbines in a gas flow with dust reached a similar conclusion. In his opinion, one of the causes for the mechanical fracture of the surface of metals in a gas flow not accompanied by the melting of their surface, is the presence of dust particles in it. Its mechanical fracture occurs as a result of the many impacts of these hard particles on the surface of the metal.

The formation of minute cracks oriented into the depths of the metal was detected on the surface of the blades. These indicate the fatigue phenomenon in the surface layer of the metal. The mechanical rupture is explained by the knocking-out of the smallest particles of the metal that are formed as a result of the microcracks which occur and the cutting of the surfaces that are formed (plateaus) bombarded by the rapidly-moving particles.

I. N. Dekhtyarev (1949) expressed the opinion that the values of the stresses in the upper layers of the metal of turbine blades may attain values which are commensurate with the fatigue limit in blade steels.

I. N. Bogachev and R. I. Mints (1958 and in later publications) concluded on the basis of the nonuniformity of the distribution of acoustical pressures during the flow past the surface of airwings of an airstream that the distribution of the pressures in the metal was nonuniform. The rapid gas flow acts mechanically on the metallic surface which, in view of the inhomogeneity of the flow, leads to considerable inhomogeneity of the stress field in the metal. The latter throws some light on one of the most important mechanism of erosion fracture. Under a local load, microvolumes can occur in some sector in which, along with the elastic deformation, a plastic deformation or even microcracks will occur. The total recorded deformation level may not be high; however, the presence of microfracture is already dangerous to some extent with respect to the sufficient performance reliability of the structure. The same authors noted the large values of the loads associated with the aerodynamic action of the gases flowing out from a jet exhaust nozzle and the pressure impulses with the high frequency oscillations formed in the process, etc. It turned out that the loads from the factors that were mentioned, which can lead to fracture during the service life of the airplanes, are encountered relatively often.

When a gas flow acts on the surface of the metal, its relief is shaped. The character of the microrelief is determined not only by the type of load, deformation, but also by the nature of the metal. The microrelief obtained when a rapid flow is acting on the surface of the metal should be considered as a characteristic of the surface which determines the performance strength of the structure. The character of the relief allows us to make preliminary conclusions about the strength of the metal under the given conditions, since the capability of the metal to form microrelief and the endurance are directly related. The usual properties of the material (the macrocharacteristics obtained on standard samples taking into account the aggressiveness of the medium and the temperature) are only a coarse preliminary criterion for estimating the endurance of the material during the contact with a rapid flow.

L. A. Urvantsev (1966) proposed, on the basis of an analysis of known theories of the erosion structure of materials caused by various reasons, that the existing concepts be generalized and introduced for this purpose the so-called "principal generalizing function" which must characterize the propoerties of the medium, of the boundary layer and of the material. The description of the erosion fracture mechanism proposed by him includes a repeated cyclic loading of the surface layer of the material and the fatigue cracks formed in it (both in the body of the grains as well as on their boundaries), chemical, thermal and electrical effects of the medium and the changes that occur in the material as a result of them. The majority of investigators see the reason for hydroerosion in corrosion and cavitation processes.

A. D. Moiseyev (1954-1956) considered hydroerosion as an electrochemical process which develops and depends on the rate at which water moves. It is assumed that when the flow moves at high velocities, there is not enough time for the formation of the oxidation film and the corrosion medium interacting with the bared surface creates conditions for an intense development of the electrochemical process.

I. N. Voskrenskiy, V. V. Fomin, et al. (1949) assumed that the fracture of metals during hydroerosion occurs under the action of the corrosion and mechanical factors and that it depends on the velocity with which the water moves. At low velocities of the flow, mainly only the electrochemical process develops. As the velocity increases, the mechanical factor begins to act and the fracture of the metal becomes corrosive-mechanical. At high flow velocities, the mechanical factor is dominant. It was shown in the study by M. G. Timerbulatov (1965) that in addition to the high mechanical strength and the high fatigue limit, the materials must have high anticorrosive properties.

L. A. Glikman (1955) has shown the subordinate role of the corrosion factor during hydroerosion. It was established that the hydroerosion rate sometimes exceeds 10^5 times the corrosion fracture rate.

The erosion fracture has essentially a heterogeneous character. The strengthening of plastic metals with time (hardening) under the effect of cavitation spreads into the depth by several microwaves (L. A. Glikman, Yu. Ye. Zobachev, et al., 1956), and relatively weak sectors on the surface of the alloys are first subjected to erosion (I. N. Bogachev, R. I. Mints, et al., 1961), on the surface of the alloys and concrete the erosion is primarily localized in the natural pores and cracks (K. K. Shal'nev, N. P. Rozanov, et al., 1965). K. K. Shal'nev, R. D. Stepanov, et al. (1966) detected the considerable effect of the load on the tested sample due to the external stressing force on the intensity of the erosion.

The study of erosion fracture on models is of interest. I. Varga, B. A. Chernyavskiy and K. K. Shal'nev (1962, 1963) studied the relation between the intensity of the corrosion and the hydrodynamic parameters and the physical properties of liquids (the velocity of the flow, the characteristic dimension of the model, the density, the surface strain, the viscosity, and the three-dimensional elasticity of the liquid).

I. R. Kryanin (1955-1962) considered the hydroerosion of metals as a corrosion-fatigue process resulting from one-sided cyclic compression. In his opinion, the reason for the unsuccessful attempts to determine a relationship between the cavitation strength of the metals and their corrosion-fatigue strength is the special character of the cycle of hydraulic shocks that occur during the cavitation which is not taken into account by many investigators. V. V. Havranek (1955) considered hydroerosion as a microfatigue process. The protrusions on the surface of the metal were considered by him as microoverhangs which experience, during the hydraulic shock loads with changing signs, and are broken off as a result of this.

V. A. Konstantinov (1947), who studied the physical nature of cavitation reached the conclusion that the fracture of the metal during cavitation is related to the electric charges which are formed during the compression of the cavitation bubbles. These electrical charges in the form of microscopic "lightening," can rupture in a short time materials with any strength. Later, in connection with the use of cathode protection of hydroturbines from cavitational erosion, additional studies of the electrical effect in the cavitation zone were made (V. I. Skorobogatov, 1960, Yu. N. Paukov, M. K. Bologa and K. K. Shal'nev, 1968). The presence of the electrical effects and the effect of the external electrical field on the intensity of the erosion were confirmed.

I. N. Bogachev and R. I. Mints, V. V. Havranek, M. Fuchs, D. K. Bol'shutkin, et al. (1955) have shown in their studies that the hydroerosion process is caused by the mechanical action of the hydraulic shocks which occur when the cavitational blisters contract. As a result of such multiple shocks, individual microvolumes are deformed and displacement and double lines occur. The hardness of the surface layer is increased during the process. X-ray analysis shows the distortion of the crystal lattice and the fragmentation of the structural mosaic blocks. The fracture of the metal is preceded by the formation of cracks and fracture foci in the surface layer.

The studies made by S. P. Kozyrev, K. K. Shal'nev and M. G. Timerbulatov (1956, 1965) in hydrodynamic pipes have shown that cavitation depressions do not only not collapse instantaneously, which follows from the Rayleigh theory, but do not collapse at all. A ripple was detected in the cavity over time, with a large frequency, and during the ripple the cavity decreases in diameter, then vanishes, is formed again, etc. The basis for the cavitation fractures are the fatigue phenomena on the surface resulting from the high frequency impulse effect.[1]

V. V. Fomin (1966) reached the conclusion on the basis of his studies of the hydroerosion of metals and the generalization of the results obtained by other authors, that as a rule it is observed at high flow rates and that it occurs mainly due to the mechanical action of the liquid. The nature of this effect is related to the qualitative change of the character of the liquid flow. Under these conditions, the impact load has an impulsive character, i.e., it is distinguished by a fast increase in the pressure, which is followed by its rapid decrease. A characteristic feature is the very small region in which the maximal stresses are acting which is commensurate with the dimensions of the individual microsectors (whose size is approximately 10^{-4} - 10^{-6} mm^2). The stresses are distinguished by being local and nonuniform, and they are formed in individual microvolumes regardless of what occurs at another point in the surface layer. During such character of the mechanical action, the fracture of metals is related to the break-off of very small particles resulting from the formation of microscopic cracks in the surface layer that are formed as a result of the

1. In 1968 S. P. Kozyrev considered as one of the most probable reasons for the strong force action on the surface the effect of the cumulative collapse of the cavitational cavity.

plastic deformation which occurs in the microvolumes. V. V. Fomin assumes that the hydroerosion of metals must be considered as a process which occurs as a result of the microimpact action of the liquid. During such character of the load, the resistance of the metal to rupture is determined not by the averaged properties of the individual microvolumes but by the properties of the metal in the microvolumes, i.e., the mechanical strength of the individual microsectors or structural components.

During the microshock action, stresses and strains are formed which are localized in the microvolumes of the surface layer, so that the rupture has a local character. The properties of the surface layer determine the erosion strength of the metal.

The state on the surface of the sample has also an effect on the formation of the rupture foci. However, the effect of the surface profile is only felt in the initial stage of the erosion process. When the deformation relief is formed, this effect is removed.

I. N. Bogachev and R. I. Mints (1958, 1964) studied several steels and resistance to cavitation-erosion rupture. As a result of these studies they found out that the cavitation-erosion strength of the steel depends on the size of the grain, the character of the boundary and the body of the grains. The intensity of the rupture is determined by combining the properties of the grains and its boundaries. It was also noted that steels in the viscous state resist erosion better than in the brittle state. The authors proposed the hypothesis that the resistance to cavitation-erosion rupture must depend on the damping capability of the material (i.e., on the magnitude of the decrement in the damping of the oscillations), provided the fracture of the metal due to erosion is considered as a fatigue phenomenon, taking into account the multiple action of the water drops on the surface of the plates.

In the mutual contraction processes of solids, studies of problems dealing with the abrasive rupture during friction on the boundary play an important role. V. D. Kuznetsov (1947) assumed that the mechanism of the abrasive wear is extremely simple and reduces to the sum of a large number of abrasion processes. A deep relationship must exist between the phenomenon of a simple scratch and the abrasive wear. However, studies have shown that no unique relation exists between abrasive wear and the mechanical properties of the metal.

Using the concept that during wear under the same condition the same degree of plastic deformation and hardening is attained, M. M. Khrushchov and M. A. Babichev (1960) proposed a theoretical relationship between the three-dimensional wear, the length of the frictional paths, the dimension of the abrasive grain, the load and the initial hardness of the metal. The tests that were made have shown that, in fact, the wear is directly proportional to the frictional path, the load and the dimension of the abrasive wear and that a critical value exists for the dimension of the grain after which the abrasive wear does not increase. At the same time, the wear is inversely proportional to the value of the hardness of the metal before the test which was confirmed experimentally for technical pure metals and steels in the annealed state.

Subsequent studies have shown that the actual contact area of two surfaces differs considerably from the contact which is blackened by the outer contour of the surfaces and conventionally used to calculate the mean specific pressures. Given the purity attained during the treatment at the present time, the actual contact area is 10^{-5} to 10^{-2} of the contact area, as a result of which specific pressures of several thousands kg/cm^2 are formed on the contact areas. Naturally this leads to a rapid plastic deformation of the microinhomogeneities and also to the rupture of individual sectors in the surface layer of the metal. The fracture occurs as a result of the micro- and macrocracks that are formed and, apparently, the main reason for the formation of the cracks are the internal and thermal stresses. The latter are formed as a result of the local temperature burst caused by the transition into the body of over 50% of the external energy used up in the irreversible plastic deformation process, and also as a result of the rapid cooling of the surface layers of the entire metal mass. Since in plastic materials under variable temperature field conditions, the stresses in the plastic region are much smaller than the elastic stresses in brittle materials, the latter resist better thermal fatigue and consequently, pitting. In addition to this it must be remembered that the microplastic deformation of grains formed during cyclic temperature changes which manifests itself in the form of slippage lines and in some metals also in twins and mosaic structures is accompanied by distortion of the crystal lattice, by a loosening of the boundaries of the grains and the formation of microvacancies, which also worsens the mechanical properties (long-term strength) and enhances the fracture of the material.

I. V. Kragel'skiy (1963 and later studies) considered the occurrence of free particles during friction on the boundary as a result of microcut processes, "depth" tear-off and repeated deformation. Microcuts are observed during the indentation of the contact protrusion at a sufficiently large depth (approximately 0.2-0.3 of the radius of the protrusion), i.e., when the external frictional threshold is exceeded. At the usual frictional nodes particles are not obtained, for all practical purposes, as a result of microcuts, since the loads for which the indentation does not attain a value necessary for a cut are selected in advance.

Depth tear-outs are formed when the external frictional threshold is violated as a result of the negative gradient of the mechanical properties, which is formed along the depth of the frictional surface or as a result of the excessively large relative indentation. It has the character of tear-out or spiking of the material not along the seam but inside one of the bodies. However, the microcut and the depth tear-out of the material are extreme cases of wear during friction.

According to I. V. Kragel'skiy, during a stationary frictional regime, the coarseness of the surface is reproduced. However, the coarseness reproduction mechanism remains unknown. Usually, the wear of frictional surfaces occurs mainly during slippage. This can only occur when films are formed on the surfaces which protect the main material from direct contact. The film which separates the surface is an absolutely necessary condition for the slippage. In its absence, depth tear-out will inevitably occur.

Under dry frictional conditions, the oxidation film which is formed on the surface increases along its thickness up to a certain value, peals off, increases again, etc. This film interacts molecularly with the film on the second surface. The films protect the main material from depth tear-out; however, they do not protect it from deformation which it undergoes during the indentation of the protrusion on it during slippage.

Each section of the worn body is successively subjected to compressive and tensile stresses. This effect was described for the first time on the basis of experimental data by A. S. Radchik and V. S. Radchik (1958), who detected a change in the sign of the deformation in a particular zone of the worn sample.

Even a small repeated load on the surface may lead to its fatigue fracture. Fatigue cracks are formed on defects which are always present in the solid. They are related both to the structure of the metal (the vacancy in the crystal lattice, the boundaries of the blocks), and to the traces from the treatment (scratches) and, finally, to the metallurgical defects (contraction pores, gas bubbles, slag inclusions, the pronounced inhomogeneity in the dimensions of the crystals, differences in hardness, etc.).

During the development of a crack, the cracks which gradually amalgamate may lead to the formation of wear particles. The ruptures formed on the frictional surface as a result of repeated actions were called by I. V. Kragel'skiy (1963 and later studies) frictional fatigue. The rupture of the material as a result of repeated deformations leading to the loosening of the metal has been described in detail by N. N. Afanas'ev (1965).

M. V. Khanin (1966 and later publications) who studied the fracture of materials in high temperature and high velocity flows of an inertial gas under conditions which excluded practically all forms of rupture except erosion rupture, has shown experimentally the presence of mechanical rupture on the surface. Microstructural studies of the surface layers of a material subjected to erosion rupture have detected characteristic fatigue changes (wide slippage strips, microcracks, etc.). This indicates the presence of cyclicly changing force action on the surface of the material from the side streamlined by a gas flow. In the uneven depressions, a vortex pulsating motion occurs, as a result of which forces which vary with time are acting on the pimples which are the causes for the erosion rupture.

M. V. Khanin reached recently the conclusion on the basis of an analysis of the fatigue theory of erosion rupture of materials during friction on the boundary developed by I. V. Kragel'skiy by comparing it with the mechanism for the erosion rupture of materials in gas flows and a liquid, that the erosion, both during friction and during the action of the liquid flow, represents a fatigue rupture process on the surface layer, occurring as a result of the forced oscillations of material particles on whose protruding parts variable forces are acting. He obtained formulas for determining the erosion rate for the fracture of the materials and the value of the coarseness on their surface.

In our country, especially in recent years, extensive studies were also made on the wear of all kinds of equipment in a liquid or gas flow with hard particles. For example, we mention the studies dealing with combatting the so-called aerosol wear in boiler equipment. Considerable attention was given to this problem in the studies by I. K. Lebedev, O. N. Murovskin, A. V. Ryabchenkov, S. N. Syrkin, and others.

The studies by Ye. I. Pazyuk (1953), Sh. M. Bilik (1960) and by other authors deal with the study of the hydroabrasive treatment of metals. The abrasive-erosion wear of equipment used in the oil and gas industry (Christmas tree and tubing head, drilling pumps, turbine drills, pipes, etc.) presents especially acute problems. Extensive studies along these lines were made by A. V. Kol'chenko (1956), L. A. Shreyner and A. I. Spivak (1958), A. A. Antonov (1963), V. N. Vinogradov (1963), and others.

One hypothesis explaining the nature of abrasive-erosion pitting was proposed by L. B.Erhlich (1950). According to this hypothesis, most adjoining parts operating under contact load conditions have certain characteristic features, namely the short duration of the loads on individual sectors of the operating surfaces, considerable local loads, multiple cycle repeated external loads, a comparatively large mass of the metal adjacent to the surface layers, the presence of the structural component in the form of a wide strip which is not etched by the usual reagents and which is only detected as a result of a metallographic analysis. L. B. Erhlich proposed a scheme which outlined a sequence of phenomena which occur in the surface layers of the working parts. According to this scheme, first instantaneous force action occurs in the surface layer, then the contact loading, next the successive plastic deformation, a temperature burst and rapid cooling. The instantaneous action of the forces is caused by the kinetic energy from the impact of the abrasive particles on the surface of the product and it depends on the mass and the speed of the particles. The contact pressure which attains extremely high values is very important.

During the multiple action of the abrasive particles on the surface of the metal a thermal load which changes sign resulting from the causes that were presented above is observed. The crack, which is formed in the process, has a fatigue character and it facilitates the stress concentration on the surface of the product and is probably one of the main reasons for the pitting of the material. Thus, it is obvious that the kinetics of the pitting process, including the abrasion-erosion rupture, includes various forms of deformation and is determined by a number of mechanical properties of the metal.

A very interesting variety of permanent rupture is the rupture of superstrong materials, whose strength approaches the theoretical strength (high-strength glasses obtained when special technological conditions are maintained, metallic "whiskers," "defect-free" glass fibers, high-strength polymer fibers, etc.). The rupture of such materials occurs in "flairs" when they are decomposed into a set of small particles. We note that the rupture of an ideal crystal structure during a sufficiently high load must occur in the limit in the form of decomposition into individual atoms.

"The flair-like" rupture of superstrong glasses was observed, for example, by M. S. Aslanova, G. M. Bartenev, F. F. Vitman, L. K. Izmaylova, and others. A theory of this phenomenon, called the selfsustaining rupture phenomenon, was proposed by L. A. Galin and G. P. Cherepanov (1967). It is based on a study of the front of the permanent surface rupture spreading due to a margin of elastic energy in the body (which is analogous to the motion of a detonation wave resulting from a margin of chemical energy). We note that such a type of rupture may also occur in the usual brittle bodies (for example, in hard rocks), provided they have a sufficient margin of elastic energy. This can be obtained, for example, as a result of compression in nearly all directions. The monograph by S. G. Avershin (1955) discusses in detail various aspects of the rupture phenomenon that was mentioned which is becoming more and more important in the mechanics of rocks (rock impact).

§11. Rupture during an Explosion

The strength problems that were discussed above pertain mainly to problems in the protection of equipment and structures whose rupture is undesirable. The study of rupture processes during an explosion is of independent interest and it determines, to a considerable extent, the effectiveness and usefulness of explosive work.

When a sufficiently large amount of energy is released very rapidly in a volume of a solid, many rupture processes occur whose character depends considerably on the total amount of energy that was released and its concentration, the source and the manner in which the energy was released, and on the physical-mechanical properties of the solid.

The sources for the explosive release of energy are variegated. They are nuclear reactions (atomic and nuclear explosions), chemical reactions (the majority of explosives used), strong electrical discharges (for example, atmospheric lightning), powerful light pulses (obtained in lasers), a margin in the kinetic or elastic energy (obtained, for example,

during the collision of fast moving bodies, during the explosions of balloons with compressed gas, during the shocks in rocks in earthquakes, during the rupture of high-strength glasses or highly brittle materials), etc.

The problem of regulating the explosion energy in order to obtain the most useful effect is one of the most important problems in engineering. A substantial contribution to the solution of this problem was made by Russian engineers and scientists, B. N. Bokiy, M. M. Boreskov, M. M. Protod'yakonov and others. However, the greatest progress in this field was made in the last few decades.

We will first discuss the main results that were obtained in the study of the effect of an explosion in rocks and soils. Explosive substances are used in mining and construction to crush and pit rocks, and also to eject or displace the rocks in order to form artificial cavities, dams, etc.

Explosion practice is based on the empirical similarity law, on the basis of which the volume of the rock that is destroyed (and also the volume of the cavity formed after the explosion) is directly proportional to the volume of the explosive charge. Now, it is difficult to determine who was the first person to formulate this law (it is mentioned, perhaps for the first time in 1628, and it is due to the Frenchman Deville). The proportionality coefficient in the law depends on the strength of the rocks, the characteristics of the explosive material, the shape and the position of the charges, etc. The similarity law that was mentioned is violated in very powerful explosions due to the comparatively high effect of gravity, and for very strong brittle rocks apparently as a result of the strength constant of the material (the critical coefficient of the intensity of the stresses), whose dimensionality is different from the dimension of the stresses.

During the study of the effect of an explosion in soils and rocks, the model of an ideal incompressible fluid has been widely used (the explosion itself is considered to be instantaneous). The distribution of the pressure pulses and of the velocities in the space immediately after the explosion is determined from solving a boundary value problem for the Laplace equation, and it can be constructed with sufficient accuracy This kind of approach was developed by M. A. Lavrent'ev and also by O. Ye. Vlasov (1945). It is justified physically, since the pressure in the explosion chamber from the usual explosives attains tens and hundreds of thousands of atmospheres, which exceeds many times the strength of rocks. Using this

frame of reference, O. Ye. Vlasov and S. A. Smirnov (1962) developed in this frame of reference a theoretical scheme for the crushing of rocks by the explosion of concentrated and elongated charges, and found the boundaries and the volume of the crushing zone, the distribution of the size of the crushed rocks, the statistical grain size distribution, the crushed part of the rock mass and estimated the duration of the crushing process. They used the concept of the critical destruction rate introduced by O. Ye. Vlasov. According to this concept, the dimension of the pieces of the rock formed as a result of the explosion is such that the difference of two neighboring pieces is equal to some critical value (each material has its own value). These calculations made it possible to obtain a general description of the character of the crushed rock during an explosion. We note that the problem of uniform crushing is exceptionally important in mining and that many experimental and theoretical studies have been devoted to it (so that pieces of rocks whose dimensions exceed some limiting volume permitted by the technological conditions are not left as a result of the explosion).

V. M. Kuznetsov (1966) used the model of the medium that was mentioned to calculate the shape of the explosion crater. He used the formulation for the problem proposed by M. A. Lavrent'ev. The model of an ideal liquid was used by G. P. Cherepanov in the solution of a number of problems on the effect of an explosion on underwater equipment and equipment below the ground (1966), and also in the construction of a hydrodynamic variant in the theory of cracks formed under the effect of the explosion (1963). The acoustical variant of the knock theory proposed earlier by V. S. Lenskiy (1958) is intimately related to this approach.

M. A. Lavrent'ev, V. M. Kuznetsov and Ye. N. Sher formulated in 1960 the problem of the directed ejection of the soil by an explosion and obtained an elegant solution for it as a reverse hydrodynamic problem. This solution was validated experimentally for soft soils. Mass explosion methods for ejection with the aid of elongated charges distributed properly in underground mines were based on it. Using chambers with an enlarged volume to improve the effectiveness of the explosion, it became evident that it was useful to fill them with water.

When the seismic effect of an explosion is studied, the soil or the rock is usually considered as an elastic body. The problem of the damping of shock and seismic waves in soft soils saturated with water was studied in the last decade by G. M. Lyakhov, V. N. Nikolayevskiy, and others.

The concept used for representing the soil was a two component medium ("double" solid medium-porous deformed solid with pores filled by a liquid or gas). These problems are discussed in the special survey by G. K. Mikhaylov and V. N. Nikolayevskiy published in the second volume of the collection, which is not mentioned here.

Many approaches using various complex models of a solid medium were proposed for the study of the destruction process and the stress waves in the explosion zone. We will first mention the main results obtained by means of a formulation of the mathematical model and the solution of certain boundary value problems for the corresponding differential equations. All these results are based on various variants of the model of an elasto-plastic body.

In 1957 Kh. A. Rakhmatulin proposed a model for "a plastic gas" which was a generalization of the model of an ideal compressible fluid. According to this model, a single valued relation exists within the pressure and the density of the gas (the tangential stresses are ignored) during the loading, which is replaced by a different law during the unloading (in the simplest case, it is assumed that the density remains constant under the unloading conditions). This model gives an ideal zed description of the properties of the soil when the mean hydrostatic pressure exceeds many times the tangential stresses.

Subsequently, Kh. A. Rakhmatulin, A. Ya. Sagomonyan and N. A. Alekseyev (1965) generalized the model to the case when tangential stresses are present using deformation concepts (the system of equations that was derived is a generalization of the Hanke-Nadai equation for the case of an arbitrary and irreversible three-dimensional compression). In their earlier studies, A. Yu. Ishlinskiy, N. V. Zvolinskiy and I. Z. Stepanenko (1954) and A. Ya. Sagomonyan (1954) considered several one-dimensional problems in the dynamics of soils under certain concrete assumptions with regard to the properties of the medium ("plastic gas"). The studies by A. S. Kompaneyts (1956), N. V. Zvolinskiy (1960), A. Ya. Sagomonyan (1961) take also into account the tangential stresses in similar one-dimensional problems (with the Prandtl plasticity condition).

The experimental studies by V. V. Adushkin and A. P. Sukhotin (1961), S. S. Grigoryan, G. M. Lyakhov, V. V. Mel'nikov and G V. Rykov (1963), M. V. Gogolev, V. G. Myrkin, G. V. Parkhomov and A. N. Khanukayev (1965), A. B. Bagdasaryan and S. S. Grigoryan (1967) should be mentioned. These studies investigated the physical pattern of the destruction process

in the nearest camouflet exposure zone (to improve the explosion in the organic glass, high speed camera photography was used and during an explosion in a forest soil the explosion cavity was analyzed after the explosion.

The problem of a camouflet explosion in an ideal plastic body was studied for different special variants by G. Taylor, R. Hill, E. Pinni, and others.

E. I. Andriankin, V. P. Koryavov (1962, 1965), V. N. Rodionov (1962), Kh. M. Aliyev (1964) introduced in their studies for the solution of the spherical symmetric problem of an explosion in a brittle body the concept of a destructive wave separating two possible states of the medium (the destroyed state and the state that was not destroyed). Generally, the stresses on this wave have a discontinuity.

In the studies by S. S. Grigoryan (1959-1967) problems in the dynamics of soils were studied in the most general formulation. The hypotheses of a mechanical and thermodynamic nature formulated by them reflect the specific properties of the soils and rocks. Models describing the deformation and rupture processes and the movement of the media under consideration under arbitrary external forces are based on these hypotheses. They constructed models for soft soils (1960) and for solid brittle rocks (1967). The authors studied the general properties of the solutions of the equations that were constructed and found the basic qualitative properties of the movements which they described and formulated the conditions and the rules for the modeling.

S. S. Grigoryan studied several problems on the effect of an explosion in soils and rocks on the basis of the models that were proposed. In particular he obtained the solution for problems dealing with the effect of an explosion from a concentrated charge in an unbounded soft soil and rock.

In addition, S. S. Grigoryan (1962) formulated and developed an algorithm for the numerical solution of problems on waves indicated in the ground halfspace by an explosion above ground. The solution of the problem provided quantitative information about the changes in the parameters of explosive waves with the distance (maximal stresses, velocities, residual and total deformations, displacements, characteristic active times of the wave, etc.), the dynamics for the elongation of the cavity and the boundaries of the destruction regions, the plastic deformations and the character of the destruction in these regions.

The real ground or rock has properties in the vicinity of a sufficiently powerful explosion, which apparently occur in all possible sets of models of a continuous medium, all the way from an ideal fluid in the immediate vicinity of the point where the explosion occurs, to an elastic body at relatively large distances from the explosion. The problem is further complicated by the development of a set of explosion displacement surfaces (cracks) in the explosion zone. Under these conditions any, even the most complex model of the medium, can only yield a very approximate description of the entire set of destruction phenomena that were mentioned. In addition, it must be taken into account that the volume and complexity of the corresponding calculations increases sharply when the model of the medium is more complex. Under these conditions, simple approximate computational techniques, based to a large extent on engineering experience, are important in practice.

It should also be mentioned that the problem of optimizing the crushing or tear-out process during an explosion is essentially a reverse problem (which is, therefore, much more complex) of the problem of calculating the deformations and stresses in a medium subjected to a force which is known in advance. Therefore, the importance of finding sufficiently simple estimates for the solution of direct problems (sometimes only of a qualitative character) is obvious.

M. V. Machinskiy (1935) posed the problem of developing a general theory for the crushing of rocks. According to his studies, the three main factors which determine the crushing, are the shock wave, the distribution of the weak spots in the rock and the rate at which the cracks propagate in it. M. V. Machinskiy studied the joint effect of a system of points and linear charges, paying particular attention to the determination of the most advantageous distance between the charges.

In the last decades, the relation between the explosive characteristics of explosives and various effects of the explosion in a rock was established.

A. F. Belyayev and M. A. Sadovskiy (1952) have shown that the disruptive characteristics of explosives caused by the head part of the explosion impulse which are related to the density of the explosive and its detonation speed, determine in advance the degree of crushing of the rock only in the immediate vicinity of the charge. The total effect of the explosion, which manifests itself in the destruction of the body at greater distances from the charge and is proportional to the total impulse of the charge, is related to the total energy of

the explosion and does not directly depend on the detonation rate. Therefore, to crush large volumes of rock, it is necessary to increase the duration of the effect of the explosion on the rock, not the peak pressure.

To solve the last problem, N. V. Mel'nikov and L. N. Marchenko (1958, 1964, 1965) proposed modifications for the design of the charges. These modifications boil down to a different ratio of the height of the charge to its diameter, the introduction of air gaps between the charges and also between the charges and the walls of the charge chamber. The proposals that were mentioned, which are also useful in tear-out explosions, were verified on a large volume of experimental material and incorporated in production. It also became possible to increase the explosion time by way of producing new less high explosive types of explosives.

The short fuse explosion method where each successive charge or series of charges is exploded after a certain time interval (on the order of 10^{-3} sec), after the explosion of the previous charge, is used for the same purpose. This method is particularly effective in combination with the rational selection of the spatial location of the charges. The theoretical and experimental studies by K. A. Berlin (1934), F. I. Kucheryavov, M. F. Drukovann and Yu. V. Gayek (1962), N. G. Petrov (1964), V. N. Mosintz (1967), who proposed a number of schemes for the positioning of the charges and calculated the delay time, played an important role in the development of this trend.

The series of studies by G. I. Pokrovskiy (1955-1958) deals with the study of the destruction of rocks by means of an explosion. According to his concepts, the medium makes the transition to the plastic stage in some vicinity of the spot where the explosion occurs, while it is only subjected to compressive stresses. The zone where the cracks are formed follows this region in which the circular tensile stresses are acting.

G. I. Pokrovskiy emphasized the impossibility of the existence of a gap in the consolidation in grounds with a smoothly increasing compressional characteristic and poirted out the great effect of the free surfaces or the artificially created free cavities on the distribution of the destructive energy in the space. As soon as the compression wave arrives at the free surface, the compressed body begins to expand and a refraction wave is formed which is caused by the tensile stresses. In the acoustical approximation, this wave corresponds to the source of the tension which is the mirror image of the

charge with respect to the free surface. The reflected wave of the tensile stresses causes incomparably greater destruction than the compression wave. This mechanism is analogous to the knock mechanism. Depending on the mechanical properties of the rocks and the location of the charges, the relative portion of the direct and reflected wave will be different in the total destruction. On the basis of the general qualitative destruction pattern in simple computational schemes, G. I. Pokrovskiy proposed several useful formulas which are widely used in explosive work in a wide range of variations of the parameters.

Different concepts about the effects of explosion were developed by S. D. Osnovin (1939), A. F. Sukhanov (1950, 1958), M. P. Brodskiy (1953) and others. They assumed that a part of the energy is used up during the explosion in tearing-off the rock from the mass that is destroyed along the lateral surface of the crater, and that the other part is used up in overcoming the gravity of the volume that is being destroyed and in the crushing of the rock inside this volume. According to these concepts (which are also based on some additional assumptions) the similarity law that was mentioned above must be replaced by a different relationship (which is a generalization of the M. M. Frolov formula proposed already in 1868).

The approach that was mentioned was further developed in the studies by A. N. Khanukayev (1958, 1962) and V. N. Mosinets (1963, 1967). In particular, A. N. Khanukayev proposed to classify the destruction of the rock on the basis of acoustical characteristics (the best known classification of rocks on the basis of strength was given by M. M. Protod'yakonov in 1911). V. N. Mosinets formulated the general energy law for the crushing of rocks by means of an explosion, in accordance with which the destruction of rocks is characterized by the presence of a definite energy content for the crushing which depends on the mechanical propertis of the rocks, the statistical distribution for the natural crack and the cracks developed during the crushing deformations. The studies of these authors are characterized by a deep analysis of the mechanism for the energy transferred from the explosion to the rock mass which takes into account the physical-mechanical properties of the rocks which make up the monolith and its natural cracks.

A promising method for controlling the explosion are special artifical cavities (low yield explosions). Such cavities can be used as a protective screen which protects useful objects from destruction and also for reflecting the compression wave (and directing the reflected tensile stress wave to the given object which must be demolished). V. N. Mosinets (1963,

1967) found certain fundamental laws for the screening of stress waves. The experiments made by him have shown that it is possible to reflect in this manner in the direction of the demolished object up to 20-25% of the energy of the waves (67-72% is used up to destroy the material near the screen and only 8-10% passes through the screen).

In connection with the problem of protecting buildings from an explosion, the studies by M. A. Lavrent'ev, V. M. Kuznetsov and Ye. N. Sherr (1962), who developed a cavitational system for the protection from an explosion should be mentioned.

Explosions are used more and more in the construction of very large buildings which require more powerful well-calculated explosions. As the scale of the explosion increases, the similarity law is violated as a result of the effect of gravity, and it becomes necessary to use more accurate computational formulas which take into account the scale factor. G. I. Pokrovskiy proposed the corresponding corrections for the M. M. Boreskov formula for the calculation of the size of the charge. The experimental studies by M. M. Dokuchayev, V. N. Rodionov and A. N. Romashov (1963) with powerful ejection explosions, made it possible to determine a formula for the calculation of large charges taking into account their scale. The theory for the formation of cavities and the theory of rock movements during ejection explosions have been the subject of many investigations in the last few years. The most important results along these lines were obtained by F. A. Baum, L. K. Belopukhov, A. F. Belyayev, V. A. Vinogradov, O. Ye. Vlasov, M. M. Dokuchayev, V. M. Kuznetsov, M. A. Lavrent'ev, G. M. Lyakhov, L. N. Marchenko, G. I. Pokrovskiy, V. N. Rodionov, A. N. Romashov, K. P. Stanyukovich, I. S. Fedorov, A. A. Chernigovskiy, Ye. N. Sher, B. I. Shekhter.

Studies of directed rock ejection by an explosion are of particular interest. G. I. Pokrovskiy, I. S. Fedorov and M. M. Dokuchayev (1963) proposed to realize directed ejection by means of creating additional free surfaces, cavities or craters on the given ejection side. M. A. Lavrent'ev, Ye. N. Sher and V. M. Kuznetsov (1964), starting with a simple exact solution of the problem in a hydrodynamic formulation, proposed to use for this purpose a nonuniform distribution of the explosive charge along the depth of the holes (the thickness of the explosive layer must increase linearly with the depth). A. A. Chernigovskiy (1965) developed a variant of this method, using a special system of plane and wedge-shaped charges. Apparently the joint utilization of the ejection explosion methods that were mentioned is most effective.

A characteristic example of a powerful directed ejection explosion is the explosion that took place in October 1966 in Meadow, as a result of which a runoff dam was formed. In the process for several seconds before the explosion of the main charge (about 3700 tons of trotyl) four axuiliary charges were exploded (total weight about 1600 tons), creating an artificial auxiliary crater which ensured the directed ejection of the rock.

The use of powerful short-term pressures formed during a directed explosion for the creation of high velocity metallic jets (the hollow charge phenomenon) is exceptionally interesting. The hollow charge effect which was already discovered in the last century consists of the fact that, for instance, if, on the external surface of a metallic cone-shaped shell, the charge is distributed uniformly and is subsequently exploded, as a result of the explosion a thin jet (thread) is formed from the metal which moves along its own axis at a tremendous speed (on the order of 2-10 km/sec). A speed of about 100 km/sec was attained in hollow charge jet experiments in a vacuum.

The studies by M. A. Lavrent'ev, who developed a hydrodynamic theory of this phenomenon are fundamental in the theory of hollow charges. He determined on the basis of this theory the speed, thickness and length of the hollow charge jet that is formed as well as the speed and depth at which the initial jet penetrates the solid which lies on its path of motion.

Many interesting aspects of the use of an explosion (for example, during pressing, hardening of the structure of metals, catalytic acceleration of chemical reactions, etc), which merit a special discussion are outside the scope of this survey.

A problem which is allied to demolition during an explosion occurs in the study of the collision of bodies moving at high relative velocities (depending on the material from hundreds of meters to cosmic speeds on the order of tens of kilometers per second). Important results along these lines were obtained by L. V. Al'tshuler, F. A. Baum, M. I. Brazhnik, F. F. Vitman, L. A. Vladimirov, L. A. Galin, N. A. Zlatin, K. K. Krupnikov, M. A. Lavrent'ev, K. P. Stanyukovich, V. A. Stepanov, G. P. Cherepanov, B. I. Shekhter, and others. The extent to which this phenomenon is understood corresponds approximately to the general level of the theory of action of an explosion on the surface of a solid.

&12. Some Special Problems in the Mechanics of Fracture

The intense development of technology poses a series of new important problems in the mechanics of fracture. We first mention the problem of the effect of irradiation on the strength and rupture of solids (neutron and proton beams, powerful photoirradiation, high frequency magnetic and magnetic fields, etc.). In connection with the needs of rocket and space technology, the destruction of solid fuels which occurs under complex combustion conditions which is often the cause for the transition of the engine operation to an unstable regime is of great importance.[1] Certain problems related to new technological processes also occur in the study of known phenomena (disintegration of fluids, the effect of residual stresses and other problems).

The study of the strength of ideal structures is of fundamental importance in the mechanics of fracture and its many applications. By an ideal structure is meant a strictly periodic arrangement of the atoms in space (an ideal crystal lattice).

---

1. Specific problems in the mechanics of rupture which are intimately related to physical chemistry and gas dynamics occur in the study of certain unstable combustion processes of solid fuels which are especially important in rocket technology (I. Ye. Sorkin, 1964).

   One of the frequent reasons why the engine does not operate in the calculated regime is the presence of crack-like defects in the solid fuel which are too large and which can lead to unstable combustion. The instability mechanism consists of the following. When the combustion front approaches the edge of the crack-shaped cavity, the combustion quickly spreads over the surface of the cavity, since the pressure in the chamber is much greater than the original pressure in the cavity. As a result of the more difficult gas transfer, the local pressures and the temperature can increase sharply, especially at the end of the cavity. In addition to this, because of the specific structure of solid fuels in the end region that was mentioned, volume combustion can occur which, in combination with the rupture mechanism in this region, may lead to burn-outs or even an explosion.

Already the first theoretical strength estimate had a revolutionary effect on the development of the physics of strength and the mechanics of rupture. Further progress in this direction can apparently be achieved by using quantum mechanics concepts from the very beginning. Here we may expect considerable successes not only in a deeper and more precise study of the theoretical strength of known crystalline structures, but also in the discovery of new structures, for example, structures with a much higher strength.

We will briefly discuss some of the problems that were mentioned above.

12.1. Disintegration of Liquids

The upper bound for the volume strength of the fluid which is used is the absolute value of the maximum negative pressure which can be applied to the fluid. The upper bound for the volume strength varies within a wide range for different liquids. For example, at 20°C, this quantity is 24,500 kg/cm^2 for mercury, 3250 kg/cm^2 for water and 600 kg/cm^2 for ethyl ether. These numbers correspond to the concept of the disintegration or decomposition of the fluid which takes place simultaneously in the entire volume. However, the disintegration mechanism for fluids is much more complex. The fluid begins to disintegrate from the "weakest" link (such a link can be, for example, a gas bubble contained in the volume of the fluid). During the expansion of the fluid, the radius of the bubble and the surface energy of the liquid are increased and the gas pressure in it drops. A simple calculation shows that the dependence of the tensile stress p on the radius of the bubble R has the form given in Fig. 3. Thus, until the hydrostatic pressure reaches the maximum value, the gas bubble is stable. The loss of stability occurs at the instant when the hydrostatic pressure attains the maximum. The disintegration mechanism of fluids is analogous to the brittle rupture mechanism in solids.

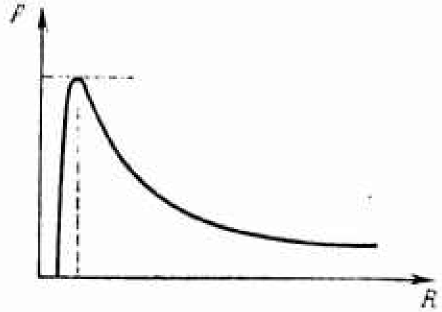

Fig. 3

A study of the strength of the fluid has shown that it decreases as the radius of the bubble increases. Hence, the gas bubble with the greatest radius from among all existing bubbles before the expansion is most dangerous. These concepts are confirmed experimentally by the considerable increase in the strength of the liquid when the gas bubbles are removed by means of applying large pressures (E. N. Harvey, V. D. MacElroy and A. G. Viteli, J. Appl. Phys., Vol. 18, No. 2, pp. 162-176).

However, the existence of small gas bubbles is difficult because of their solubility in the surrounding liquid, and, therefore, in this connection gas bubbles must be considered which are formed in the fluid during favorable fluctuations. Such a bubble increases when the elasticity of the liquid vapor p_v is greater than the external pressure p (the sum of the hydrostatic pressure and the surface tension force), and is reduced in the contrary case. The critical dimension of the bubble is

$$R^* = \frac{2\sigma}{p_{\text{п}_a} - p}. \qquad (12.1)$$

Key: a. v

Thus, the disintegration of the fluid subjected to the pressure $p_v - p$ occurs at the instant when gas bubbles with a radius $R > R^*$ are formed.

Experimental tests have shown that the strength of liquids is 5-6 orders of magnitude lower than the theoretical strength. One reason for this divergence is the fact that the disintegration of fluids occurs not in the volume but on the separation boundary of the liquid and any solid surface (the particles suspended in the liquid, the walls of the vessel, etc.).

Ya. I. Frenkel (1945) who introduced into the discussion the value of the angle on the edge (the damping angle) calculated the strength on the "solid-liquid" boundary during the rupture of the vapor bubble. The calculations have shown that the surface strength is reduced as the damping angle increases and that it becomes much smaller than the volume strength.

The volume strength of liquids increases monotonically with an increase in the temperature and becomes zero near the critical temperature, which agrees relatively well with experiments for

the majority of liquids, except water for which the maximum volume strength is attained at 6°C. The similarity between the disintegration mechanism in fluids and solids served as an impetus for making a number of experiments which have shown at sufficiently high deformation rates that the liquid behaves like a solid which can undergo brittle fracture.

M. O, Kornfeld and M. M. Ryvkin (1939) made a series of original experiments on a mixture of rosin and transformer oil in which the desired viscosity of the mixture was obtained by varying the rosin concentration. The disintegration of the jet flowing out through the bottom of the vessel was obtained with the aid of a Guillery impact machine. The experiments have shown that for small rates the deformation of the jet is plastic (laminar) and when the impact rate is about 23 m/sec brittle rupture of the jet occurs and the brittle rupture phenomenon can be observed at viscosities on the order of 10^3 poise, i.e., in substances which are known to be liquid.

We also mention here the interesting features of the unstable flow of polymer systems which leads, under certain conditions, to brittle rupture.

Experiments made on various polymer substances (solutions and melts elastomers, rubber mixtures, etc.), which were pressed from smooth samples have shown that the surface of the jet can vary, depending on the flow rate, from light turbidity (coarseness) to a complete decomposition of the jet into individual irregularly shaped pieces. To describe this phenomenon, various terms are used in the literature which attempt to emphasize the peculiar behavior of the surface of the jet in the given case under consideration ("shagreen," "orange skin," "disintegration" or "break-up" of the melt, "elastic turbulence" or "unstable flow"). By the break-up of the melt is meant the appearance of sharp fluctuations in the flow at the entry to the capillary which leads to considerable defectiveness. The study of the break-up of a melt is of great importance, for example, in the manufacturing technology of polymer materials, where the defectiveness of the jet hampers considerably the productivity of such processes as the extrusion of fibers, insulation for cables, etc.

In the experimental studies the flow occurred predominantly under isothermal conditions with a constant volume rate or drop in the pressure in which the shearing stress and the shearing rate were determined. The instant at which the unstable flow regime occurred corresponded to some critical values of the rate and shear stress.

G. V. Vinogradov, M. L. Fridman, et al. (1962) have shown that two critical points exist, the first of which corresponds to the beginning of the unstable flow and the second to the appearance on the jet of large periodic defects (at higher velocities). The dependence of these critical values on various parameters was studied extensively (such as the temperature, the relative length of the capillary and molecular characteristics of the polymer).

In the majority of studies in which the effect of the temperature was studied, it was shown that the critical value of the shear stress varies little when the unstable flow begins. In this case the dependence of the critical shear rate on the temperature was determined by the activation energy calculated from the largest Newtonian viscosity (G. V. Vinogradov and A. Ya. Malkin, 1965).

A study of the effect of the length of the capillary on the condition for the beginning instability has shown that the critical values of the velocity and shear stress increase as the length increases at constant pressure, which, in the opinion of many authors, is related to the disturbing effect of the inlet.[1]

When theoretical schemes were constructed, which take into account the phenomenon that was described above, a large number of experimental factors was taken into account which have shown that in many cases the unstable flow of polymers is related to their elasticity. The hypothesis on the main role played by the high elasticity when instability of the flow begins was proposed for the first time by G. P. Tordell (Trans. Soc. Rheol., Vol. 1, 1957, pp. 203-212) and E. B. Begli (Trans. Soc. Rheol., Vol. 5, 1961, pp. 355-368).

1. G. V. Vinogradov, A. Ya. Malkin and V. F. Shumskiy proposed in 1968 the hypothesis that this is caused by the excessively slow stabilization of the stationary values of the normal stresses and deformations in the displacement flow in comparison with the tangential stresses. However, the effect of the geometry of the inlet zone is rather qualitative than quantitative. In other words, the critical conditions remain the same regardless of the angle of the inlet cone (in a sufficiently wide range of variation) and the intensity with which the defects appear in the jet depends on the flow conditions at the inlet to the capillary.

G. V. Vinogradov, A. I. Leonov and A. Ya. Malkin (1963), without considering the detailed instability mechanism, proposed as the condition for the beginning instability, the attainment of a certain ratio between the elasticity and viscosity forces in the flow. For a system with one relaxation time, this criterion represents the product of the characteristic relaxation time and the displacement rate, and in the simplest case, it is equal to the critical value of the elastic deformation. Experimental studies made on different polymer systems have shown that the beginning instability is described satisfactorily by this criterion. Thus, the highly elastic deformation which accumulates in the flow determines the critical flow conditions.

Recently attempts were made in the studies by V. A. Gorodtsov, A. V. Karakin, A. I. Leonov and S. A. Regirer to obtain a solution of the stability problem of flow of elasto-viscous media and to solve individual special problems.

We note that under certain conditions some analogy was noted between the brittle rupture phenomenon in bodies and the beginning instability of the flow.

G. F. Hutton (Nature, Vol. 200, No. 4907, 1963, pp. 646-648, Proc. Roy. Soc., London, A287, No. 1409, 1965, pp. 222-239) proposed a criterion for the beginning instability of the flow which is analogous to the Griffith criterion during the formation of cracks in an elastic medium. However, E. B. Bagley and coauthors (Nature, Vol. 203, No. 4941, 1964, pp. 175-176) subjected this criterion to criticism, since the Griffith criterion does not take into account the energy dissipation during irreversible deformations, which plays an important role in elasto-plastic media.

12.2. Effect of Residual Stresses and Loading Rates on the Strength of Solids[1]

It is known that residual stresses[2] exist in bodies regardless of the external effects (forces and temperature effects)

---

1. A detailed discussion of the problems that are touched on here can be found in the monographs by F. F. Vitman (1933), I. Ye. Kantorovich and L. S. Livshits (1943), P. M. Gur' (1947), I. V. Kudryavtsev (1951), Ya. B. Fridman (1952), M. A. Babichev (1955), B. A. Kravchenko (1962), A. D. Monasevich (1962), V. V. Abramov (1963), I. A. Birger (1963).

2. They are also called internal, natural or original stresses.

which are found as a result of the nonuniformity of the linear or three-dimensional deformations in adjacent volumes of the material. Depending on the dimensions of the latter, we distinguish macro- micro- and ultramicroscopic stresses (stresses of the first, second and third kind). The first scientific studies on residual stresses were made by H. Rodman (1857), I. A. Umnov (1871) and N. V. Kalkutskiy (1887), who were the first to propose a method for the measurement of internal stresses. However, for a long time these studies went unnoticed and only beginning inthe 20's of our century serious attention was given to the study of problems related to internal stresses.

Apparently, no standard classification of internal stresses exists. The most complete and accurate classification was proposed by N. N. Davidenkov (1936), which was made more precise by B. M. Rovinskiy (1948, 1949). The report by E. Orowan at the Symposium on Internal Stresses in Metals and Alloys (London, 1948) in which the definition of internal stresses corresponded to the concept of an "internal stress" introduced already by N. V. Kalakutskiy, is devoted to the problem of the "classification and nomenclature of internal stresses." Without analyzing these problems, we can arbitrarily divide the residual stresses into macrostresses and microstresses, depending onthe rate of change of the stresses along the space coordinate. Macrostresses are stresses in the material which vary negligibly within the dimensions of the grain. Below, we will touch on certain problems connected with the effect of the residual stresses on the strength and deformations in parts which take into account the action of macroscopic stresses.

A necessary condition for the formation of internal stresses is the occurrence of the non-uniformity of the deformed state at various points of the body (violation of the compatibility condition of the deformations). This nonuniformity may be the result of various causes: the nonuniform thermal stressing or compression during the nonuniform heating or cooling of the body, phase conversions leading to nonuniform volume changes (tempering, hardening, cooling after welding, etc.), nonuniform plastic deformation, etc.

The complexity of studying the laws for the appearance of residual stresses is related to the necessity of taking into account the mechanical, thermal and physical-chemical factors, which have an effect on the behavior of the technological process.

The problem of determining the residual stresses formed during the metallurgical and technological processes is extremely complex since its solution requires a theoretical study of the associated physical-chemical processes. Promising studies along these lines were begun by Ya. S. Podstrigach (1964 and later publications). Generally the operational residual stresses can be calculated within the frame of reference of the corresponding mechanical models of a continuous medium with an irreversible reaction.

We will consider the effect of residual stresses on the strength during static and variable loads. A large amount of experimental data indicates the strong effect of the residual stresses on the reliability and endurance of structures and buildings. The fracture of the latter (often in the beginning of the operation) at a relatively low value of the acting stresses is sometimes explained by the disadvantageous distribution of the residual stresses. Experiments have shown that for plastic materials, the residual stresses do not have a great effect on the magnitude of the rupturing force, and that the plastic deformation formed by the single external loads, leads to a reduction or even complete vanishing of the residual stresses.

A study of the effect of the residual stresses on the static strength of brittle materials has shown that the magnitude of the fracturing load is usually lower than the value of the same load in the absence of residual stresses. The small plastic deformations which occur before the rupture do not eliminate the residual stresses, and given the tendency of the material to brittle rupture, the effect of the residual stresses may be considerable.

To reduce the residual stresses and harmful effect on the brittle strength, the parts are usually subjected to special thermal treatment.

Various parts are often subjected during the operation to the action of variable stresses. Then in the general case the expression for the stresses which vary according to the asymptotic cycle has a constant and variable component:

$$\sigma = \sigma_m + \sigma_v f(\tau). \qquad (12.2)$$

where $f(\tau)$ is a periodic function of dimensionless time $(-1 \leq f(\tau) \leq 1$, σ_v is the variable stress, and σ_m is the constant stress.

It is known that the residual stresses may vary under the action of cyclic loads. If the sum of the residual stresses and the variable stresses is larger than the elasticity limit of the material, plastic deformations which reduce the residual stresses are formed on the cyclic load. In addition, in the case when the plastic deformation caused by the variable stresses exceeds the magnitude of the original residual stresses, the residual stresses may change sign (L. A. Glikman, 1956).

In the case when the sum of the stresses (the residual and variable stresses) is smaller than the elasticity limit of the material, the residual stresses vary little under the action of variable loads.

Experimental studies have shown that the residual stresses are reduced in the surface layers, which are weaker by their physical nature. In this case, to preserve the residual stresses the surface layers are hardened and shot blating is used, which leads to the formation of residual compression stresses and an increase in the fatigue strength of the parts (S. V. Serensen, 1950, I. V. Kudryavtsev, 1951, M. M. Kobrin, 1954, M. M. Severin, 1955). Thus, the residual compression stresses increase the fatigue strength while the tensile stresses have an unfavorable effect. An increase in the fatigue strength during considerable residual stresses manifests itself to a large extent in less plastic materials in the presence of stress concentrations.

The phenomenon of residual deformations during thermal cycles was studied in the studies by A. A. Bochvar, et al. (1957), A. A. Bochvar and G. I. Thomson (1957), N. N. Davidenkov and V. A. Likhachev (1960).

The effect of the residual stresses on the rupturing process was studied by V. A. Lomakin, Ya. S. Podstrigach, S. F. Yur'ev (mechanical studies which determine the values of the residual stresses connected with volume changes), by S. P. Borisov, N. A. Borodin (relaxation problems in residual stresses), by V. P. Kogayev, M. N. Stepnov (quantitative laws for the increase and decrease of the load-bearing capacity related to the stress fields, the temperature-time factor) and many others.

In strength calculations, sometimes it is necessary to take into account the loading rates, since, under real conditions, the deformation processes occur at various rates, all the way from the smallest rates (for example, under long-term creep conditions) to very high rates (for example, in those

cases when the plastic deformation and rupture end in a minute fraction of a second). The bulk of the studies along these lines are the experimental studies related to the determination of the mechanical properties of dynamically deformable materials. The most complete survey of the studies made along these lines can be found in the monograph by L. P. Orlenko (1964) and also in the book by P. M. Ogibalov and I. A. Kiyko (1966), which provide information about the behavior of the materials under superintense effects.

The static and dynamic characteristics of Armco iron and different steels under impact and wrapping loading are described in the monograph by Yu. Ya. Voloshenko-Klimovitskiy (1965).

Generally the effect of the loading rate reduces to the following: as the loading rate increases the relative role played by the plastic effect is reduced and the rupture becomes more brittle.

The effect of the loading rate in carbon steels was investigated most extensively. Two yield points are introduced (the upper and lower) and it turns out that the upper yield point is most sensitive to a change in the loading rate. In order to have an idea about the magnitude of this effect, we note that when the loading rate is increased by one order of magnitude, the upper yield point in carbon steels is increased approximately by 4 kg/mm^2. Thus, when the loading rate is increased by 5 orders of magnitude, which corresponds roughly to a transition from static to impact loading, the upper yield point increases by 20 kg/mm^2. It is clear that for soft (low carbon) steels this effect is considerable and for high strength (high carbon) steels it can be ignored.[1]

1. A survey of experimental studies on lagged flow in steels was given in 1968 by Yu. V. Suvorova. At the same time Yu. N. Rabotnov proposed a general three-dimensional model for an elastoplastic medium with delayed flow. Within the frame of reference of this model he and Yu. V. Suvorova solved several concrete dynamic problems.

P. I. Skokov and V. P. Belyayev (1966), applying the statistical theory of strength studied the problem of the increased resistance to deformation with the increase in the deformation rate. The authors have shown that an increase in the deformation rate leads to a more uniform distribution of the stresses over the cross section of the sample.

G. I. Pogodin-Alekseyev and B. A. Artamonov (1964) compared the stress-strain diagrams constructed using the method of deformation characteristics and oscillographs from samples that were not cut during the study of structural steels subjected to different kinds of thermal treatment. V. I. Bugay and V. P. Troshchenko (1966), who continued the investigations along these lines, taking into account the experimental data have shown that the dissipation of the energy under elasto-plastic deformation conditions depends on the degree of heterogeneity and the type of stressed state (for example, according to the data of F. S. Savitskiy (1964), the energy losses under elasto-plastic shock conditions in bending are approximately 7%, and under impact stressing, 5%).

V. Ya. Moroz, A. V. Popov and Yu. P. Sogrishin (1964) studied the effect of the deformation rate on the plasticity of different steels and also on aluminum and other alloys. The authors divided these materials into three groups, depending on the reaction to an increase in the deformation rate.

The static and dynamic tests of copper samples made by A. G. Bobrov, A. I. Nikolayeva and Ye. O. Shvaykovskaya (1964) have shown that during a dynamic load, the crystal lattice is less distorted than in the case of the static tests.

Unlike in metals and the majority of natural materials, polymers are characterized by a stronger effect of such factors as the deformation rate, the temperature and time effects.

The study by K. A. Kerimov (1965) has shown on the example of rubber and polyvinyl that the dynamic "stress-strain" curves which are nearly lines lie above the static curves and that in the region of stresses, which are nearly zero, the residual deformations from the dynamic loads may exceed the static deformations by a factor of three.

The study of various plastics (epoxypolyester resin, polyvinyl-butyryl, glass-textolite) enables N. P. Ivanov and V. A. Stepanov (1965) to detect a relatively high correlation between the impact strength and the temperature (an increase in the impact rate by a factor of 10^7 under raised temperature

conditions leads to an increase in the strength by a factor of 3-6, whereas at a temperature of -196° for the same increase in the rate, the strength is only increased by 25%).

The study by S. M. Kokoshvili and V. P. Tamuzh (1966) investigated the effect of the deformation rate on polyformaldehyde samples. The results have shown that an increase in the deformation rate leads to an increase in the strength of the materials whereas the ductility does not change, for all practical purposes, which leads to an increase in the rupture energy.[1]

12.3. Fracture under the Effect of High Frequency Electrical and Magnetic Fields

Recently, both in the Soviet Union and abroad, electrophysical methods used in the rupture of various materials have been investigated intensely. These required a detailed study of electro-physical and other properties of the materials subjected to the effect of electromagnetic fields and electric charges. The first studies along these lines deal with the electrothermal destruction of dielectric materials during dielectric heating. In the 40's G. I. Babat, A. V. Varzin, et al., studied on the example of rocks the behavior of dielectric materials in high frequency electric fields between flat electrodes. It was shown that the high voltage electric fields that are formed under the electrodes lead to the dielectric heating of the regions adjacent to the electrodes and the formation of temperature stresses (for rocks, this leads to the cracking of large pieces and the splitting of the rock from the monolith). The effectiveness of the rupture depends considerably on the agreement in the parameters of the oscillatory circuit of the generator with the properties in the material (the load). In addition, the electric field which is formed in a certain way may ensure the splitting of the material along a given splitting line ("directed high frequency rupture"). Many experiments in this field were made on rocks by V. S. Kravchenko, A. P. Obraztsov, V. M. Semenov, and others (1961-1963). The directed high frequency rupture may be particularly valuable, for example, in such problems as the dielectric heating of rocks containing valuable inclusions, which requires

---

1. Sufficiently complete information dealing with studies of the effect of the rate on the mechanical properties and strength of metals and polymers is available in the published 1968 survey by M. I. Reitman and G. S. Shapiro.

a selective destruction of empty rocks and preservation of valuable rocks.

The attempts to intensify the destructive process relative to simple dielectric heating, led to the discovery of the high frequency thermal breakdown phenomenon.

The theory of high frequency thermal breakdown and electrothermal fracture must relate the physical properties of the material subjected to the crushing to the parameters of the field which determines the thermal breakdown and fracture conditions. Attempts to construct such a theory for rocks (ferrous, granular quartz, and similar rocks) were made by V. D. Itskhakin, A. P. Obraztsov and V. V. Ustinov (1962-1964) but the complexity of the structure and the anisotropy of the rocks, the variability of their characteristics and the complex dependence of the latter on the temperature, the stressed field and the frequency for different samples, made it only possible to obtain some qualitative results.

Recently, in connection with the successes in the field of high power electronics, which made it possible to design powerful high frequency generators, wave guides and radiators the study of the fracture of materials by super high frequency (SHF) radio waves began. In the 50's G. I. Babat, A. V. Varzin, et al., have shown that the super high frequency radio wave (on the order of 3000 MHz) incident to sandstone causes the splitting off of thin layers from its surface. Using a 5 kwt magnetron, an impulsive splitting takes place in intervals of several tens of seconds and when the power of the magnetron is increased 3 times, the destruction productivity is increased 6 times. The depth and the distribution of the electromagnetic energy flow in the material depends on the length of the electromagnetic waves, the manner in which they are supplied and the electrophysical properties of the material. During the action of electromagnetic waves from a distance they can be focused at some depth which may lead to the phenomenon of the splitting of the material ("radio wave breakage"). This phenomenon was verified experimentally on a granite sample by V. S. Kravchenko, A. P. Obraztsov, et al., (1965).

The thermal factor is not only the cause of the fracture in the case of a heat flux (jet burner or plasmatron) but also in the case of radio wave destruction. However, radio wave heating occurs in the entire volume pierced by the electromagnetic flux while, during thermal heating, the heat propagates into the depth mainly as a result of thermal conductivity.

Conversions leading to a weakening of the strength, all the way to complete destruction of the grain of the material, or to the weakening of the intercrystalline bonds under the effect of high frequency electromagnetic fields and also the effect of the splitting of mica in high frequency electric fields, are of particular importance in rock science. The search for the most effective methods advanced the problems of the destruction of rocks by industrial-frequency currents (thermal breakdown, the method of an electric arc and melting of the rock) and impulsive electrical charges which differ in their dynamic and explosive character.

12.4. Effect of Neutron Radiation on the Mechanical Properties, Strength and Rupture of Solids[1]

Many experiments dealing with radioactive irradiation have shown considerable changes both in the chemical and mechanical properties of materials, where in many cases these changes can only be restored with difficulty and are preserved for a long time. This fact required not only the development of technological protective measures from harmful effects, but also the development of new methods for the calculation of structural elements and buildings subjected to radioactive radiation (atomic reactors, artificial satellites, spacecraft and space stations).

The main method for the experimental study of radioactive radiation having an effect on the strength characteristics of the material, is determining the spectrum of the natural frequencies of the sample and the change in the logarithmic damping decrement. A large amount of experimental data on radioactive radiation has shown the insignificant change in the modulus of elasticity, while the strength (and especially the yield) are exceptionally sensitive to radiation. The common factor for metals during irradiation is the heterogeneity of the elasto-plastic properties, the outward displacement of the stress-strain diagram, the tendency towards embrittlement, and, in the majority of cases, the reduced strength in the plastic masses.

The effect of the radiation on high-molecular substances is of particular interest. For moderate radiation doses, plastics (for example, polyethylene) are strengthened whereas other substances lose strength and become brittle all the way to conversion into powder. However, under large radiation doses

1. More detailed information on these problems is available in the monograph by P. M. Ogibalov and I. A. Kiyko (1966).

almost all plastics are destroyed which brings to the fore the problem of reinforcing the plastics and producing radiation-resistant polymers.

The most complete list of studies dealing with the changes in the properties of materials and under the effect of radioactive radiation which include a description of some physical mechanisms for this phenomenon is available in the surveys by F.Bowie (1959) and V. S. Lenskiy (1960). The survey by V. S. Lenskiy proposed a generalization of the theory of small elasto-plastic deformations to the case of an heterogeneous medium caused by the nonuniformity of the radiation.

The irradiation of the substance by a neutron flux causes a series of complex structural changes and conversions. The primary effects consist of the displacement of the atoms from the lattice points and also of the excitation of atoms and electrons without displacement and nuclear conversions, and the secondary effect is the ionization effect.

Internal stresses may occur in the body due to different physical processes (for example, as a result of the three-dimensional elongation that is formed in the body. Generally, the different three-dimensional elongation that occurs at various points of the body leads to the formation of internal stresses even in the absence of external loads.

Yu. I. Remnev (1958, 1959) considered the relation between the stresses and the small deformations in a crystalline solid during three-dimensional elongation caused by irradiation by heavy particles and proposed a number of hypotheses which made it possible to determine this elongation. He studied neutron irradiation as well as the bombarding neutron which passed through the crystal lattice without interacting with the atoms of the Coulomb forces, causing the greatest damage. It was assumed that as a result of the irradiation, the mechanical properties of the material (the Young modulus, the yield point, etc.) may vary and that the isotropy of the material is preserved. A. A. Il'yushin and P. M. Ogibalov (1960) proposed methods for calculating the strength of shells in a thick-walled cyclinder and a smooth sphere. As in the studies by Yu. I. Remnev, it was assumed here that the drop in the neutron flux is proportional to the energy and the thickness of the layer and that the properties of the body at a given point depend on the radiation dose at this point.

A. G. Zhuravlev (1961, 1962) assumed in studies dealing with the determination of the stressed and deformed state of light metals during irradiation, in addition to the assumption that there are no nuclear reactions and that the two hypotheses that were mentioned above are satisfied, that the nonuniformity of the elastic properties in the body can be ignored. This is due to the presence of the experimentally verified facts that the elastic properties change little in comparison with the changes in the plasticity and strength characteristics, which makes it possible to use the ordinary equations of elasticity theory to calculate the stresses and strains.

12.5. Fracture under the Effect of Powerful Photoirradiation

New aspects in the study of the strength and fracture of solids were discovered in connection with the use of lasers which can generate very powerful laser beams. The studies of the stressed state during the passage of a laser ray were begun in transparent polymers (organic glasses). The passage of the laser ray is accompanied by complex physical phenomena and for a particular power of the pulse it leads to the rupture of the transparent material. At the present time relatively few physical-mechanical studies of this phenom[illegible]re available. In this regard, the problems of determining [illegible] parameters having an effect on the fracture as well as [illegible]rmination of the laws for the conversion of the energy of [illegible] electro-magnetic oscillations into mechanical stress are [illegible] from completed.

The first study along these lines was the study by B. M. Ashkinadze, V. I. Vladimirova, V. A. Likhachev et al., (1966), who determined the formation of plane cracks with a circular contour approximately at a 45° angle to the ray axis. The cracks are also formed on the path of the laser beam reflected from the boundary, when the latter is internally fully reflected. The exceptionally high energy concentration during the formation of the cracks which is generally comparable to the energy concentration must be mentioned. In the opinion of the authors of the study that was mentioned, the fracture occurs under the effect of the coherent hypersound generated by the laser. The rupture of the material at a 45° angle occurs under the effect of the limiting and transverse phonons which are directly generated by the light wave.

At the same time V. V. Volkov, V. A. Likhachev, et al. (1967), V. A. Likhachev, V. M. Salganov, et al. (1968) note that the main effect in these processes is heat liberation.

The studies by G. I. Barenblatt, N. N. Vsevolodov, O. Ye. Marin, et al. (1967, 1968) proposed the hypothesis that dynamic stresses are formed at the initial fracture instant in the region of the channel as a result of heating and the hypersound effect. The formation of the stresses leads to the formation of tiny shear defects in the plane of the largest tangential stresses which are at a 45° angle to the ray axis. The formation of these defects is accompanied by the light energy absorption which leads to the formation of gas bubbles with a high internal pressure and temperature which determines in advance the subsequent development of the cracks.

We note that the rupture of transparent dielectric substances under the effect of powerful laser radiation occurs in a time period on the order of 10^{-5} - 10^{-8} sec. The rupture begins at the local inhomogeneities and it depends both on the power and on the energy of the light wave.

Some concepts about the character of the rupture of transparent polymers have also been proposed in the studies by B. M. Ashkinadze, V. A. Likhachev, et al. (1966, 1967), B. F. Ponomarenko, V. I. Samoylov, et al. (1968).

END